Modular Representation Theory and the CDE Triangle

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$$\begin{array}{ccc} P_{\overline{\mathbb{F}}_p}(G) & \xrightarrow{\quad c \quad} & R_{\overline{\mathbb{F}}_p}(G) \\ & {\scriptstyle e}\searrow \qquad \nearrow{\scriptstyle d} & \\ & R_{\overline{\mathbb{Q}}_p}(G) & \end{array}$$

SENIOR HONORS THESIS

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Introduction

"Representation theory is like a George R.R. Martin novel. There are lots and lots of characters; most of them are complex, and many of them are unfaithful. But everything gets a lot tensor: there are a lot of duals, and some of the characters end up decomposing!"

This paper is my senior honors thesis, written under the direction of Professor Akshay Venkatesh. The senior thesis is a significant part of the requirements for the honors major in mathematics at Stanford, and serves the dual purpose of educating a student on an advanced topic in mathematics and producing an exposition of that topic. My thesis will discuss modular representation theory and a collection of results on modular representations encapsulated in a diagram called the CDE triangle (1.5.1), the figure on the title page. Specifically, in this paper, I will present the results from modular representation theory needed to state and prove these results; then, I will use these results in several explicit examples, calculating modular character tables and the matrices defining the CDE triangle for several groups.

Though this is a senior honors thesis, the prerequisites will be relatively benign. In order to talk about modular representation theory, one of course has to use results from the representation theory of finite groups, and this prerequisite will be pretty important. I will also use some module theory (e.g. projective modules, simple modules, and tensor products) in an important way. There will also be scattered other dependencies from algebra, such as some basic properties of the p-adic numbers; these are less central than having already seen representation theory of finite groups in characteristic 0.

Overview and Major Results. Though representation theory is the study of group actions on vector spaces, the most common choices for these vector spaces are those over fields of characteristic zero, or positive characteristic that doesn't divide the order of the group G. Modular representation theory concerns itself with the other choices, where the characteristic of the base field divides the order of the group.

This seemingly small change makes a world of difference, as Maschke's theorem (Theorem 1.1.1), the cornerstone of representation theory in characteristic 0, no longer holds, as explored in Section 1.1. This makes it much more complicated to describe how a representation breaks down into irreducible representations. In order to have a general theory, we need ways of working around this, such as working with semisimplifications (which do decompose cleanly) and the modular characters discussed in Section 1.3. Then, we can use the CDE triangle to connect the characteristic 0 theory and the characteristic p theory for a particular group G.

This paper is divided into two chapters; Chapter 1 provides an exposition of the theory, and Chapter 2 provides examples of the modular characters and explicit calculations of the morphisms in the CDE triangle.

Here's a summary of the important results contained in the first chapter. Let G be a finite group and p be a prime dividing $|G|$.

(1) Though not all representations in positive characteristic are semisimple, taking the semisimplification, defined in Section 1.1, still provides a lot of information about finitely generated $\overline{\mathbb{F}}_p[G]$-modules. Specifically, the following are equivalent for two such modules M and N.
   - M and N have the same semisimplification.
   - For any $g \in G$, the actions of g on M and N have the same characteristic polynomial. (Proposition 1.2.7)
   - The modular characters of M and N are the same. (Corollary 1.3.5)
   - M and N define the same class in the Grothendieck group $R_k(G)$. (Corollary 1.5.2)

(2) Brauer's theorem (Theorem 1.3.4), that the modular characters of the simple $\overline{\mathbb{F}}_p[G]$-modules are a basis for the space of class functions on p-regular elements of G, and therefore (Corollary 1.3.6) there are as many simple $\overline{\mathbb{F}}_p[G]$-modules as there are conjugacy classes of G with order not divisible by p.

(3) The results on the CDE triangle, including:
- Given a representation in characteristic 0, there's a way to produce a representation in characteristic p that's well-defined up to semisimplification (Section 1.2); moreover, reduction induces a surjective morphism on Grothendieck groups (Theorem 1.5.9), which means that every $\overline{\mathbb{F}}_p[G]$-module can be lifted to characteristic zero in the form of a "virtual module," i.e. a linear combination of simple modules in characteristic zero.
- To obtain all of the irreducible modular characters, it suffices to take the irreducible characters in characteristic zero, reduce them mod p, and then decompose them into irreducibles; in this sense, the characteristic 0 theory contains all of the information from characteristic p, and also makes calculating modular character tables much simpler.

Then, in the second chapter, we work this out in several explicit cases: the cyclic group $\mathbb{Z}/p$; the symmetric groups S_3, S_4, and S_5; the alternating group A_4; the dihedral group D_{10}; the group $\mathrm{GL}_2(\mathbb{F}_3)$; and finally p-groups such as D_8 and Q_8. In each of these cases, we use reduction of characters from characteristic 0 to obtain the table of modular characters, and then use this to calculate the three maps in the CDE triangle; in the simpler cases, we can also explicitly describe the projective indecomposable modules, rather than just their characters.

A standard reference for this subject is Serre's *Linear Representations of Finite Groups* [**5**].

Some Applications. Representation theory over $\mathbb{R}$ or $\mathbb{C}$ is sometimes motivated by its applications to physics, as it is very useful in quantum mechanics and particle physics. However, this application goes away when one passes to positive characteristic, since the symmetries physicists describe with representation theory tend to be as Lie groups acting on Euclidean space. Nonetheless, modular representation theory is useful in several areas of mathematics.

For example, results from modular representation theory are used in the classification of the finite simple groups. The proof of the Brauer-Suzuki theorem [**1**] rests on the relationship between the ordinary and modular characters through the decomposition homomorphism as well as the linear independence of the modular characters (Theorem 1.3.4). The Z^* theorem [**2**], another important result in the theory of finite simple groups, also depends on calculations with modular characters.

Results from modular representation theory also appear in algebraic topology: the Adams conjecture is a statement about real vector bundles over CW complexes, and yet its proof in [**4**] involves the characteristic p representation theory of the finite groups $\mathrm{GL}_n(\mathbb{F}_p)$ and $\mathrm{O}_n(\mathbb{F}_p)$.

Notational Conventions. In this paper, we will use the following notational conventions.

- G will always denote a finite group, and $|G|$ will denote its cardinality. p will be a prime number dividing $|G|$, and $G_{\mathrm{reg}}^{(p)}$ is the set of p-regular elements of G, as defined in Section 1.3.
- $\mathbb{Z}/p$ denotes the cyclic group of order p, and $\mathbb{Z}_p$ denotes the p-adic integers.
- S_n denotes the symmetric group on n letters; A_n denotes the alternating group on n letters; D_{2n} denotes the dihedral group with $2n$ elements; and Q_8 denotes the quaternion group.
- K will denote a field of characteristic 0, often $\overline{\mathbb{Q}}_p$, and k will denote a field of characteristic p, often $\overline{\mathbb{F}}_p$. L will be a field of any characteristic.
- The ring of integers of K is denoted $\mathscr{O}_K$. In this paper, $\mathscr{O}_K$ will always be a discrete valuation ring, so $\mathfrak{m}$ will denote its maximal ideal, and π will denote a uniformizer for $\mathfrak{m}$.
- Λ will denote a lattice inside K, as defined in Section 1.2.
- If A is a ring, $A[G]$ will denote the group algebra.
- All modules in this paper are left modules, and will usually be denoted M and N.
- $R_L(G)$ and $P_k(G)$ are the Grothendieck groups defined in Section 1.5.
- χ will denote the character of a representation in characteristic 0; ϕ_i will be used to denote the modular characters of the simple $k[G]$-modules; and Φ_i will be used for the characters of the projective indecomposable $k[G]$-modules.
- $C\ell(G,K)$ denotes the space of K-valued class functions on G.

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CHAPTER 1

The Theory Behind Modular Representations

"As much as we often prefer to work over $\mathbb{C}$, modular representation theory has some very positive characteristics."

1.1. Irreducibility and Indecomposability in Positive Characteristic

In representation theory, it's often extremely useful to know how to build a representation out of smaller or less complicated representations. Irreducibility and indecomposability are two basic examples of this.

Definition. Let V be a representation of a group G. Then, a subspace W of V is *G-stable* if the action of G sends W to itself, i.e. $g \cdot w \in W$ for all $g \in G$ and $w \in W$.

Definition.

- A G-representation V is *irreducible* if there are no G-stable subspaces W of V other than 0 and V itself.
- A G-representation V is *indecomposable* if it is not isomorphic to a direct sum of G-representations $W_1 \oplus W_2$.
- A representation that isn't irreducible is called *reducible*, and similarly, one that is not indecomposable is called *decomposable*.

Here are some quick properties.

- Irreducibility implies indecomposability: if $V \cong W_1 \oplus W_2$ as G-representations, then $W_1 \oplus \{0\} \subsetneq V$ is G-stable.
- An irreducible representation of G over a field k is a *simple* $k[G]$-module (a module whose only submodules are 0 and itself).
- If V is reducible, with W a G-stable subspace, then $0 \to W \to V \to V/W \to 0$ is exact.

Furthermore, in characteristic 0, these two notions are equivalent.

Theorem 1.1.1 (Maschke)**.** *Let G be a finite group and k be a field such that* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) = p \nmid |G|$. *If V is a representation of G over k and V has a G-stable subspace W, then there exists a representation W' of G over k such that $V = W \oplus W'$ (as G-representations).*

This theorem is proven in [**3**, Ch. XVIII, § 1]; the key step in the proof of this theorem is averaging an action of the elements of G, which involves dividing by $|G|$. Thus, this doesn't work when $\mathrm{char}(k)$ divides $|G|$. Moreover, when $\mathrm{char}(k) = p$ does divide $|G|$, there are representations that are reducible, but indecomposable.

Example 1.1.2. Let V be the two-dimensional representation of $\mathbb{Z}/p = \langle x \rangle$ over $\mathbb{F}_p$, where x^n acts as $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$; this is a representation because $0 \mapsto I$ and

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m+n & 1 \end{pmatrix}.$$

Then, the subspace

$$W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{F}_p \right\}$$

is $\mathbb{Z}/p$-stable, but V is not decomposable: if it were, then there would exist some other $\mathbb{Z}/p$-stable subspace W', so that $V = W \oplus W'$, but suppose that

$$\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} cx \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cnx + y \end{pmatrix} = \begin{pmatrix} \lambda cx \\ \lambda y \end{pmatrix}$$

for some $\lambda \in \mathbb{F}_p$ and all $n \in \mathbb{F}_p$; then, since $cx = \lambda cx$, we have $\lambda = 1$, and therefore $cx = 0$ (when $n = 1$), but this means that $c = 0$, so we recover W as the only stable subspace. In particular, this means V is indecomposable, but not irreducible.

This discrepancy between indecomposability and irreducibility is very important in modular representation theory.

Another basic fact about irreducibility is the Jacobson density theorem.

Theorem 1.1.3 (Jacobson density theorem)**.** *Let k be an algebraically closed field and G be a finite group. Then, the $k[G]$-module map*

$$k[G] \longrightarrow \bigoplus_{M \in S_k} \operatorname{End}_k(M),$$

where S_k is the set of isomorphism classes of simple $k[G]$-modules, is surjective.

For a proof, see [**3**, Ch. XVII, § 3].

1.1.1. The Semisimplification of a Module. In addition to simple modules, we can also consider *semisimple modules*, which are defined to be those modules isomorphic to a direct sum of simple modules. A representation V over a field k is a semisimple $k[G]$-module if every G-stable subspace is a direct summand, i.e. if $W \subseteq V$ is G-stable, then there exists a $W' \subseteq V$ such that $V \cong W \oplus W'$. Thus, Maschke's theorem informs us that all representations in characteristic zero are semisimple modules, and Example 1.1.2 demonstrates that not all representations over fields of positive characteristic are semisimple.

Nonetheless, if V is any finite-dimensional representation and $W \subset V$ is G-stable, then V/W is also a G-representation, and, assuming W isn't trivial and isn't all of V, then, both have dimension strictly less than V. Thus, these can be thought of as less complicated representations that build together to form V. If one keeps repeating this process on W and V/W, it terminates eventually (since one-dimensional representations are irreducible), providing a finite set of irreducible G-representations known as its *composition factors*.

Lemma 1.1.4. *The composition factors of an f.g. $k[G]$-module V are unique up to isomorphism and reordering.*

Proof. Let $\{M_1, \dots, M_m\}$ and $\{N_1, \dots, N_n\}$ be two sets of composition factors for V. It's sufficient to prove that $M_1 \cong N_j$ for some j; then, induction takes care of the rest.

The key observation is that if M and N are two G-stable subspaces of a $k[G]$-module V, then it doesn't matter whether one quotients by M before $N/(M \cap N)$ or quotients by N followed by $M/(M \cap N)$. Notice that $M \cap N$ is also G-stable, since if $x \in M \cap N$, then $g \cdot x \in M$ and $g \cdot x \in N$ for all $g \in G$, and therefore, e.g. $M/(M \cap N)$ is a G-stable subspace of V/N.

- If we consider M first, the components are M and V/M, so then taking N, which has become $N/(M \cap N)$ in the quotient, we get M, $N/(M \cap N)$, and $(V/M)/(N/(M \cap N))$. Finally, we can decompose M into $M \cap N$ and $M/(M \cap N)$.
- If one chooses N first, the same factors result, but with M and N switched: $M \cap N$, $N/(M \cap N)$, $M/(M \cap N)$, and $(V/N)/(N/(M \cap N))$.

Three of these are the same: $M \cap N$, $M/(M \cap N)$, and $N/(M \cap N)$. The remaining ones are isomorphic as well: by the second isomorphism theorem of modules, $M/(M \cap N) \cong (M+N)/N$, so $(V/M)/(N/(M \cap N)) \cong (V/M)/((M+N)/M) \cong V/(M+N)$ by the third isomorphism theorem of modules, and applying it to the module arising from the other choice (with M and N switched) also produces $V/(M+N)$, so they're isomorphic.

