Deformation Theory

Undergraduate Honor's Thesis

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Contents

1 Introduction

The goal of this honor's thesis is to provide an introduction to deformation theory noting general definitions, important theorems and applications. The study of deformations arise when studying various questions. One important example is deforming an object from characteristic p to characteristic 0 can allow information about objects over one characteristic to be transferred to the study of objects over the other. Furthermore, deformations are useful in studying properties of moduli schemes.

We begin with a discussion of infinitesimal deformations and give an example of a non-deformizable scheme. We include two examples of deformation of curves as an exercise in and illustration of the ideas. After, we move into a discussion of moduli schemes. The thesis ends with a discussion of abelian varieties, which gives a family of examples which are particularly well-behaved under deformations. The examples and counterexamples which are included are done to either illustrate the general idea or to show the that the property being considered is actually a special one.

The proofs of the section on formal schemes and the section on moduli schemes are not as illustrative of the general ideas found the other sections, and thus are left particularly bare of details. For both of these sections, I have followed [3] which has discussions of the proofs. Would I to continue work on this thesis, I would like to discuss more applications to deformation theory.

2 Formal schemes

We briefly discuss motivation for the forthcoming definitions:

We let $i : X \hookrightarrow Y$ be a closed immersion. Affine locally on Y, i looks like a like a map $\operatorname{Spec} A/I \to \operatorname{Spec} A$ where I is an ideal of A. It is clear that $\operatorname{Spec} A/I^n$ has the same underlying topological space as $\operatorname{Spec} A/I$ as the elements of I are all nilpotent in A/I^n, but the structure ring A/I^n is a fattening of the old structure ring, A/I, and thus captures more information about the infinitesimal neighborhood of A. We may put all of these together by considering $\varprojlim_n A/I^n$.

The following will be a toy example used throughout the section to illustrate definitions and constructions. Set for the rest of the section B to be $\mathbb{Z}_p[x_1, \ldots, x_n]/f$ for $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$, we see that $\operatorname{Spec} \mathbb{F}_p[x_1, \ldots]/f$ is a closed subscheme of $\operatorname{Spec} B$ with defining ideal $J = (p)$. B/J^n this becomes $(\mathbb{Z}/p^n)[x_1, \ldots]/f$. This example is helpful in that it reminds us that one case of formal spectra is that of transforming a characteristic p object into a characteristic zero object.

With this in mind we give the following definition:

Definition 2.1. Let A be an I-adic noetherian ring, by which we mean separated and complete with respect to the I-adic topology. We denote X_n for $n \geq 0$ to be $\operatorname{Spec} A/I^{n+1}$. The *formal spectrum* of A, here denoted $\mathcal{X}$ is a locally ringed space of topological rings whose underlying topological space is that of $\operatorname{Spec} A/I$ and such that $\Gamma(U, \mathscr{O}_{\mathcal{X}}) = \varprojlim_n \Gamma(U, \mathscr{O}_{X_n})$ as a topological ring where the $\Gamma(U, \mathscr{O}_{X_n})$ are given the discrete topology. We denote this object $\operatorname{Spf} A$

Returning to $B = \mathbb{Z}_p[x_1, \ldots, x_n]/f$ and $J = (p)$, we note that $\operatorname{Spec} B$ is an object over $\operatorname{Spec} \mathbb{Z}_p$ and the topological space of $\operatorname{Spf} A$ corresponds only to the fiber over the closed point. However, as ringed spaces $\operatorname{Spf} A$ includes information about the object over every $\mathbb{Z}/p^n$ as opposed to just over $\mathbb{Z}/p$.

With this definition of affine formal spectrum we can define:

Definition 2.2. An **affine noetherian formal scheme** is a topologically ringed space isomorphic to the formal spectrum of a noetherian I-adic ring, and a **locally noetherian formal scheme** is a topologically ringed space covered by open sets which are affine noetherian formal schemes.

We recall that the morphisms of topologically ringed spaces are morphisms of ringed spaces $f : \mathcal{X} \to \mathcal{Y}$ such that for every open the map $\Gamma(U, \mathscr{O}_{\mathcal{Y}}) \to \Gamma(U, f_* \mathscr{O}_{\mathcal{X}})$ is continuous.