To obtain a set of composition factors of V, one chooses a G-stable submodule M and quotients by it, and then repeats. Without loss of generality, we can assume that M is simple; if not, it contains a simple submodule that the algorithm will eventually get to, and we just showed that order doesn't matter in this algorithm, so we may place the simple submodule first, and get the same answer.

Thus, there's an ordered list of G-stable simple modules which, when one applies this algorithm to it, produces $\{M_1, \dots, M_m\}$, and another such ordered list which yields $\{N_1, \dots, N_n\}$. But we just showed that the order doesn't matter, so we can regard the M_i and N_j are drawn from the same finite set of G-stable simple submodules of V (which is finite because V is finitely generated). But since each of the M_i and N_j are simple, M_1 must be a submodule or a quotient of one of the N_j. If it's a submodule, then $M_1 = N_j$, since N_j is simple, and if it's a quotient, then we proceed one step further in the algorithm, so M_1 must be a submodule or quotient of another $M_{j'}$. Since the algorithm terminates after a finite number of steps, then $M_1 = N_j$ for some j. ⊠

Definition. The *semisimplification* of a finitely generated $k[G]$-module is the direct sum of its composition factors.

Lemma 1.1.4 guarantees this is well-defined up to isomorphism. Moreover, since two semisimple modules are isomorphic iff they have the same composition factors, then the semisimplification of a module M is the unique semisimple module that has the same composition factors as M. Additionally, we know that if M is a semisimple module, then M is its own semisimplification.

1.2. Reduction of Representations

The simplest way to obtain representations of a group in positive characteristic is from a representation in characteristic zero. We want to take a representation "mod p," which is possible when it's represented by matrices with integer entries, but for anything else there's a problem. Additionally, it'll be better to have a description that is independent of basis. So we need a systematic way of sending representations from an algebraically closed field of characteristic zero to one of characteristic p. This suggests that the base field should be the p-adics $\mathbb{Q}_p$, because it's relatively easy to describe how $\mathbb{Z}_p$ sits inside $\mathbb{Q}_p$, and $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$.

Thus, the formal process of reducing a representation looks like this: let G be a finite group, p be a prime dividing $|G|$, and ρ be a complex (finite-dimensional) representation of G. Then, since G is finite, ρ may be realized over $\overline{\mathbb{Q}}$.

Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, so since we have an embedding $\iota : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$, we can take $\overline{\mathbb{Q}}$ to be the algebraic closure of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}_p$ with respect to this embedding, and therefore ρ is also a representation over $\overline{\mathbb{Q}}_p$ (that is, it's a map into $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$). Since ρ is in fact a representation over a finite extension over $\mathbb{Q}$, it is also a representation over a finite extension of $\mathbb{Q}_p$.

For the rest of this section, let K denote a finite extension of $\mathbb{Q}_p$, and let $\mathscr{O}_K$ denote its ring of integers. Thus, $\mathscr{O}_K$ is a discrete valuation ring, and K is its fraction field; let π be a uniformizer for $\mathscr{O}_K$ (i.e. a generator for its maximal ideal).

Definition. A *lattice* in a K-vector space V is the $\mathscr{O}_K$-span of a K-basis for V.

This is the same idea as a $\mathbb{Z}$-lattice inside a $\mathbb{Q}$-vector space, and the geometric intuition carries over, e.g. given two lattices, there's a common lattice contained in both.

Lemma 1.2.1. *The sum of two lattices is a lattice; that is, if V is a K-vector space and L_1 and L_2 are lattices in V, then $L_1 + L_2 = \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\}$ is also a lattice.*

Proof. TODO ⊠

Applying the lemma n times shows that any finite sum of lattices is also a lattice.

Proposition 1.2.2. *G-stable lattices exist over $\mathbb{Q}_p$. That is, if $\rho : G \to \mathrm{GL}(V)$ is a representation over K, then there exists a lattice $L \subseteq V$ such that $g \cdot L \subseteq L$ for all $g \in G$.*

Proof. Let L be any lattice in V, and for any $g \in G$, define $gL = \{gv \mid v \in L\}$. Since $\rho(g)$ is invertible, then it must have full rank, and in particular sends bases of V to bases of V, so gL is still a lattice.

Then, let
$$L' = \sum_{g \in G} gL.$$
By Lemma 1.2.1, L' is a lattice. An arbitrary element of L' has the form
$$\ell = \sum_{g \in G} g v_g \text{ for some } v_g \in L,$$
so
$$h \cdot \ell = \sum_{g \in G} (hg) v_g.$$
Multiplication by h is a (setwise) bijection $G \to G$, so as g ranges over all elements of G, so does hg. In particular, $h \cdot \ell \in L'$, so L' is G-stable. ⊠

Now, returning to ρ and V, choose a G-stable lattice L in V. Since L is G-stable, the action of ρ on L fixes it, which means that ρ is actually a map into $\mathrm{GL}_n(\mathscr{O}_K)$. This means that, in a basis, the entries of the matrices of ρ are in $\mathscr{O}_K$, so, reducing them mod p yields matrices with coefficients in $k = \mathscr{O}_K/\pi\mathscr{O}_K$, which is a finite field extension of $\mathbb{F}_p$. Thus, we have obtained a representation $\overline{\rho} : G \to \mathrm{GL}_n(k)$.

This operation, called *reduction* of a representation mod p, depends on the lattice chosen.

Example 1.2.3. Let's explicitly reduce a representation of S_3, the symmetric group on three elements. S_3 acts on 3-tuples of elements of $\mathbb{Q}_3$ by $\sigma \cdot (a_1, a_2, a_3) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. Since this action preserves the sum $a_1 + a_2 + a_3$, then it preserves the subspace of 3-tuples summing to zero, so restricting to the vector space V of 3-tuples summing to zero, we obtain a two-dimensional representation.

If $v_1 = (1, -1, 0)$ and $v_2 = (0, 1, -1)$, then $\{v_1, v_2\}$ is a basis for V. In this basis, this representation can be described in terms of matrices.

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (a\ b) \longmapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (a\ b\ c) \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{1.2.1}$$

In particular, $L_1 = \mathbb{Z}_3 \cdot \{v_1, v_2\}$ is a lattice in V. Since the action of each element of S_3 maps v_1 and v_2 to $\mathbb{Z}_3$-linear combinations of v_1 and v_2, then L_1 is S_3-stable, so we can reduce.

In order to take the reduction, we simply need to reduce mod 3, so the group elements have the same matrix representations as in (1.2.1), though now the matrices are in $\mathrm{GL}_2(\mathbb{F}_3)$ rather then $\mathrm{GL}_2(\mathbb{Q}_3)$.

However, if one chooses a different lattice, the resulting representation might not be isomorphic. Consider $v_3 = (-2, 1, 1)$ and $v_4 = (1, -1, 0)$. Then, $\{v_3, v_4\}$ is a $\mathbb{Q}_3$-basis for V, so its $\mathbb{Z}_3$-span is a lattice; call it L_2. In this basis, our representation takes on the following form.

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (a\ b) \longmapsto \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} \qquad (a\ b\ c) \longmapsto \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix} \tag{1.2.2}$$

Just as above, since the coefficients of these matrices are in $\mathbb{Z}_3$, then the action of each element of S_3 sends v_3 and v_4 to $\mathbb{Z}_3$-linear combinations of them, so L_2 is S_3-stable. Thus, we may once again reduce mod 3, producing a matrix representation of S_3 over $\mathbb{F}_3$:

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (a\ b) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (a\ b\ c) \longmapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{1.2.3}$$

However, it turns out these two representations aren't conjugate, and thus are actually nonisomorphic. If they were, then there would be some 2×2 matrix that simultaneously conjugates the matrices in (1.2.1) into (1.2.3). The identity matrix is preserved by all conjugation, so let's consider the matrices for $(a\ b\ c)$. If we know that

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

then

$$\begin{pmatrix} -c & -d \\ a-c & b-d \end{pmatrix} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix},$$

so $b + c = a - b = -d$. Solving these equations in $\mathbb{F}_3$ yields four matrices:

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$-A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad -B = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Since the sign of the conjugating matrix makes no difference, we really only have two options. However, out of these two options, only B conjugates $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ to $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. However, it doesn't work for the matrices for $(a\ b)$:

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \text{but} \qquad \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Thus, there is no matrix that conjugates the first representation to the other, so they're not actually isomorphic.

Though the above representations are nonisomorphic, the characteristic polynomial of the automorphism associated to each group element is independent of the lattice chosen: if $\chi_{1,\sigma}$ denotes the characteristic polynomial of $\sigma \in S_3$ acting as in the first representation, given by the matrices in (1.2.1), and $\chi_{2,\sigma}$ denotes that of the second representation, given by the matrices in (1.2.3), then:

- $1 \in S_3$ is sent to the same matrix by each representation, so $\chi_{1,1} = \chi_{2,1}$.
- $\chi_{1,(a\ b)}(\lambda) = (1+\lambda)(1-\lambda) + (0)(1) = 1 - \lambda^2$, and $\chi_{2,(a\ b)}(\lambda) = (1+\lambda)(1-\lambda) - (0)(0) = 1 - \lambda^2$, so they're equal.

- $\chi_{1,(a\ b\ c)}(\lambda) = \lambda(\lambda+1)+1 = \lambda^2+\lambda+1$, and $\chi_{2,(a\ b\ c)}(\lambda) = (\lambda-1)^2-(1)(0) = \lambda^2-2\lambda+1 = \lambda^2+\lambda+1$ in $\mathbb{F}_3$, so they're equal.

This is no accident.

Proposition 1.2.4. *The characteristic polynomial of the reduction of a representation does not depend on the choice of lattice.*

Proof. Let ρ be a representation of G over K, g be an element of G, and L be a G-stable lattice. Let $\chi_{g,K}$ denote the characteristic polynomial of the representation over K; then, since L is G-stable, then in the basis given by L, the matrix for g has coefficients in $\mathscr{O}_K$, and in particular, the coefficients for $\chi_{g,K}$ are in $\mathscr{O}_K$. However, the coefficients of a characteristic polynomial don't depend on the choice of basis, and therefore don't depend on the choice of lattice. Thus, when the matrix M for g is reduced, the characteristic polynomial is the determinant of $M - \lambda I$, but since this is a polynomial in λ and the entries of M, then when these entries are reduced mod p (i.e. are sent to k by modding out by $\pi\mathscr{O}_K$), the new determinant is just the original determinant with all of the coefficients taken mod p as well. This also doesn't depend on the lattice chosen, so the characteristic polynomial of the reduced representation is independent of the lattice chosen. ⊠

In the above proof, we reduced the character of a representation along with the representation itself, by taking the coefficients of the characteristic polynomial modulo $\pi\mathscr{O}_K$. We'll end up using this again.

Proposition 1.2.5. *If $\chi_1, \dots, \chi_\ell$ are linearly independent class functions for a group G over K, then if $\overline{\chi}_i$ denotes the reduction of χ_i modulo $\pi\mathscr{O}_K$, then the $\overline{\chi}_1, \dots, \overline{\chi}_\ell$ are linearly independent over k.*

Proof. Suppose there is a dependence relation between them:

$$\sum_{i=1}^{\ell} \alpha_i \chi_i = 0,$$

with $\alpha_i \in K$. Since K is the fraction field of $\mathscr{O}_K$, and there are finitely many α_i, we can find a common denominator and multiply by it, making everything in $\mathscr{O}_K$. Thus, without loss of generality, we can assume that $\alpha_1, \dots, \alpha_n \in \mathscr{O}_K$. Moreover, we can assume at least one isn't in $\pi\mathscr{O}_K$: if all of them are, then divide by the highest power of π that divides all of them.

The next step is to reduce mod π; let $\overline{\alpha}_i = \alpha_i \bmod \pi$, and $\overline{\chi}_i$ denote the function $g \mapsto \chi_i(g) \bmod \pi$. Since at least one of the α_i isn't in $\pi\mathscr{O}_K$, we still have a nontrivial linear dependence relation as k-valued class functions on G. ⊠

Corollary 1.2.6. *The reductions of the irreducible characters for a group G are linearly independent over k.*

This is because the irreducible characters are linearly independent.

Proposition 1.2.4 tells us that the characteristic polynomial of the reduction is well-defined, but it also completely determines the semisimplification.

Proposition 1.2.7. *Let M and N be f.g. $k[G]$-modules such that for every $g \in G$, the characteristic polynomials for $(g\cdot) \in \mathrm{End}_k(M)$ and $(g\cdot) \in \mathrm{End}_k(N)$ are equal. Then, the semisimplifications of M and N are isomorphic.*

Proof. Let $\{S_i\}_{i\in I}$ denote the set of simple f.g. $k[G]$-modules, up to isomorphism, and assume that there exist some pair of modules M and N whose characteristic polynomials are equal for all $g \in G$, but that are not themselves isomorphci. Then, let m_i be the multiplicity of S_i in the composition factors for M, and n_i be the multiplicity of S_i in the composition factors for N. In particular, choose M and N to be minimal over all such pairs of $k[G]$-modules, as a pair that minimizes $\sum_{i\in I} m_i + n_i$. Since M and N are finitely generated, then this is a finite sum, so such a minimum exists.

Since each $g \in G$ has the same characteristic polynomial for M and for N, then the characters of M and N are the same; in particular, if $\theta_i(g)$ denotes the characteristic polynomial of g acting on S_i, then the characters are

$$\chi_M(g) = \sum_{i\in I} m_i \operatorname{Tr}(\theta_i(g)) = \sum_{i\in I} n_i \operatorname{Tr}(\theta_i(g)) = \chi_N(g).$$

Thus, by Proposition 1.2.5, the irreducible characters are independent, so $n_i \equiv m_i \bmod p$ for each i.

Now, let's look at the characteristic polynomials themselves. Since they're equal, their ratio is 1, i.e.

$$\prod_{i\in I} \theta_i(g)^{n_i - m_i} = 1.$$

But since $n_i - m_i \equiv 0 \bmod p$, then each term in the product is a p^{th} power. And since we're in characteristic p, if $f(x)^p = 1$, then $f(x) = 1$ (since there are no other p^{th} roots), and therefore

$$\prod_{i\in I} \theta_i(g)^{(n_i - m_i)/p} = 1.$$

But this means that we can choose smaller values for the sum of the m_i and n_i and still have the characteristic polynomials for the resulting M and N be equal, which is a contradiction, because we assumed we chose the minimal one. ⊠

In summary, reduction is a way of obtaining representations over $\overline{\mathbb{F}}_p$ from representations over $\overline{\mathbb{Q}}_p$. It will be very useful for concrete calculations.

1.3. Modular Characters

As in the previous section, let G be a finite group, and p be a prime dividing $|G|$. We'll again be interested in reducing representations from $K = \overline{\mathbb{Q}}_p$ to $k = \overline{\mathbb{F}}_p$.

Definition. An element g of G is *p-regular* if $p \nmid |g|$. Similarly, a conjugacy class is p-regular if its elements are (since they all have the same order). The set of p-regular elements of G is denoted $G^{(p)}_{\text{reg}}$.

Let m denote the least common multiple of the orders of the elements of $G^{(p)}_{\text{reg}}$, and let $\mu_K \subset K$ and $\mu_k \subset k$ denote the sets of m^{th} roots of unity in each respective field. Then, since $p \nmid m$, μ_k and μ_K both have m elements, and reduction mod p is an isomorphism $r : \mu_K \to \mu_k$.

If M is an n-dimensional representation of G, then any $g \in G^{(p)}_{\text{reg}}$ defines an action $(g\cdot) \in \operatorname{Aut}(M)$. Its eigenvalues $\lambda_1, \dots, \lambda_n$ must lie in μ_k, as $g^m = \text{id}$, and if λ is an eigenvalue of g, then λ^m must be an eigenvalue of id, i.e. 1. Thus, one can define

$$\phi_M(s) = \sum_{i=1}^{n} r^{-1}(\lambda_i),$$

which defines a function $\phi_M : G^{(p)}_{\text{reg}} \to K$.

Definition. This function ϕ_M is called the *Brauer character* or *modular character* of M.

Here are some quick but important properties.

Proposition 1.3.1.

(1) $\phi_M(1) = \dim(M)$.
(2) ϕ_M *is a class function on* $G^{(p)}_{\text{reg}}$*, i.e. if* $g \in G^{(p)}_{\text{reg}}$ *and* $h \in G$*, then* $\phi_M(hgh^{-1}) = \phi_M(g)$.
(3) If $0 \to L \to M \to N \to 0$ *is a short exact sequence of* $k[G]$*-modules, then* $\phi_M = \phi_L + \phi_N$*. Thus, as a special case,* $\phi_{M\oplus N} = \phi_M + \phi_N$ *for* $k[G]$*-modules* M *and* N.
(4) If M *and* N *are* $k[G]$*-modules, then* $\phi_{M\otimes N} = \phi_M \cdot \phi_N$.

Proof. TODO, or see Serre, pp. 147-48.

Since the list of composition factors of a module is generated by taking submodules and quotients, then applying part (3) of the above proposition yields the following.

Corollary 1.3.2. *Let* M *be an f.g.* $k[G]$*-module and* $N_1, \dots, N_m$ *be its composition factors. Then,* $\phi_M = \phi_{N_1} + \cdots + \phi_{N_m}$.

Then, there are a few more results which tie these characters more explicitly to representations.

Proposition 1.3.3. *Let* M *be a* $K[G]$*-module with character* χ_M*, and let* $\overline{M}$ *denote its reduction as a* $k[G]$*-module. Then,* $\phi_{\overline{M}} = \chi_M|_{G^{(p)}_{\text{reg}}}$.

That is, reduction sends ordinary characters to modular characters!

Proof. Let $g \in G^{(p)}_{\text{reg}}$. Then, the action of g in $\operatorname{Aut}(M)$ has trace $\chi_M(g) = \lambda_1 + \cdots + \lambda_n$ for $\lambda_i \in \mu_K$, since this action has order dividing m and is invertible, as per the above discussion.

Thus, $\phi_{\overline{M}}(g) = \chi_M(g) \bmod p$, because the characteristic polynomial for g acting on $\overline{M}$ is just that of g acting on M modulo p. But reduction mod p is an isomorphism $\mu_K \to \mu_k$, so $\phi_{\overline{M}}(g) = \chi_M(g)$. ⊠

Theorem 1.3.4 (Brauer)**.** *The irreducible modular characters $\{\phi_M \mid M$ is a simple $k[G]$-module$\}$ are a basis for the K-vector space $C\ell(G_{\mathrm{reg}}^{(p)}, K)$ of class functions $G_{\mathrm{reg}}^{(p)} \to K$.*

Proof. First, we must show that they span $C\ell(G_{\mathrm{reg}}^{(p)}, K)$. Given any class function $f : G_{\mathrm{reg}}^{(p)} \to K$, it's possible to extend it to a class function $\widetilde{f} : G \to K$ (e.g. letting $\widetilde{f}(g) = 0$ if $g \notin G_{\mathrm{reg}}^{(p)}$). Then, since the characters of simple $K[G]$-modules are a basis for $C\ell(G, K)$, then there's a linear combination

$$\widetilde{f} = \sum_i \lambda_i \chi_{M_i}$$

where each M_i is a simple $K[G]$-module and $\lambda_i \in K$, and if we restrict to the p-regular conjugacy classes, we get the same relation for f. Then, reduce each M_i mod p to get a $k[G]$-module $\overline{M}_i$; by Proposition 1.3.3 we now have the relation

$$f = \sum_i \lambda_i \phi_{\overline{M}_i}.$$

For a given i, $\overline{M}_i$ may not be a simple $k[G]$-module, but it has some finite list of composition factors $N_{i,1}, \dots, N_{n,m_i}$, which are simple $k[G]$-modules. Then, by Corollary 1.3.2,

$$f = \sum_i \sum_{j=1}^{m_i} \lambda_i \phi_{N_j},$$

so f is a linear combination of characters of simple $k[G]$-modules.

Next, we will show that the irreducible modular characters $\phi_1, \dots, \phi_\ell$ are linearly independent. Since these are class functions $G_{\mathrm{reg}}^{(p)} \to K$, it's possible to extend each one to a class function $\widetilde{\phi}_i : G \to K$ by letting $\widetilde{\phi}_i(g) = 0$ if $g \notin G_{\mathrm{reg}}^{(p)}$, and $\widetilde{\phi}_i(g) = \phi_i(g)$ otherwise. In particular, the $\widetilde{\phi}_i$ are linearly independent if and only if the ϕ_i are.

By Proposition 1.2.5, the $\widetilde{\phi}_i$ are linearly independent iff their reductions $\overline{\phi}_i$ are, so suppose there is a dependence relation between the $\overline{\phi}_i$:

$$\sum_i \alpha_i \overline{\phi}_i = 0,$$

with the $\alpha_i \in k$.

Let j be such that $\alpha_j \neq 0$. Then, by the Jacobson density theorem (Theorem 1.1.3), there exists an $x \in k[G]$ such that $\overline{\phi}_i(x) = 0$ when $i \neq j$ and is nonzero when $i = j$, i.e. x acts as a projection. With this x, the dependence relation is $\alpha_j \overline{\phi}_j(x) = \alpha_j \cdot c_j = 0$ for a nonzero c_j, which is a contradiction. In particular, the $\overline{\phi}_i$ are linearly independent, so the $\widetilde{\phi}_i$ are, and therefore the irreducible modular characters ϕ_i are linearly independent as well. ⊠

In characteristic zero, two representations of the same group G with the same character are isomorphic. This isn't true in positive characteristic: the two $k[S_3]$-representations from Example 1.2.3 aren't isomorphic, but have the same characteristic polynomials, and therefore the same characters. The key is once again semisimplification.

Corollary 1.3.5. *If two $k[G]$-modules M and N have the same modular character, then their semisimplifications are isomorphic.*

Proof. Since M and N have the same character, then by Theorem 1.3.4, this character is a linear combination of the irreducible modular characters in a unique way. However, it is also the sum of the characters of the composition factors of M and the sum of the characters of the composition factors of N, so the composition factors of M and N must be the same up to isomorphism. Thus, since they're semisimple, M and N are both isomorphic to direct sums of the same set of composition factors, and thus are isomorphic themselves. ⊠

In particular, when building character tables, the following corollary is extremely useful.

Corollary 1.3.6. *The number of irreducible modular characters, and therefore the number of simple $k[G]$-modules up to isomorphism, is equal to the number of p-regular conjugacy classes of G.*

Proof. If there are ℓ classes of G that are p-regular, then a class function on $G_{\mathrm{reg}}^{(p)}$ is a choice of ℓ elements of K, and therefore a basis for this space has ℓ elements. ⊠

It'll also be useful to have the following criterion.

Theorem 1.3.7 (Brauer-Nesbitt)**.** *Let n be the largest power of p dividing $|G|$ and M be a simple $K[G]$-module such that $n \mid \dim(M)$. Then, if $\overline{M}$ is a reduction of M, $\overline{M}$ is a simple $k[G]$-module.*

(Sometimes, the theorem is stated differently, so as to include another result, but this part is the only bit we'll need.)

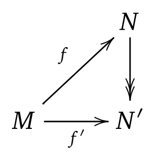
PROOF. TODO ⊠

1.4. Projective Modules

Projective modules are important in modular representation theory: when k has characteristic dividing $|G|$, projective $k[G]$-modules are particularly well-behaved, and if $\mathrm{char}(k) = 0$ or doesn't divide $|G|$, all $k[G]$-modules are projective, so the existence of nonprojective $k[G]$-modules is another important difference that arises in modular representation theory.

First of all, let's recall the definition of projective.

Definition. If A is a ring, then a left A-module M is a *projective module* if maps out of M can be lifted across surjections; that is, for every surjection $N \twoheadrightarrow N'$ of left A-modules and A-module homomorphism $f' : M \to N'$, there exists an A-linear $f : M \to N$ such that the following diagram commutes.



The following equivalent criterion will also be useful.

Lemma 1.4.1. *An A-module M is projective iff it is a direct summand in a free module.*

TODO: may as well cite.

Definition.

- A surjection $f : M' \twoheadrightarrow M$ of A-modules is *essential* if there is no proper submodule $M'' \subsetneq M'$ such that $f(M'') = M$.
- A *projective envelope* of an A-module M is the data of a projective A-module P and an essential homomorphism $f : P \to M$.

Proposition 1.4.2 (Serre, §14.3, Prop. 41(a))**.** *If A is Artinian, then every A-module has a projective envelope.*

TODO: possibly prove.

In particular, $\mathscr{O}_K[G]$ and $k[G]$ are both Artinian, so $\mathscr{O}_K[G]$-modules and $k[G]$-modules have projective envelopes.

Proposition 1.4.3 (Serre, §14.4, Prop. 42(b))**.** *If M is a projective $k[G]$-module, then there is a unique (up to isomorphism) projective $\mathscr{O}_K[G]$-module P such that $P/\pi P \cong M$.*

PROOF. First, let $\mathfrak{m} = \pi\mathscr{O}_K$ denote the maximal ideal of $\mathscr{O}_K$, and $A_n = \mathscr{O}_K/\mathfrak{m}^n$, so that $\mathscr{O}_K = \varprojlim A_n$.

- A_n and $A_n[G]$ are Artinian rings. TODO

Since $A_n[G]$ is Artinian, then by Lemma 1.4.2, M has a projective envelope P_n as an $A_n[G]$-module. Let $\varphi_n : P_n \twoheadrightarrow M$ denote the essential map; then, $\varphi_n(\mathfrak{m}P_n) = 0$, because M is a $k[G]$-module, so the action of $\mathfrak{m}\mathscr{O}_K$ is trivial on M, and therefore the action of $\mathfrak{m}P_n$ is as well. Thus, $\varphi_n' : P_n/\mathfrak{m}P_n \twoheadrightarrow M$ is surjective as well.

Since M is projective, then the identity map $M \to M$ lifts across the surjection φ_n' to a map $\psi_n : M \to P_n/\mathfrak{m}P_n$ such that $\varphi_n' \circ \psi_n = \mathrm{id}$; thus, ψ_n must be injective. In particular, its image M' is is isomorphic to M, and $\varphi_n'(M') = M$. However, if P' denotes the inverse image of M' under reduction mod $\mathfrak{m}$, then $\varphi_n(P') = M$ too, so since φ_n is essential, then $P' = P_n$, so $M' = P_n/\mathfrak{m}P_n$, so φ_n' is an isomorphism $P_n/\mathfrak{m}P_n \xrightarrow{\sim} M$. Thus, if $P = \varprojlim_n P_n$, then

$$P/\mathfrak{m}P \cong \varprojlim P_n/\mathfrak{m}P_n = \varprojlim M = M.$$

For uniqueness, let P_1 and P_2 be projective $\mathscr{O}_K[G]$-modules and let $M_1 = P_1/\mathfrak{m}P_1$ and $M_2 = P_2/\mathfrak{m}P_2$, so that M_1 and M_2 are projective $k[G]$-modules. Let pr_1 denote reduction $P_1 \twoheadrightarrow M_1$, and pr_2 be analogous for P_2 and M_2. Then, suppose there is an isomorphism $\overline{\varphi} : M_1 \xrightarrow{\sim} M_2$; then, we will show that there is an isomorphism $\varphi : P_1 \xrightarrow{\sim} P_2$.

First, let's lift $\overline{\varphi}$ to a map $P_1 \to P_2$. In the diagram

$$\begin{array}{ccc} P_1 & & P_2 \\ \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ M_1 & \xrightarrow{\overline{\varphi}} & M_2, \end{array}$$

P_1 is projective, so we may lift $\overline{\varphi} \circ \mathrm{pr}_1$ across pr_2 to obtain an $\mathscr{O}_K[G]$-module homomorphism $\varphi : P_1 \to P_2$.