Returning again to our example of $B = \mathbb{Z}_p[x_1, \ldots, x_n]/f$, $J = (p)$, we can see that the formal scheme $\operatorname{Spf} B$ is clearly related to the traditional scheme $\operatorname{Spec} B$. In the general the process of associating to a scheme to a locally noetherian formal scheme will be called algebraization. Before we discuss this, we must make more precise some necessary details.

Return to the case of $i : X \hookrightarrow Y$ a closed immersion where Y is a locally noetherian scheme, and let this closed subscheme be defined by the ideal $\mathscr{I}$. We consider the system

$$X_0 \to X_1 \to \cdots \to X_n \to \cdots$$

where X_n is the closed subscheme defined $\mathscr{I}^n$.

Definition 2.3. The **formal completion of** Y **along** X is defined to be the normal scheme

$$Y_{/X} := \varinjlim_n X_n.$$

This will also be denoted $\hat{X}$.

Of course, we see that $\operatorname{Spf} B$ is the formal completion of $\operatorname{Spec} \mathbb{Z}_p[x_1, \ldots, x_n]/f$ along $\operatorname{Spec} \mathbb{F}_p[x_1, \ldots, x_n]/f$

It is useful to define many of the familiar objects from schemes in terms of formal schemes. The following construction is reminiscent of the construction in scheme theory. For a noetherian ring R and finite type R module N, we will denote $\widetilde{N}$ to be the associated coherent sheaf on $\operatorname{Spec} R$. If M is a finite type A module, where A is an I-adic noetherian ring, then we let $M_n = M/I^{n+1}M$ and $M^{\Delta} = \varinjlim_n \widetilde{M_n}$

If $\mathcal{X}$ is a noetherian formal scheme, we will sketch why the coherent sheaves on $\mathcal{X}$ are those sheaves affine locally such for affine opens $\operatorname{Spf} A \subseteq \mathcal{X}$ the sheaf is of the form M^{Δ} for a finite type A module M.

To show this, it suffices to consider $\mathcal{X} = \operatorname{Spf} A$. In this case we first note that

$$\Gamma(\mathcal{X}, M^{\Delta}) = M. \tag{2.4}$$

We let $X = \operatorname{Spec} A$, and consider $i : \mathcal{X} \to X$ the natural morphism of ringed spaces. Krull's theorem on modules over an I-adic ring says that if R is a ring and $\widehat{R}$ its I-adic completion, the completion of M is the same as $M \otimes \widehat{R}$, which is precisely the statement that

$$M^{\Delta} = i^* \widetilde{M}. \tag{2.5}$$

From equations 2.4 and 2.5, it follows $\mathscr{O}_{\mathcal{X}}$ is coherent, and the coherent modules on $\mathcal{X}$ are those locally of the form M^{Δ} for M of finite type over A. Furthermore, from this it follows that vector bundles of rank r on $\operatorname{Spf} A$ correspond precisely to vector bundles on $\operatorname{Spec} A$ of rank r.

Now let $\mathcal{X}$ be an I-adic formal scheme, and $\mathscr{F}$ a coherent sheaf on $\mathcal{X}$. We say the support of $\mathscr{F}$ is proper over $\mathcal{X}$ if the support of $F_0 = \mathscr{F} \otimes \mathscr{O}_{X_0}$ is proper over X_0. Furthermore, if $\mathcal{X} = \widehat{X}$ then we have a map $i : \widehat{X} \to X$ and for any coherent sheaf $\mathscr{F}$ on X, we define the $\widehat{\mathscr{F}}$ to be the coherent sheaf $i^*\mathscr{F}$. With this we are able to state the following important theorem relating coherent sheaves on X and on $\widehat{X}$:

Theorem 2.6 ([3] Theorem 8.4.2)**.** *Let X be a noetherian scheme which is separated and finite type over Y, and let $\widehat{X}$ be its $\mathscr{I}$-adic completion (cf Definition 2.3). Then we have that the map $\mathscr{F} \mapsto \widehat{\mathscr{F}}$ from the category of coherent sheaves on X whose support is proper over Y to the category of coherent sheaves on $\widehat{X}$ whose support is proper over $\widehat{Y}$ is an equivalence of categories.*