Choose a basis $\{\overline{x}_1, \dots, \overline{x}_n\}$ for P_1, and a preimage $x_i \in \mathrm{pr}_1^{-1}(\overline{x}_i)$ for each i. Then, $\mathrm{pr}_1 \circ \varphi$ sends $\{x_1, \dots, x_n\}$ to a basis of M_2, so, by Nakayama's lemma, $\{\varphi(x_1), \dots, \varphi(x_n)\}$ generates P_2. Thus, φ is surjective.

Repeating the above construction with P_1 and P_2 switched and M_1 and M_2 switched means there is also a surjection $P_2 \twoheadrightarrow P_1$; thus, P_1 and P_2 are isomorphic.

To apply this to the proposition, let $M_1 = M_2$, $\overline{\varphi} = \mathrm{id}$, and let P_1 and P_2 be two lifts of M_1. Then, the above argument implies there is an isomorphism $\varphi : P_1 \xrightarrow{\sim} P_2$ such that reduction commutes with φ, and so the lift P of a projective $k[G]$-module is unique. ⊠

Though not all simple $k[G]$-modules are projective, there's a similar notion that will play an important role in later sections.

Definition. A *projective indecomposable* is a projective $k[G]$-module that is indecomposable (i.e. it doesn't split as the direct sum of two $k[G]$-modules).

1.5. The CDE Triangle

Some of the most powerful statements in modular representation theory, relating representations in characteristic zero to those in characteristic p, can be encapsulated in a commutative diagram called the CDE triangle.

Let G, K, k, p, and $\mathscr{O}_K$ be as in the previous section. To have maps between representations in characteristic p and those in characteristic zero, i.e. $k[G]$-modules and $K[G]$-modules, there needs to be some sort of structure that parameterizes these modules, and we'd like reduction to be well-defined on this structure, so two reductions of the same representation should have the same representative. This motivates the following definition.

Definition.

- Let R be a ring and $\mathscr{C}$ be a full subcategory of the category of finitely generated left R-modules. Then, the *Grothendieck group* of $\mathscr{C}$, denoted $\mathrm{GG}(\mathscr{C})$, is the abelian group with the following generators and relations.
  - For each object M in $\mathscr{C}$, there is a generator $[M] \in \mathrm{GG}(\mathscr{C})$.
  - For each short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{C}$, add the relation $[M] = [M'] + [M'']$.

  The class of an R-module M in the Grothendieck group is denoted $[M]$.
- If L is a field, G is a group, and $\mathscr{F}$ denotes the category of finitely generated $L[G]$-modules, the Grothendieck group $\mathrm{GG}(\mathscr{F})$ is denoted $R_L(G)$. If A is a ring and $\mathscr{P}$ denotes the category of finitely generated projective $A[G]$-modules, the Grothendieck group $\mathrm{GG}(\mathscr{P})$ is denoted $P_A(G)$.

Notice that not every element of $\mathrm{GG}(\mathscr{C})$ is the class of a module, e.g. $-[M]$. However, they're linear combinations of classes of modules.

Lemma 1.5.1. *If $N_1, \dots, N_m$ are the composition factors of a module $M \in \mathscr{C}$, then in $\mathrm{GG}(\mathscr{C})$, $[M] = [N_1] + \cdots + [N_m]$.*

Proof. Let's induct on m, the number of composition factors; this is vacuously true if $m = 1$.

More generally, suppose M has m composition factors, where $m > 1$. Then, there exists a nontrivial submodule $N \subsetneq M$, and therefore M/N is nontrivial as well. Then, the set of composition factors of M is the union of those of N and those of M/N, and since each is nontrivial, then each has strictly fewer composition factors than M does, so by induction, it's true for N and M/N. Thus, $[M] = [N] + [M/N] = [N_1] + \cdots + [N_m]$. ⊠

Corollary 1.5.2. *If two modules $M_1, M_2 \in \mathscr{C}$ have the same compsition factors up to isomorphism, then $[M_1] = [M_2] \in \mathrm{GG}(\mathscr{C})$.*

Proof. Let $\{N_1, \dots, N_m\}$ be these composition factors; then, $[M_1] = [N_1] + \cdots + [N_m] = [M_2]$. ⊠

Finally, it'll be useful to have a criterion for checking when a map is well-defined on a Grothendieck group.

Lemma 1.5.3. *Let H be an abelian group and $\Phi : \mathscr{C} \to H$ be a function. If for every exact sequence $0 \to M' \to M \to M'' \to 0$, we have $\Phi(M) = \Phi(M') + \Phi(M'')$, then there is a unique group homomorphism $\Phi^* : \mathrm{GG}(\mathscr{C}) \to H$ such that $\Phi^*([M]) = \Phi(M)$ for all $M \in \mathscr{C}$.*

In this case, Φ is called *additive*.

Proof. The relations for $\mathrm{GG}(\mathscr{C})$ are all of the form $[M] = [M'] + [M'']$ for such short exact sequences, and so such a Φ^* commutes with all relations in $\mathrm{GG}(\mathscr{C})$, so it is well-defined. Thus, such a Φ^* exists; it is unique because it's specified on the basis for $\mathrm{GG}(\mathscr{C})$. ⊠

Proposition 1.5.4. *If L is a field, the set S_L of isomorphism classes of simple f.g. $L[G]$-modules is a basis for $R_L(G)$.*

Proof. Let R be a free abelian group with S_L as a basis. Then, there's a map $\alpha : R \to R_L(G)$ given by sending a simple module $M \in S_L$ to its class $[M]$.

Given any finitely generated $L[G]$-module M, let $N_1, \dots, N_m$ be its composition factors, with multiplicity, so that $[M] = [N_1] + \cdots + [N_m]$ in $R_L(G)$ and each N_j is simple. Then, let $\beta : R_L(G) \to R$ be the map sending $[M] \mapsto N_1 + \cdots + N_m$ (and then extending linearly to the entire group). This is an additive map, because if $0 \to N \to M \to M/N \to 0$ is a short exact sequence, then the set of composition factors of M is the union of those of N and of M/N, up to isomorphism, so by Lemma 1.5.3, β is a well-defined group homomorphism.

Finally, α and β are inverses: if $M \in S_L$, then $\beta(\alpha(M))$ is equal to M again, since it's simple, so M is its only composition factor, and if $[M] \in R_L(G)$ has composition factors $N_1, \dots, N_m$, then $[M] = [N_1] + \cdots + [N_m] = \alpha(N_1 + \cdots + N_n) = \alpha(\beta(M))$. ⊠

Proposition 1.5.5. *The set S_P of classes of f.g. projective indecomposables is a basis for $P_k(G)$.*

Proof. TODO: this doesn't depend on too much more. Specifically, I would want to say that:

- Taking projective envelopes commutes with finite direct sums.
- Semisimplification doesn't change the projective envelope.
- The projective envelope (of a simple $k[G]$-module, or generally) is unique up to isomorphism.
- The projective indecomposables are the projective envelopes of the simple $k[G]$-modules.

This would probably go into the previous section somewhere. ⊠

Now, we can introduce the CDE triangle, which is the following diagram of abelian groups.

$$\begin{array}{ccc} P_k(G) & \xrightarrow{\quad c \quad} & R_k(G) \\ & {}_{e}\searrow \quad \nearrow_{d} & \\ & R_K(G) & \end{array} \tag{1.5.1}$$

We already know what the objects are, so now we must define c, d, and e.

1.5.1. Definition of $c : P_k(G) \to R_k(G)$. Let P be an f.g. projective $k[G]$-module; then, sending P to its class in $R_k(G)$, i.e. as an f.g. $k[G]$-module, is additive, because if $0 \to P' \to P \to P'' \to 0$ is short exact, then in $R_k(G)$, $[P] = [P'] + [P'']$. Thus, by Lemma 1.5.3, c is a well-defined group homomorphism $c : P_k(G) \to R_k(G)$; often, it is called the *Cartan homomorphism*.

1.5.2. Definition of $d : R_K(G) \to R_k(G)$. Given a finitely generated $K[G]$-module M, i.e. a representation of G in characteristic zero, we defined how to reduce it to a representation in characteristic p, i.e. a $k[G]$-module. There are two issues of well-definedness to check.

First, as we saw in Example 1.2.3, the reduction of a $K[G]$-module depends on the choice of lattice. However, by Theorem 1.2.4, its characteristic polynomial is well-defined, and therefore its semisimplification is well-defined too. Thus, any two reductions of the same $K[G]$-module have the same composition factors up to isomorphism, so reduction creates a well-defined class in $R_k(G)$.

Secondly, we have to deal with reduction.

Lemma 1.5.6. *Reduction of a $K[G]$-module into an element of $R_k(G)$ is additive.*

Proof. TODO ⊠

Thus, reduction is a homomorphism $d : R_K(G) \to R_k(G)$. This is called the *decomposition homomorphism*.

1.5.3. Definition of $e : P_k(G) \to R_K(G)$. By Proposition 1.4.3, given a projective $k[G]$-module M, there's a unique projective $\mathscr{O}_K[G]$-module P up to isomorphism such that its reduction mod $\pi\mathscr{O}_K$ is equal to M. Then, $P \otimes_{\mathbb{Z}_p} K$ is a $K[G]$-module.

We will define $e : P_k(G) \to R_K(G)$ by sending $[M] \mapsto [P \otimes_{\mathbb{Z}_p} K]$; however, in order to show that this extends to a well-defined homomorphism on the Grothendieck groups, we'll need to show that it's additive.

By Proposition 1.4.3, reduction has an inverse; since reduction from $\mathscr{O}_K[G]$-modules to $k[G]$-modules is additive by Lemma 1.5.6, then this inverse of reduction must also be additive; thus, this inverse defines a map $e_1 : P_k(G) \to P_{\mathscr{O}_K}(G)$, which is the first part of e.

The second part of e is $-\otimes_{\mathbb{Z}_p} K$, but since K is a flat $\mathbb{Z}_p$-module, then this functor is exact, and therefore additive, so it defines a map $e_2 : P_{\mathscr{O}_K}(G) \to R_K(G)$. Thus, the composition of these two maps is $e_2 \circ e_1 = e$, and therefore e is well-defined.

1.5.4. Properties of the CDE Triangle.

Theorem 1.5.7. *The CDE diagram* (1.5.1) *commutes.*

Proof. Using the definitions of c, d, and e, the theorem statement is equivalent to stating that if one chooses a projective $k[G]$-module P and lifts it to a $K[G]$-module M as described in Section 1.5.3, the reduction of M has the same semisimplification as P.

P is lifted to M by way of an intermediate $\mathscr{O}_K[G]$-module P', which has the properties that $P' \otimes_{\mathscr{O}_K} K = M$ and $P'/\mathfrak{m}P' \cong P$. The first property means there's a natural inclusion $i : P' \hookrightarrow M$ of $\mathscr{O}_K[G]$-modules defined by $i(x) = x \otimes 1$, so that $i(P')$ is isomorphic to P'. Furthermore, $i(P')$ is a G-stable lattice in M, since it is an $\mathscr{O}_K[G]$-module that generates all of M when tensored with K.

Thus, we may as well use $i(P')$ as our lattice for reducing M, since the class of the reduction in $R_k(G)$ doesn't depend on the choice of lattice. But $i(P') \cong P'$, so the reduction is just $P'/\mathfrak{m}P'$, which is isomorphic to P again. Thus, any reduction of M has the same semisimplification as P, so the diagram commutes. ⊠

Proposition 1.5.8. *In the bases established above, $E = D^{\mathrm{T}}$, and therefore C is a symmetric matrix.*

Proof. TODO ⊠

Theorem 1.5.9. *d is surjective.*

Corollary 1.5.10. *e is injective.*

Proof. Since $E = D^{\mathrm{T}}$ and D is surjective, then if D is $m \times n$, then its rank is m, so E is $n \times m$ and has the same rank, so the dimension of its kernel is $m - \operatorname{rank}(E) = 0$. ⊠

Theorem 1.5.11. *c is injective.*

This can be restated by applying the definition of the Grothendieck groups.

Corollary 1.5.12. *If two projective $k[G]$-modules have the same composition factors, they are isomorphic.*

TODO: Figure out how to state that every irreducible modular character arises from the reduction of an irreducible characteristic 0 representation.

1.6. The CDE Triangle on the Character Level

The Grothendieck groups in the CDE triangle are closely related to spaces of K-valued class functions; for example, $R_K(G)$ has the set of classes of simple modules as a basis, and the space of class functions $K \to G$ has the characters of these simple modules as a basis.

TODO: I can't seem to get things to come out correctly. The matrix interpretation seems to disagree with the CDE triangle interpretation.

TODO, in which I want to cover:

- The character interpretation of the CDE triangle. That is, if $C\ell(G,K)$ denotes the vector space of K-valued class functions on G, then we have the triangle

$$\begin{array}{ccc} \{f \in C\ell(G,K) \text{ zero away from } G^{(p)}_{\text{reg}}\} & \xrightarrow{\ \mathrm{id}_K \otimes c\ } & C\ell(G^{(p)}_{\text{reg}},K) \\ & {\scriptstyle \mathrm{id}\otimes e} \searrow \qquad \nearrow {\scriptstyle \mathrm{id}_K \otimes d} & \\ & C\ell(G,K) & \end{array}$$

  (from Serre, p. 151), obtained by tensoring with K. In particular, this explains why the characters of the projective indecomposables are determined from the modular characters with coefficients from C.

Finally, the following lemma will be used several times when calculating character tables, so it'll be useful to prove it here.

Lemma 1.6.1. *The trivial representation is the only one-dimensional representation of $\mathbb{Z}/p$ in characteristic p.*

Proof. Let k be a field of characteristic p. A one-dimensional representation of $\mathbb{Z}/p$ over k is a group homomorphism $\mathbb{Z}/p \to k^\times$, and since $\mathbb{Z}/p$ is generated by 1, such a homomorphism is uniquely determined by a choice of element $x \in k^\times$ with order p, i.e. a root of $x^p - 1$.

However, in characteristic p, this polynomial factors as $(x-1)^p$, and therefore the only solution is $x = 1$. In this case, the homomorphism $\mathbb{Z}/p \to k^\times$ sends everything to 1, and therefore is the trivial representation. ⊠

After this, everything that's needed for the character table calculations should be at least stated.

CHAPTER 2

Modular Representations of Some Small Groups

2.1. The Modular Representation Theory of $\mathbb{Z}/p$

Let p be prime. Then $\mathbb{Z}/p$, the cyclic group of order p, has p elements, so its modular representation theory is only interesting over characteristic p.

Character Table in Characteristic Zero. Since $\mathbb{Z}/p$ is abelian, then each element is in its own conjugacy class, so there are p conjugacy classes and p irreducible representations. Furthermore, since $\mathbb{Z}/p$ is abelian, each representation is one-dimensional. Each representation is specified by sending the generator $1 \in \mathbb{Z}/p$ to a p^{th} root of unity. Thus, the character table is given in Table 1.

	0	1	2	$\cdots$	$p-1$
χ_1	1	1	1	$\dots$	1
χ_2	1	ζ_p	ζ_p^2	$\dots$	ζ_p^{p-1}
χ_3	1	ζ_p^2	ζ_p^4	$\dots$	ζ_p^{2p-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_p	1	ζ_p^{p-1}	ζ_p^{2p-2}	$\dots$	ζ_p

TABLE 1. Character table for $\mathbb{Z}/p$ in characteristic 0, where p is prime and ζ_p is a fixed primitive p^{th} root of unity.

What Happens In Characteristic p. Since the order of every non-identity element in $\mathbb{Z}/p$ is p, then $\{0\}$ is the sole p-regular conjugacy class. Thus, by Corollary 1.3.6, there's only a single irreducible modular character, which is therefore given by the trivial representation. All of the other irreducible representations in characteristic 0 are equal to the trivial one on the p-regular class. Thus, the character table is given in Table 2.

	0
ϕ	1

TABLE 2. Character table for $\mathbb{Z}/p$ in characteristic p.

We can also calculate the CDE triangle. The decomposition matrix is the $1 \times p$ matrix

$$D = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} p \end{bmatrix}.$$

Thus, the character of the sole projective indecomposable is

$$\Phi = \chi_1 + \chi_2 + \cdots + \chi_p$$

or in terms of ϕ,

$$\Phi = p\phi.$$

However, we can go one step better and compute the projective indecomposable for $\mathbb{Z}/p$ itself.

Proposition 2.1.1. *The sole projective indecomposable module for $\mathbb{Z}/p$ in characteristic p is the group algebra $k[\mathbb{Z}/p]$.*

Proof. First of all, why is $k[\mathbb{Z}/p]$ indecomposable? As a $k[\mathbb{Z}/p]$-module, it's generated by the identity e of $\mathbb{Z}/p$, so if it's possible to write $f : k[\mathbb{Z}/p] \xrightarrow{\sim} M \oplus N$, then what is $f(e)$? It must be of the form (m,n) where m generates M and n generates N, or else f wouldn't be surjective. Moreover, the orders of m and n must divide the order of e, which is p, so they must be either 1 or p. They can't both be 1 (or $k[\mathbb{Z}/p]$ would be trivial), and if one is 1 and the other is p, then the direct sum is trivial, so both m and n must have order p. If this is the case, however, then $(m,0)$ isn't generated by (m,n), but (m,n) is supposed to generate all of $M \oplus N$. Thus, $k[\mathbb{Z}/p]$ is indecomposable.

Since $k[\mathbb{Z}/p]$ is free of dimension 1 over itself, then it is a projective $k[\mathbb{Z}/p]$-module. Thus, it is the projective indecomposable; we know there cannot be any more because there is only one p-regular conjugacy class. ⊠

2.2. The Modular Representation Theory of S_3

S_3, the symmetric group on three elements, has 6 elements, so its modular reperesentation theory breaks down into the cases $p = 2$ and $p = 3$.

2.2.1. Character Table in Characteristic Zero. As conjugacy type is equivalent to cycle type in symmetric groups, there are three conjugacy classes, 1, $(a\ b)$, and $(a\ b\ c)$. The character table for S_3 is given in Table 3.

	1	$(a\ b)$	$(a\ b\ c)$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 3. Character table for S_3 in characteristic zero.

Here, χ_1 is the character of the trivial representation and χ_2 is that of the sign representation. Then, χ_3 can be found by the orthogonality relations (it must be two-dimensional, since $|S_3| = 6 = 1^2 + 1^2 + \chi_3(1)^2$, and therefore its values on the remaining two classes are given by taking dot products); explicitly, it is the permutation representation on three-tuples summing to zero, since that sum is preserved by the action of S_3. Specifically, if $\sigma \in S_3$, $\sigma \cdot (a_1, a_2, a_3) \mapsto (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. The space of 3-tuples whose sum is zero is two-dimensional, and we obtain the irreducible two-dimensional representation whose character is χ_3.

2.2.2. The case $p = 2$. The 2-regular conjugacy classes are 1 and $(a\ b\ c)$; thus, in this characteristic, χ_1 and χ_2 coincide, reducing to the trivial character ϕ_1. Then, there can be no more irreducible one-dimensional representations in this characteristic, because such a representation must factor through the abelianization, $\mathbb{Z}/2$, but by Lemma 1.6.1, the only one-dimensional representation of $\mathbb{Z}/2$ in characteristic 2 is trivial.

As a consequence, χ_2 is sent to an irreducible representation in this characteristic, because if it were reducible, then it would decompose as a sum of two one-dimensional representations. Since there's only one 1-dimensional representation, $\chi_2(a\ b\ c)$ would have to be $2\phi_1(a\ b\ c) = 2$, but instead it's -1. Thus, χ_2 reduces to an irreducible representation ϕ_2 in characteristic 2.

Since there are two 2-regular conjugacy classes, this is the complete list of irreducible representations in this characteristic, and the character table is presented in Table 4.

	1	$(a\ b\ c)$
ϕ_1	1	1
ϕ_2	2	-1

Table 4. Character table for S_3 in characteristic 2.

The decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\Phi_1 = \chi_1 + \chi_2,$$
$$\Phi_2 = \chi_3,$$

or in terms of the ϕ_i,

$$\Phi_1 = 2\phi_1.$$
$$\Phi_2 = \phi_2.$$

2.2.3. The case $p = 3$. The 3-regular conjugacy classes are 1 and $(a\ b)$. Thus, in this characteristic, χ_1 and χ_2 are distinct, so since they're one-dimensional, then they become irreducible representations, respectively ϕ_1 and ϕ_2, in characteristic 3. Since there are only two 3-regular conjugacy classes, then these must be all of the irreducible representations (and χ_3 decomposes to $\phi_1 + \phi_2$), so the character table is given in Table 5.

	1	$(a\ b)$
ϕ_1	1	1
ϕ_2	1	-1

TABLE 5. Character table for S_3 in characteristic 3.

The decomposition matrix is

$$D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\Phi_1 = \chi_1 + \chi_3,$$
$$\Phi_2 = \chi_2 + \chi_3,$$

or in terms of the ϕ_i,

$$\Phi_1 = 2\phi_1 + \phi_2.$$
$$\Phi_2 = \phi_1 + 2\phi_2.$$

We can also compute the two projective indecomposables for S_3 in characteristic 3. Let V denote the permutation representation on k^3, where $\sigma \cdot (a_1, a_2, a_3) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$, and let S denote the sign representation, whose character is ϕ_2.

Proposition 2.2.1. *The projective indecomposables are V and $V \otimes_k S$.*

PROOF. First of all, since tensoring with the sign representation preserves invariant subspaces, then $V \otimes S$ is indecomposable iff V is, and it also sends projective modules to projective modules: if W is a projective $k[S_3]$-module, then given $\pi : N \twoheadrightarrow N'$ and a map $W \otimes S \to N'$, we know $N \otimes S \twoheadrightarrow N' \otimes S$, since the tensor product is right exact. Since $W \otimes S \otimes S = W$ and W is projective, then the induced map $W \to N' \otimes S$ lifts to a map $W \to N \otimes S$, but applying $-\otimes S$ once again produces a map $W \otimes S \to N$ that commutes with π, because $-\otimes S$ applied twice is the identity functor. Thus, $W \otimes S$ is also projective. In particular, it suffices to prove that V is a projective indecomposable, and then $V \otimes S$ must be the other one.

Suppose $\pi : N \twoheadrightarrow N'$ and $f' : V \to N'$ are maps of $k[S_3]$-modules. We know that for any $k[S_3]$-module X,

$$\mathrm{Hom}_{k[S_3]}(V, X) \xrightarrow{\sim} \{x \in X \mid (1\ 2)x = x\},$$

by the map $\Phi : \varphi \mapsto \varphi(e_3)$, which is preserved by $(1\ 2)$ because e_3 is and φ must be S_3-equivariant. This assignment is an isomorphism because we can map in the reverse direction: given an x fixed by $(1\ 2)$, define $\varphi(e_3) = x$; then, because e_3 generates V as an S_3-module, this defines a map $\varphi : V \to X$ of $k[S_3]$-modules. This is an inverse to Φ because given such an x, the φ we get has been defined to satisfy $\varphi(e_3) = x$. Thus, specifying a map $f : V \to N$ is equivalent to finding an $x \in N$ fixed by $(1\ 2)$. Moreover, even though we want $\pi \circ f = f'$, all we have to require is that $\pi(x) = f'(e_3)$; then, for any other $v \in V$, $v = \sum_{g \in S_3} \lambda_g g \cdot e_3$, so since f is S_3-equivariant, then

$$\pi(f(v)) = \sum_{g \in S_3} \lambda_g g \pi(f(e_3)) = \sum_{g \in S_3} \lambda_g g f'(e_3) = f'(v).$$

Thus, to show that V is projective, it suffices to find an $x \in \pi^{-1}(f(e_3))$ that is fixed by $(1\ 2)$.

Choose any $y \in \pi^{-1}(f(e_3))$ and let $x = (y + (1\ 2)y)/2$, which we can do because we're in characteristic 3. Then, $(1\ 2)x = ((1\ 2)y + y)/2 = x$, and

$$\pi(x) = \frac{\pi(y) + (1\ 2)\pi(y)}{2} = \frac{f(e_3) + (1\ 2)f(e_3)}{2} = \frac{f(e_3) + f(e_3)}{2} = f(e_3),$$

since f' is S_3-equivariant, so $x \in \pi^{-1}(f(e_3))$ and therefore f' lifts.

Finally, we must show that V is indecomposable. TODO ⊠

2.3. The Modular Representation Theory of S_4

S_4, the symmetric group on four elements, has 24 elements, so its modular representation theory breaks down into the cases $p = 2$ and $p = 3$.

2.3.1. Character Table in Characteristic Zero. As conjugacy type is equivalent to cycle type in symmetric groups, there are five conjugacy classes, 1, $(a\ b)$, $(a\ b)(c\ d)$, $(a\ b\ c)$, and $(a\ b\ c\ d)$. The character table for S_4 is given in Figure 6. Here, χ_1 is the character for the trivial representation and χ_2 that for the sign

	1	$(a\ b)$	$(a\ b)(c\ d)$	$(a\ b\ c)$	$(a\ b\ c\ d)$
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	2	0	2	−1	0
χ_4	3	1	−1	0	−1
χ_5	3	−1	−1	0	1

TABLE 6. Character table for S_4 in characteristic zero.

representation. χ_4 comes from the representation ρ_4 where S_4 acts on four-tuples summing to zero: $\sigma \cdot (a_1, a_2, a_3, a_4) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)})$. These tuples are a three-dimensional vector space, so ρ_4 is a three-dimensional representation, and its character is χ_4. Then, χ_5 arises from tensoring ρ_4 with the sign representation; in particular, since tensoring with a one-dimensional representation doesn't change the invariant subspaces, then over any characteristic, χ_3 is irreducible iff χ_4 is.

2.3.2. The Case $p = 2$. The 2-regular conjugacy classes are 1 and $(a\ b\ c)$, so there should be two irreducible representations.

The trivial representation and the sign representation coincide, providing an irreducible character ϕ_1, with dimension 1. Thus, we need only one more representation. There can't be any more one-dimensional ones, because a one-dimensional representation must factor through the abelianization of S_4, which is $\mathbb{Z}/2$. However, by Lemma 1.6.1, the only such representation is the trivial representation, so in particular, any one-dimensional representation of S_4 in characteristic 2 must be trivial.

Suppose that χ_3 doesn't reduce to an irreducible representation in characteristic 2, so that it splits on the 2-regular elements. Since it's two-dimensional, then it must split as the sum of two one-dimensional representations, i.e. twice the trivial representation. But then, $\chi_3(1\ 2\ 3) = -1$ would have to be twice that of $\chi_1(1\ 2\ 3) = 1$, so it's not $\chi_1 + \chi_1$. Thus, χ_3 reduces to an irreducible ϕ_2.

These are the two irreducible representations in characteristic 2; the character table is presented in Table 7.

	1	$(a\ b\ c)$
ϕ_1	1	1
ϕ_2	2	−1

TABLE 7. Character table for S_4 in characteristic 2.

Now, it's possible to fill in the CDE triangle. The remaining irreducibles from characteristic 0 must decompose as sums of ϕ_1 and ϕ_2 on the 2-regular elements; looking at the character table, one sees that $\chi_1, \chi_2 \mapsto \phi_1$ and $\chi_3 \mapsto \phi_2$, as established, and that $\chi_4, \chi_5 \mapsto \phi_1 + \phi_2$. Thus, the decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}.$$

Associated to ϕ_1 and ϕ_2 are the indecomposable projective $\overline{\mathbb{F}}_2[S_4]$-modules Φ_1 and Φ_2, whose characters are given by the matrix E:

$$\Phi_1 = \chi_1 + \chi_2 + \chi_4 + \chi_5$$
$$\Phi_2 = \chi_3 + \chi_4 + \chi_5.$$

And after decomposing the χ_i, the characters for the indecomposable projective modules are expressed in terms of ϕ_1 and ϕ_2 by the matrix C, as

$$\Phi_1 = 4\phi_1 + 2\phi_2$$
$$\Phi_2 = 2\phi_1 + 3\phi_2.$$

2.3.3. The Case $p = 3$. The 3-regular conjugacy classes are 1, $(a\ b)$, $(a\ b)(c\ d)$, and $(a\ b\ c\ d)$.

Thus, the trivial and sign representations are distinct in this characteristic (e.g. since they differ on $(a\ b)$), so they are two of the Brauer characters: $\chi_1 \mapsto \phi_1$ and $\chi_2 \mapsto \phi_2$; they must be irreducible, because they are one-dimensional.

Just as with characteristic 2, there can be no more irreducible one-dimensional representations of S_4 in characteristic 3, because they would have to factor through the abelianization $\mathbb{Z}/2\mathbb{Z}$. Once again, a one-dimensional representation of $\mathbb{Z}/2$ is a choice of an element squaring to 1, but in characteristic 3, there are two: 1, corresponding to the trivial representation, and -1, corresponding to the sign representation. But both of these have already been accounted for in S_4, so there can be no others.

On the 3-regular elements, χ_3 splits as $\phi_1 + \phi_2$, so it's reducible.

Since χ_5 is obtained from χ_4 by tensoring with the sign representation, then as stated above one is irreducible iff the other is. Since there are four 3-regular conjugacy classes, then there will be four modular characters, and two have already been accounted for. Thus, if χ_4 and χ_5 reduce to irreducible representations, then we will be done.

We know that $\dim(\rho_4) = \chi_4(1) = 3$. Since $|S_4| = 24$, then $3 \mid 24$, but $3^2 \nmid 24$, and $3 \mid \dim(\rho_4)$. Thus, by Theorem 1.3.7, ρ_4 passes to an irreducible representation in characteristic 3, and so $\chi_4 \mapsto \phi_3$, the next modular character. Then, the same argument works for χ_5, so its character on the 3-regular elements is also irreducible, and will be called ϕ_4.

Thus, we have the four irreducible characters in characteristic 3. Table 8 shows the character table.

	1	$(a\ b)$	$(a\ b)(c\ d)$	$(a\ b\ c\ d)$
ϕ_1	1	1	1	1
ϕ_2	1	-1	1	-1
ϕ_3	3	1	-1	-1
ϕ_4	3	-1	-1	1

TABLE 8. Character table for S_4 in characteristic 3.

With the character table in place, the next step is the CDE triangle. To calculate the decomposition matrix, $\chi_1 \mapsto \phi_1$ and $\chi_2 \mapsto \phi_2$ as noted above; then, $\chi_3 \mapsto \phi_1 + \phi_2$, and $\chi_4 \mapsto \phi_3$ and $\chi_5 \mapsto \phi_4$. Thus,

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$ as usual, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 1 & & \\ 1 & 2 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix}.$$

Associated to ϕ_1, ϕ_2, ϕ_3, and ϕ_3 are the indecomposable, projective $\overline{\mathbb{F}}_2[S_4]$-modules $\Phi_1, \dots, \Phi_4$, whose description in terms of the χ_i is given by E:

$$\begin{aligned} \Phi_1 &= \chi_1 + \chi_3 \\ \Phi_2 &= \chi_2 + \chi_3 \\ \Phi_3 &= \chi_4 \\ \Phi_4 &= \chi_5. \end{aligned}$$

Then, decomposing the right-hand side means the Φ_i can be expressed in terms of the ϕ_i using the matrix C.

$$\begin{aligned} \Phi_1 &= 2\phi_1 + \phi_2 \\ \Phi_2 &= \phi_1 + 2\phi_2 \\ \Phi_3 &= \phi_3 \\ \Phi_4 &= \phi_4. \end{aligned}$$

2.4. The Modular Representation Theory of A_4

A_4, the alternating group on 4 elements, has 12 elements, so its modular representation theory breaks down into two cases, $p = 2$ and $p = 3$.

Character Table in Characteristic Zero. In alternating groups, cycle type does not determine conjugacy class; there are four conjugacy classes, given by 1, $(a\ b)(c\ d)$, and two conjugacy classes of 3-cycles:

$$\begin{aligned} c_3 &= \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\} \\ c_4 &= \{(1\ 3\ 2), (4\ 1\ 2), (2\ 3\ 4), (3\ 1\ 4)\}. \end{aligned}$$

The character table is presented in Table 9.

	1	$(a\ b)(c\ d)$	c_3	c_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

TABLE 9. Character table for A_4 in characteristic zero. Here, ω is a primitive cube root of unity.

These representations can all be explicitly constructed. Since the abelianization of A_4 is $\mathbb{Z}/3$, then the three one-dimensional representations of $\mathbb{Z}/3$ produce the one-dimensional χ_1, χ_2, and χ_3 in Table 9. Specifically, a one-dimensional representation of $\mathbb{Z}/3$ is a choice of an element whose cube is 1, and there are three of these, corresponding to the three cube roots of unity in an algebraically closed field of characteristic zero. χ_1 is the trivial representation, and then the two others should send $1 \in \mathbb{Z}/3$ to an element of order 3 in $\mathbb{C}^\times$, i.e. a cube root of unity. Then, the remaining, three-dimensional representation χ_4 is the same as χ_4 for S_4 in Table 6: the action of S_4 on 4-tuples adding to zero can also be thought of as an action of A_4 on the same space, and is irreducible by the orthogonality relations.

The case $p = 2$. The 2-regular conjugacy classes are 1, c_3, and c_4, so there will be 3 modular characters. This ends up being really easy: all three one-dimensional representations are still distinct on the 2-regular conjugacy classes of A_4, and they must be irreducible, so these are all of them. That's all, folks! See Table 10 for the character table.

Notationally, we'll let ϕ_1 be the reduction of χ_1, and similarly with ϕ_2 and ϕ_3 from χ_2 and χ_3, respectively. Then, χ_4 decomposes as $\phi_1 + \phi_2 + \phi_3$ in this characteristic, because the sum of all of the n^{th} roots of unity is zero.

	1	c_3	c_4
ϕ_1	1	1	1
ϕ_2	1	ω	ω^2
ϕ_3	1	ω^2	ω

TABLE 10. Character table for A_4 in characteristic 2. Once again, ω is a primitive cube root of unity.

Thus, the decomposition matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Thus, the projective indecomposable $\overline{\mathbb{F}}_2[A_4]$-modules Φ_1, Φ_2, and Φ_3 are given by

$$\begin{aligned} \Phi_1 &= \chi_1 + \chi_4 \\ \Phi_2 &= \chi_2 + \chi_4 \\ \Phi_3 &= \chi_3 + \chi_4, \end{aligned}$$

and in terms of the ϕ_i,

$$\begin{aligned} \Phi_1 &= 2\phi_1 + \phi_2 + \phi_3 \\ \Phi_2 &= \phi_1 + 2\phi_2 + \phi_3 \\ \Phi_3 &= \phi_1 + \phi_2 + 2\phi_3. \end{aligned}$$

The case $p = 3$**.** The 3-regular conjugacy classes are 1 and $(a\ b)(c\ d)$. Thus, we should expect two irreducible representations in this characteristic.

However, all three one-dimensional representations coincide on these conjugacy classes; they're all trivial. Thus, $\chi_1, \chi_2, \chi_3 \mapsto \phi_1$.

There will be one more representation, but it cannot be one-dimensional: any one-dimensional representation must factor through the abelianization of A_4, which is $\mathbb{Z}/3$. But by Lemma 1.6.1, the only one-dimensional representation of $\mathbb{Z}/3$ in characteristic 3 is trivial. Since this was already accounted for, so $\mathbb{Z}/3\mathbb{Z}$ has no more irreducible representations in this characteristic, and therefore A_4 has no more one-dimensional representations in characteristic 3.

The Brauer-Nesbitt theorem, Theorem 1.3.7, directly proves that χ_4 reduces to an irreducible representation: $|A_4| = 12$, so 3 divides $|A_4|$, but 9 doesn't. Then, $3 \mid \dim(\rho_4) = 3$, so the theorem is satisfied, and $\chi_4 \mapsto \phi_2$, an irreducible representation. Thus, we've found all of the irreducible modular characters; the character table is given in Table 11.

	1	$(a\ b)(c\ d)$
ϕ_1	1	1
ϕ_2	2	-1

TABLE 11. Character table for A_4 in characteristic 3.

Thus, we've found both irreducibles, so the decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the projective indecomposables are

$$\Phi_1 = \chi_1 + \chi_2 + \chi_3$$
$$\Phi_2 = \chi_4,$$

or in terms of ϕ_1 and ϕ_2,

$$\Phi_1 = 3\phi_1$$
$$\Phi_2 = \phi_2.$$

2.5. The Modular Representation Theory of S_5

S_5, the symmetric group on 5 elements, has 60 elements, so its modular representation theory breaks down into the cases $p = 2$, $p = 3$, and $p = 5$.

2.5.1. Character Table in Characteristic Zero. Since conjugacy type is equivalent to cycle type in symmetric groups, there are seven conjugacy classes: 1, $(a\ b)$, $(a\ b\ c)$, $(a\ b\ c\ d)$, $(a\ b\ c\ d\ e)$, $(a\ b)(c\ d)$, and $(a\ b)(c\ d\ e)$. Its character table is given in Table 12.

	1	$(a\ b)$	$(a\ b\ c)$	$(a\ b\ c\ d)$	$(a\ b\ c\ d\ e)$	$(a\ b)(c\ d)$	$(a\ b)(c\ d\ e)$
χ_1	1	1	1	1	1	1	1
χ_2	1	−1	1	−1	1	1	−1
χ_3	4	2	1	0	−1	0	−1
χ_4	4	−2	1	0	−1	0	1
χ_5	5	1	−1	−1	0	1	1
χ_6	5	−1	−1	1	0	1	−1
χ_7	6	0	0	0	1	−2	0

TABLE 12. Character table for S_5 in characteristic 0.

Here, χ_1 is the character of the trivial representation and χ_2 is that of the sign representation. χ_3 comes from the representation ρ_3 where S_5 acts on five-tuples summing to zero via the permutation representation

$$\sigma \cdot (a_1, a_2, a_3, a_4, a_5) = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}, a_{\sigma(5)}).$$

Since the space of 5-tuples summing to zero is four-dimensional, this is a four-dimensional representation, and its character is χ_3. Then, χ_4 arises by tensoring ρ_3 with the sign representation. Since tensoring with a one-dimensional representation also doesn't change which subspaces are invariant, then this representation is irreducible in a given characteristic iff χ_3 is(though, as discussed below, it's the same as χ_3 in characteristic 2); a similar relationship holds for χ_5 and χ_6.

2.5.2. The Case $p = 2$. The 2-regular conjugacy classes are those with odd order, i.e. 1, $(a\ b\ c)$, and $(a\ b\ c\ d\ e)$. Thus, there should be three modular characters, corresponding to three irreducible representations.

In characteristic 2, the trivial representation and sign representation coincide, to provide an irreducible representation ϕ_1 of dimension 1. Furthermore, by Theorem 1.3.7, since $|S_5| = 60$, then 4 divides $|S_5|$ but $8 \nmid |S_5|$,and therefore χ_3 and χ_4 correspond to irreducible representations in characteristic 2, because they are four-dimensional. However, they coincide on the 2-regular elements of S_5, so we get only one more irreducible character, which we'll call ϕ_2.

Since the five-dimensional representations χ_5 and χ_6 came from tensoring with the sign representation, they're also identical in characteristic 2. And on the 2-regular conjugacy classes, $\chi_7 = \chi_5 + \chi_1$, so χ_7 isn't irreducible. Thus, the last irreducible representation in characteristic 2 comes either from χ_5 or a component of it. **TODO**

2.5.3. The Case $p = 3$. In characteristic 3, the trivial representation and the sign representation differ, so χ_1 and χ_2 correspond to irreducible modular representations ϕ_1 and ϕ_2, respectively.

The 3-regular conjugacy classes are 1, $(a\ b)$, $(a\ b\ c\ d)$, $(a\ b\ c\ d\ e)$, and $(a\ b)(c\ d)$, since $(a\ b\ c)$ and $(a\ b)(c\ d\ e)$ have orders divisible by 3.

Since $3 \mid 60$ but $9 \nmid 60$, then by Theorem 1.3.7, whenever $3 \mid \dim(\rho_i)$, where ρ_i is irreducible in characteristic 0, the reduction of ρ_i mod 3 is still irreducible. Thus, χ_7 reduces to an irreducible, as it is six-dimensional.

Since there are five 3-regular conjugacy classes and we've uncovered three irreducible representations, there must be two more.

Claim. χ_3 (and therefore also χ_4) reduces to an irreducible representation in characteristic 3.

Proof. We will prove this by starting with a single vector, and acting on it by group elements in order to generate the entire space.

First, though, recall how χ_3 is defined: since we're not in characteristic 5, it's a permutation action on 5-tuples adding to zero: $\sigma \cdot (a_1, \dots, a_5) = (a_{\sigma(1)}, \dots, a_{\sigma(5)})$.

The space of 5-tuples summing to zero is four-dimensional, so take the following basis:

$$e_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, $1 \in S_5$ acts as the identity; $(1\ 2)$ sends $e_1 \mapsto -e_2$, $e_2 \mapsto e_1 + e_2$, and fixes e_3 and e_4; $(1\ 2\ 3)$ sends $e_1 \mapsto e_2$, $e_2 \mapsto -e_1 - e_2$, $e_3 \mapsto e_1 + e_2 + e_3$, $e_4 \mapsto e_4$; and so on.

Since S_5 acts on these tuples by permutation, we only need to consider types of tuples up to reordering, and since we're looking at an $\mathbb{F}_3$-vector space, scalar multiples come for free; then, given any tuple of a given type, the action of S_5 generates all of the others of that type. In particular, one can ignore sign. Thus, ignoring $(0,0,0,0,0)$, there are four types:

$$e_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_1 - e_2 + e_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \quad e_1 - e_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_1 - e_2 + e_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Each of these sums to 0 mod 3, and these must be the only four types, because once you specify the number of 1s, only these types appear: e.g., if there's exactly one 1, then it's either $(1,-1,0,0,0)$ or $(1,-1,-1,-1,-1)$, both of which are the same types as above; if there are two, we get the second type; if there are three, we get the third type; and if there are four, the last type is forced.

So we know that if we have any invariant subspace and it contains a given tuple, it must contain all tuples of that type. Thus, the next step is to show that, given any type, we can obtain all of the others; we'll go between the first and the second, then between the second and the third, and then between the third and the fourth, so that given a tuple of any type (and therefore all tuples of that type), it's possible to generate all of the others.

Let's start with the first type, e_1. Then, $(1\ 3)(2\ 4) \cdot e_1$ is $(0,0,1,-1,0)$, so $e_1 + (1\ 3)(2\ 4) \cdot e_1 = (1,-1,1,-1,0)$, so we've gotten the second type. In the other direction, $v_1 = (1,1,-1,-1,0)$ and $v_2 = (1,-1,1,-1,0)$ are both of the second type, and $v_1 + v_2 = (-1,0,0,1,0)$ is first type. Thus, if an invariant subspace contains a tuple of either first or second type, it contains all vectors of both types.

If $v_1 = (1,1,-1,-1,0)$ and $v_2 = (0,-1,-1,1,1)$, so they're both second type, then $v_1 + v_2 = (1,0,1,0,1)$, which is third type; then, if $w_1 = (1,1,1,0,0)$ and $w_2 = (1,1,0,1,0)$, then they're both third type, but $w_1 + w_2 = (-1,-1,1,1,0)$, which is second type. Thus, one can go between the second and third types.

Let $v_1 = (1,1,1,0,0)$ and $v_2 = (0,0,1,1,1)$, which are both of third type, and $v_1 + v_2 = (1,1,-1,1,1)$, which is fourth type. Then, if $w_1 = (1,1,1,1,-1)$ and $w_2 = (1,1,1,-1,1)$, which are both fourth type, then $w_1 + w_2 = (-1,-1,-1,0,0)$, which is third type. Thus, it's possible to go between the third and fourth types.

Thus, given any tuple, it is possible to generate all tuples of its type, and therefore to generate the entire space, so any invariant subspace of this representation is equal to the whole space, so it is irreducible. ⊠

Since χ_3 reduces to an irreducible representation, there's a corresponding modular character ϕ_3, and as noted above, χ_4 is irreducible iff χ_3 is, so it gives us another modular character ϕ_4. Thus, we've found all five irreducible characters (since χ_7 is also irreducible, so we've found ϕ_5). This is all the information we need to make the character table, which appears in Table 13.

	1	$(a\ b)$	$(a\ b\ c\ d)$	$(a\ b\ c\ d\ e)$	$(a\ b)(c\ d)$
ϕ_1	1	1	1	1	1
ϕ_2	1	−1	−1	1	1
ϕ_3	4	2	0	−1	0
ϕ_4	4	−2	0	−1	0
ϕ_5	6	0	0	1	−2

TABLE 13. Character table for S_5 in characteristic 3.

Next, the CDE triangle: the decomposition matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\begin{aligned} \Phi_1 &= \chi_1 + \chi_6 \\ \Phi_2 &= \chi_2 + \chi_5 \\ \Phi_3 &= \chi_3 + \chi_5 \\ \Phi_4 &= \chi_4 + \chi_6 \\ \Phi_5 &= \chi_7, \end{aligned}$$

or in terms of the ϕ_i,

$$\begin{aligned} \Phi_1 &= 2\phi_1 + \phi_4 \\ \Phi_2 &= 2\phi_2 + \phi_3 \\ \Phi_3 &= \phi_2 + 2\phi_3 \\ \Phi_4 &= \phi_1 + 2\phi_4 \\ \Phi_5 &= \phi_5. \end{aligned}$$

2.5.4. The Case $p = 5$. The only conjugacy class with order dividing 5 is $(a\ b\ c\ d\ e)$, so the 5-regular conjugacy classes are 1, $(a\ b)$, $(a\ b\ c)$, $(a\ b\ c\ d)$, $(a\ b)(c\ d)$, and $(a\ b)(c\ d\ e)$. There are six conjugacy classes, and therefore six irreducible representations.

Since χ_1 and χ_2 are distinct on the 5-regular classes of S_5, then they correspond to distinct irreducibles in characteristic 5, respectively ϕ_1 and ϕ_2.

Since 5 divides $|S_5| = 60$ but 25 doesn't, then Theorem 1.3.7 implies that any five-dimensional irreducible representations remain irreducible in characteristic 5; thus, χ_5 and χ_6 remain irreducible.

Claim. χ_3 (and therefore χ_4) are *not* irreducible in characteristic 5.

PROOF. Suppose that they were; then, since there are six 5-regular classes, these would be all of the modular characters in this characteristic. In particular, that means it would be possible to describe the remaining characteristic-0 character, χ_7, in terms of the others on the 5-regular classes.

$(1\ 2)(3\ 4)$ is 5-regular, and for $i = 1, \ldots, 6$, $\chi_i((1\ 2)(3\ 4)) \geq 0$. However, $\chi_7((1\ 2)(3\ 4)) = -2$, so no positive combination of $\chi_1, \ldots, \chi_6$ can create χ_7 on the 5-regular classes, so this is a contradiction. ⊠

The reason the proof given in characteristic 3 doesn't work is that, since we can't divide by 5, this representation isn't on 5-tuples summing to 0.

In order to classify the representations, we'll need to find an invariant subspace.

Claim. If V denotes the subspace of tuples that sum to 0, V is an irreducible, invariant subspace.

Note that the sum of the entries of a tuple is well-defined, even though we quotiented by $(1,1,1,1,1)$, because it sums to zero, so adding or subtracting it doesn't affect the sum.

Proof. If $v_1, v_2 \in V$, then the sum of the entries in $v_1 + v_2$ is the sum of all 10 entries of both v_1 and v_2, i.e. $0 + 0 = 0$, and taking the sum of the entries commutes with scalar multiplication, so the scalar multiple of a $v \in V$ still sums to zero. Thus, V is a subspace. It's S_5-stable because permuting the entries of a tuple doesn't change the sum. Thus, V is invariant.

To show that it's irreducible, we'll once again sort the elements of V into types, and then show that it's possible to go from any type to any other type.

First, it's possible to realize each element of V as a 4-tuple whose entries sum to zero in a unique way, by subtracting off $(1,1,1,1,1)$ times the fifth entry, and then taking only the first four entries; for example $(0,1,2,3,4)$ is represented by $(1,2,3,4,0)$. Thus, since the last entry is zero, the sum of the first four entries is unchanged, so V is realized as those 4-tuples summing to zero.

This space is three-dimensional, with $e_1 = (1,-1,0,0)$, $e_2 = (0,1,-1,0)$, and $e_3 = (0,0,1,-1)$ forming a basis (since any $(a,b,c,-a-b-c)$ can be written as $ae_1 + (a+b)e_2 + (a+b+c)e_3$, and they're linearly independent).

Once again, we'll consider the types of 4-tuples, i.e. equivalence classes under permutation of elements (by S_4) and scalar multiplication. In particular, any nonzero tuple has a nonzero element somewhere, and without loss of generality it can be permuted into the first index, and then by scalar multiplication of the whole tuple, the first element can be made to be a 1. TODO: finish proof that these are the five types.

We end up with the following five types of tuples.

$$\begin{aligned}
\tau_1 &= (1,4,0,0)\\
\tau_2 &= (1,1,3,0)\\
\tau_3 &= (1,1,1,2)\\
\tau_4 &= (1,4,1,4)\\
\tau_5 &= (1,4,2,3).
\end{aligned}$$

From any type we can get to any other type (TODO: I should probably make this nicer to read):

- Since $2\tau_1 = (2,3,0,0)$, then we can go from the first type to the second: $(1,4,0,0)+(0,2,3,0) = (1,1,3,0)$.
- Going from type 2 to type 3: $(1,1,3,0)+(1,0,3,1) = (2,1,1,1)$.
- Going from type 3 to type 4: $(1,1,1,2)+(1,1,2,1) = (2,2,3,3)$; scalar multiplication by 3 turns this into $(1,1,4,4)$.
- Going from type 4 to type 1: $(1,4,1,4)+(1,4,4,1) = (2,3,0,0) = 2\tau_1$.

Thus, everything can be generated from everything else save the last type, which we can also account for.

- Going from type 1 to type 5: $(1,4,0,0)+(0,0,2,3) = (1,4,2,3)$.
- Going from type 5 to type 2: $(1,4,2,3)+(1,2,3,4) = (2,1,0,2)$, which is a scalar multiple of $(1,3,0,1)$.

Thus, every type can generate every other type, so every element of V can generate the entire space from the action of $\mathbb{F}_3[S_5]$. Thus, V must be irreducible. ⊠

When we quotient the original representation (that is, the one reduced from χ_3) by V, the result is therefore a one-dimensional representation; we can figure out which by calculating the character of V. TODO: I believe I am computing the character wrong, since I get the wrong answer on (1 2)(3 4 5) and the answer is worryingly choice-dependent on representatives modulo 5. Also, I may want to flesh out these calculations slightly. Use the basis e_1, e_2, e_3 from above.

- 1 acts as the identity matrix, and, since this is a three-dimensional representation, has trace 3.
- (1 2) sends $e_1 \mapsto -e_1$, $e_2 \mapsto e_1 + e_2$, and $e_3 \mapsto e_3$. Thus, it has trace 1.
- (1 2 3) sends $e_1 \mapsto e_2$, $e_2 \mapsto -e_1 - e_2$, and $e_3 \mapsto e_1 + e_2 + e_3$. This means its trace is 0.
- (1 2 3 4) sends $e_1 \mapsto e_2$ and $e_2 \mapsto e_3$, but $e_3 \mapsto -e_1 - e_2 - e_3$. Its trace is -1.

- (1 2)(3 4) sends $e_1 \mapsto -e_1$ and $e_3 \mapsto -e_3$, but $e_2 \mapsto e_1 + e_2 + e_3$; thus, its trace is also -1.
- (1 2)(3 4 5) sends $e_1 \mapsto e_1$ and $e_2 \mapsto e_1 + e_2 + e_3$. However, e_3 is sent to $(0, 0, 0, 1, -1)$, which means we have to fiddle with it to get back to its 4-tuple representation; this is equivalent to $(1, 1, 1, 2, 0)$, which is $e_1 + 2e_2 - 2e_3$, so the trace of (1 2)(3 4 5) is 0. TODO: this seems to depend on how you choose to do modular arithmetic (e.g. what if it's $3e_3$?). Moreover, as we'll see just below, it isn't right to obtain a one-dimensional representation on the quotient.

Thus, on the 5-regular elements, $\chi_3 - \chi_V$ must be one of the one-dimensional representations we've already identified. In fact,

	1	$(a\ b)$	$(a\ b\ c)$	$(a\ b\ c\ d)$	$(a\ b)(c\ d)$	$(a\ b)(c\ d\ e)$	
	4	2	1	0	0	-1	
$-($	3	1	0	-1	-1	0	$)$
	1	1	1	1	1	-1	

TODO: clearly, this is not one of our one-dimensional representations. Oops! I'm going to go ahead and assume I've made an error on (1 2)(3 4 5), so that I do get the trivial representation, and calculate the CDE triangle.

So then χ_3 decomposes as $\chi_V + \phi_1$ on the 5-regular elements, and since V is irreducible, call its character ϕ_3. Note that ϕ_3 is not invariant under tensoring with the sign representation, so the resulting representation is another irreducible representation in characteristic 5: $\phi_4 = \phi_3 \otimes \phi_1$, so we can calculate its character. In particular, since tensor products and direct sums commute (or just by checking the character tables), χ_4, which must also be reducible in this characteristic, decomposes as $\phi_2 + \phi_4$.

Then, we saw already that χ_5 and χ_6 reduce to irreducible representations, which we will call ϕ_5 and ϕ_6, respectively. And now that we've found ϕ_3 and ϕ_4, we can see that $\chi_7 = \phi_3 + \phi_4$ on the 5-regular elements. We now know enough to write down the character table in characteristic 5, and do so in Table 14.

	1	$(a\ b)$	$(a\ b\ c)$	$(a\ b\ c\ d)$	$(a\ b)(c\ d)$	$(a\ b)(c\ d\ e)$
ϕ_1	1	1	1	1	1	1
ϕ_2	1	-1	1	-1	1	-1
ϕ_3	3	1	0	-1	-1	0
ϕ_4	3	-1	0	-1	-1	0
ϕ_5	5	1	-1	-1	1	1
ϕ_6	5	-1	-1	1	1	-1

TABLE 14. Character Table of S_5 in characteristic 5.

The decomposition matrix is

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\begin{aligned}
\Phi_1 &= \chi_1 + \chi_3\\
\Phi_2 &= \chi_2 + \chi_4\\
\Phi_3 &= \chi_3 + \chi_7\\
\Phi_4 &= \chi_4 + \chi_7\\
\Phi_5 &= \chi_5\\
\Phi_6 &= \chi_6,
\end{aligned}$$

or in terms of the ϕ_i,

$$\begin{aligned}
\Phi_1 &= 2\phi_1 + \phi_3\\
\Phi_2 &= 2\phi_2 + \phi_4\\
\Phi_3 &= \phi_1 + 2\phi_3 + \phi_4\\
\Phi_4 &= \phi_2 + \phi_3 + 2\phi_4\\
\Phi_5 &= \phi_5\\
\Phi_6 &= \phi_6.
\end{aligned}$$

2.6. The Modular Representation Theory of $\mathrm{GL}_2(\mathbb{F}_3)$

The general linear group of degree 2 with coefficients in $\mathbb{F}_3$, $\mathrm{GL}_2(\mathbb{F}_3)$ (also written $\mathrm{GL}(2,\mathbb{F}_3)$ or $\mathrm{GL}(2,3)$) is the group of invertible 2×2 matrices with coefficients in $\mathbb{F}_3$. It has 48 elements, so its modular representation theory breaks down into two cases, $p=2$ and $p=3$.

2.6.1. Character Table in Characteristic Zero. $\mathrm{GL}_2(\mathbb{F}_3)$ has eight conjugacy classes.

- The identity I, with order 1.
- $-I$, with order 2.
- c_3, those matrices conjugate to $\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$, which have order 4.
- c_4, those matrices conjugate to $\begin{pmatrix} 0 & 1\\ 1 & -1\end{pmatrix}$, which have order 8.
- c_5, the matrices conjugate to $\begin{pmatrix} 0 & 1\\ 1 & 1\end{pmatrix}$, which have order 8.
- c_6, the matrices conjugate to $\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$, which have order 3.
- c_7, the matrices conjugate to $\begin{pmatrix} -1 & 1\\ 0 & -1\end{pmatrix}$, which have order 6.
- c_8, the matrices conjugate to $\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$, which have order 2.

The character table for $\mathrm{GL}_2(\mathbb{F}_3)$ is given in Table 15. Notice that $\chi_5 = \chi_4\cdot\chi_2$ and $\chi_7 = \chi_6\cdot\chi_2$.

	I	$-I$	c_3	c_4	c_5	c_6	c_7	c_8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	1	1	-1
χ_3	2	2	2	0	0	-1	-1	0
χ_4	2	-2	0	$i\sqrt{2}$	$-i\sqrt{2}$	-1	1	0
χ_5	2	-2	0	$-i\sqrt{2}$	$i\sqrt{2}$	-1	1	0
χ_6	3	3	-1	-1	-1	0	0	1
χ_7	3	3	-1	1	1	0	0	-1
χ_8	4	-4	0	0	0	1	1	0

TABLE 15. Character table for $\mathrm{GL}_2(\mathbb{F}_3)$ in characteristic 0, as proven in [**3**, Ch. XVIII, § 12].

The abelianization of $\mathrm{GL}_2(\mathbb{F}_3)$ is $\langle -I\rangle \cong \mathbb{Z}/2$.

2.6.2. The Case $p = 2$. The 2-regular classes of $\mathrm{GL}_2(\mathbb{F}_3)$ are I and c_6. Thus, by Corollary 1.3.6, there are exactly two irreducible modular characters in this characteristic.

The reductions of χ_1 and χ_2 coincide on I and c_6 as the trivial character ϕ_1; then, there can be no more one-dimensional representations in this characteristic, because they would factor through the abelianization $\mathbb{Z}/2$, which has no more representations by Lemma 1.6.1.

This means that the reduction of χ_3 (which coincides with the reductions of χ_4 and χ_5) is also irreducible: if it were reducible, it would have to be the sum of two one-dimensional characters, and therefore would equal $2\phi_1$ on I and c_6. However, $\chi_3(c_6) = -1$ and $\phi_1(c_6) = 1$, so this isn't the case, and χ_3 reduces to an irreducible character ϕ_2.

Thus, that's all of the irreducible modular characters. The character table is given in Table 16.

	I	c_1
ϕ_1	1	1
ϕ_2	2	-1

TABLE 16. Character table for $\mathrm{GL}_2(\mathbb{F}_3)$ in characteristic 2.

The decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 8 & 4 \\ 4 & 6 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\Phi_1 = \chi_1 + \chi_2 + \chi_6 + \chi_7 + 2\chi_8$$
$$\Phi_2 = \chi_3 + \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8,$$

or in terms of the ϕ_i,

$$\Phi_1 = 8\phi_1 + 4\phi_2$$
$$\Phi_2 = 4\phi_1 + 6\phi_2.$$

2.6.3. The Case $p = 3$. The 3-regular conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_3)$ are I, $-I$, c_3, c_4, c_5, and c_8. Thus, by Corollary 1.3.6, there are exactly six irreducible modular characters in this characteristic.

The reductions of χ_1 and χ_2 are distinct in this characteristic, and since they're one-dimensional, then they're irreducible, so their reductions ϕ_1 and ϕ_2, respectively, are two irreducible modular characters. However, there can be no more one-dimensional representations, since such a representation would have to factor through the abelianization $\mathbb{Z}/2$, and a one-dimensional representation of $\mathbb{Z}/2$ is given by sending $e \mapsto 1$ and $1 \mapsto \alpha$, where $\alpha^2 = 1$. Thus, $\alpha = \pm 1$, but both of these possibilities have already been accounted for, so there are no more one-dimensional representations of $\mathrm{GL}_2(\mathbb{F}_3)$ in this characteristic.

χ_4 is two-dimensional, so if it were reducible, then on the 3-regular elements, it would be a sum of two one-dimensional characters, which are therefore either ϕ_1 or ϕ_2. Since $\phi_1(-I) = \phi_2(-I) = 1$, then this would force $\chi_4(-I) = 2$ (since it would have to decompose as $2\phi_1$, $\phi_1 + \phi_2$, or $2\phi_2$), but instead, $\chi_4(-I) = -2$, so this cannot happen. Thus, χ_4 reduces to an irreducible modular character ϕ_3.

Since χ_5 is two-dimensional, distinct from χ_4 on the 3-regular elements, and has $\chi_5(-I) = -2$, then precisely the same argument works for it; thus, its reduction ϕ_4 is irreducible.

Finally, let's use Theorem 1.3.7. Since 3 divides $|\mathrm{GL}_2(\mathbb{F}_3)| = 48$ but 9 doesn't divide the order of the group, then any three-dimensional representation of $\mathrm{GL}_2(\mathbb{F}_3)$ that's irreducible in characteristic zero reduces to an irreducible representation in characteristic 3. In particular, this means the two three-dimensional characters, χ_6 and χ_7, reduce to irreducible modular characters ϕ_5 and ϕ_6, respectively, and so we've found all six irreducibles. The character table is presented in Table 17.

	I	$-I$	c_3	c_4	c_5	c_8
ϕ_1	1	1	1	1	1	1
ϕ_2	1	1	1	-1	-1	-1
ϕ_3	2	-2	0	$i\sqrt{2}$	$-i\sqrt{2}$	0
ϕ_4	2	-2	0	$-i\sqrt{2}$	$i\sqrt{2}$	0
ϕ_5	3	3	-1	-1	-1	1
ϕ_6	3	3	-1	1	1	-1

TABLE 17. Character table for $\mathrm{GL}_2(\mathbb{F}_3)$ in characteristic 3.

The decomposition matrix is

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\begin{aligned} \Phi_1 &= \chi_1 + \chi_3 \\ \Phi_2 &= \chi_2 + \chi_3 \\ \Phi_3 &= \chi_4 + \chi_8 \\ \Phi_4 &= \chi_5 + \chi_8 \\ \Phi_5 &= \chi_6 \\ \Phi_6 &= \chi_7, \end{aligned}$$

or in terms of the ϕ_i,

$$\begin{aligned} \Phi_1 &= 2\phi_1 + \phi_2 \\ \Phi_2 &= \phi_1 + 2\phi_2 \\ \Phi_3 &= 2\phi_3 + \phi_4 \\ \Phi_4 &= \phi_3 + 2\phi_4 \\ \Phi_5 &= \phi_5 \\ \Phi_6 &= \phi_6. \end{aligned}$$

2.7. The Modular Representation Theory of D_{10}

D_{10}, the dihedral group of 10 elements (symmetries of the regular pentagon), has ten elements, so its modular representation theory breaks down into two cases, $p = 2$ and $p = 5$. Note that if q is an odd prime, the modular representation theory of D_{2q} looks pretty similar to that of D_{10}, laid out below.

2.7.1. Character Table in Characteristic Zero. D_{10} has the presentation $\langle r, s \mid r^5 = s^2 = 1, srs = r^{-1} \rangle$; using this notation, its conjugacy classes are 1, $c_2 = \{r, r^4\}$, $c_3 = \{r^2, r^3\}$, and $c_4 = \{sr^n \mid n = 0, \dots, 4\}$ (i.e. all of the order-2 elements).

Thus, there are four irreducible representations, and the only way to write 10 as a sum of four nonzero squares is $10 = 1^2 + 1^2 + 2^2 + 2^2$, so two are one-dimensional and two are two-dimensional. The two one-dimensional ones are given by the trivial representation and a "sign representation" which sends an $s^m r^n \in D_{10}$ to $(-1)^m$. Then, the two-dimensional representations are the standard action of the dihedral group: r acts by rotation and s by reflection; however, r may rotate through one-fifth of a circle or two-fifths, and these produce the two remaining irreducible representations. Thus, the character table is as in Table 18.

	1	c_1	c_2	c_3
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$	0
χ_4	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$	0

TABLE 18. The character table for D_{10} in characteristic zero.

2.7.2. The Case $p = 2$. The 2-regular classes of D_{10} are 1, c_2, and c_3, on which χ_1 and χ_2 coincide as the trivial representation ϕ_1. There can be no more one-dimensional representations, as a one-dimensional representation must factor through the abelianization of D_{10}, which is $\mathbb{Z}/2$, and by Lemma 1.6.1, there are no more one-dimensional representations of $\mathbb{Z}/2$.

In particular, this means that χ_3 and χ_4, which are distinct in this characteristic, must remain irreducible: if either were reducible, it would split as a sum of two one-dimensional representations, and therefore as twice the trivial representation. However, then its character would be 2 on c_2, instead of $4\cos(2\pi/5)$. Thus, this doesn't work, so χ_3 and χ_4 reduce to irreducible representations in this characteristic, denoted ϕ_2 and ϕ_3 respectively.

Since there are three 2-regular conjugacy classes, and we've described three irreducible representations, then there aren't any more. The character table is presented in Table 19.

	1	c_2	c_3
ϕ_1	1	1	1
ϕ_2	2	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$
ϕ_3	2	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$

TABLE 19. Character table for D_{10} in characteristic 2.

Thus, the decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\begin{aligned} \Phi_1 &= \chi_1 + \chi_2, \\ \Phi_2 &= \chi_3, \\ \Phi_3 &= \chi_4, \end{aligned}$$

or in terms of the ϕ_i,

$$\begin{aligned} \Phi_1 &= 2\phi_1 \\ \Phi_2 &= \phi_2 \\ \Phi_3 &= \phi_3. \end{aligned}$$

	1	c_4
ϕ_1	1	1
ϕ_2	1	-1

TABLE 20. Character table for D_{10} in characteristic 5.

2.7.3. The Case $p = 5$. The 5-regular conjugacy classes are 1 and c_4, so by Corollary 1.3.6, there are two irreducible representations in this characteristic. However, χ_1 and χ_2 are distinct one-dimensional representations of D_{10} in characteristic 5, so call them ϕ_1 and ϕ_2, respectively. Thus, we're done, and indeed, χ_3 and χ_4 both split as $\phi_1 + \phi_2$. Table 20 contains the character table.

The decomposition matrix is

$$D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Thus, the characters of the projective indecomposables are

$$\Phi_1 = \chi_1 + \chi_3 + \chi_4,$$
$$\Phi_2 = \chi_2 + \chi_3 + \chi_4,$$

or in terms of the ϕ_i,

$$\Phi_1 = 3\phi_1 + 2\phi_2$$
$$\Phi_2 = 2\phi_1 + 3\phi_2.$$

2.8. A Sillier Example: D_8 and Q_8

Just as in characteristic zero, where D_8 and Q_8 have the same character table (though, since they're non-isomorphic as groups, the representations themselves are different), their modular representation theories also behave very similarly.

In fact, there's not very much to say about it in general: since each group is order 8, the only interesting case is $p = 2$; then, the only 2-regular subgroup of either must be the identity. Since there's only one 2-regular class, there's only one irreducible representation, which therefore must be trivial. The resulting character table, if one can even call it that, is given in Table 21.

	1
ϕ	1

TABLE 21. The character table of D_8 or Q_8 in characteristic 2, or more generally, of any p-group in characteristic p.

Nonetheless, we may calculate the CDE triangles for these two groups. Since their character tables are the same in characteristic 0 and 2, then the CDE triangles will also be identical. Since D_8 has five conjugacy classes, then it has five irreducible representations over $\mathbb{C}$, and since the sums of the squares of their dimensions must be $|D_8| = 8$, then their dimensions must be 1, 1, 1, 1, and 2. In particular, these are their characters on the identity, even after reducing to characteristic 2, so the decomposition matrix is

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \end{bmatrix},$$

and $E = D^{\mathrm{T}}$, so

$$C = DD^{\mathrm{T}} = \begin{bmatrix} 8 \end{bmatrix}.$$

Thus, the character of the lone projective indecomposable module is

$$\Phi = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5,$$

or in terms of ϕ,

$$\Phi = 8\phi,$$

and as noted above, Q_8 has the same CDE triangle.

The reason these examples were uninteresting were because all elements of D_8 and Q_8 are 2-regular save for the identity, and $p = 2$ is the only interesting case; thus, the same can be said for modular representations of any finite p-group: all elements except the identity have order dividing p, so there is only one p-regualar conjugacy class, and this is the only interesting positive characteristic. Thus, for any p-group (e.g. D_{16} or the Heisenberg group over $\mathbb{F}_p$), the only irreducible representation will be trivial, and the CDE triangle looks similar to the one above.

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