Another notion which is useful and familiar from scheme theory is the notion of a closed formal subscheme. If $\mathcal{X}$ is a locally noetherian formal scheme and $\mathcal{A}$ a coherent ideal we have that the topologically ringed space $\mathcal{Y}$ whose underlying topological space is the support of $\mathscr{O}_{\mathcal{X}}/\mathcal{A}$ and whose structure sheaf is $\mathscr{O}_{\mathcal{X}}/\mathcal{A}$ is a locally noetherian formal scheme. If X is a locally noetherian scheme and $\mathcal{X} = \widehat{X}$

its completion along a closed subscheme X_0, then $\mathcal{Y} = \widehat{Y}$ is its completion along $Y_0 = X_0 \times_X Y$ ([3] 8.4.4)

Returning to our example of $B = \mathbb{Z}_p[x_1, \ldots, x_n]/f$ and $J = (p)$. We have that $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} \mathbb{Z}_p[x_1, \ldots, x_n]$ is a closed subscheme. Similar, we have that $\operatorname{Spf} B \hookrightarrow \operatorname{Spf} \mathbb{Z}_p[x_1, \ldots, x_n]$

We define properties on formal schemes as by properties on the schemes in the defining limit:

Definition 2.7. Let us have $\mathcal{X} = \varinjlim X_n$ and $\mathcal{Y} = \varinjlim Y_n$ locally noetherian formal schemes with defining ideals $\mathscr{I}$ and $\mathscr{J}$ respectively, and let $f : \mathcal{X} \to \mathcal{Y}$ be an adic map of formal schemes. Then:

1. We call f **proper** if $f_0 : X_0 \to Y_0$ is proper.

2. We call f **finite** if $f_0 : X_0 \to Y_0$ is finite.

Theorem 2.8 will justify the definition of proper.

Theorem 2.8 ([3], 8.4.6)**.** *Let X be a noetherian scheme which is separated and finite type over Y and $\hat{X}$ its completion with respect to some ideal $\mathscr{I}$. The we have that $Z \mapsto \hat{Z}$ is a bijection from the set of closed subschemes of X proper over Y to the set of formal subschemes $\hat{X}$ proper over $\hat{Y}$, where proper for formal schemes means that for every n, X_n is proper over Y_n*

Next let $f : X \to Y$ be a map of locally noetherian schemes and X' and Y' closed subsets respectively of each such that $f(X') \subseteq Y'$. We take these closed subsets to be defined by ideals $\mathscr{I}$ and $\mathscr{J}$ respectively. Consider their completions $\mathcal{X}$ and $\mathcal{Y}$ defined those respective ideals. Note that the condition implies that $f^{\#}(\mathscr{J})\mathscr{O}_X \subseteq \mathscr{I}$ (where we use $f^{\#}$ to denote the map on structure sheaves). Therefore, we get that $f^{\#}(\mathscr{J}^n)\mathscr{O}_X \subseteq \mathscr{I}^n$, and the map on structure sheaves actually descends to a continuous map $\mathscr{O}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{X}}$. We call this map $\hat{f}$. We actually have the following ([3], 8.4.7)

Theorem 2.9. *Let X be a proper Y-scheme and Z a noetherian scheme, which is separated and of finite type over Y. The map $\operatorname{Hom}_Y(X, Z) \to \operatorname{Hom}_{\hat{Y}}(\hat{X}, \hat{Z})$ given by $f \mapsto \hat{f}$ is bijective.*

In other words, the functor $X \to \hat{X}$ form the category of proper Y-schemes to the category of $\hat{Y}$ schemes is fully faithful.

Definition 2.10. If $\mathcal{X}$ is such that there exists a scheme Y such that $\widehat{X} = \mathcal{X}$, we call $\mathcal{X}$ **algebraizable**

Theorem 2.6, Theorem 2.8, and Theorem 2.9 show already how much information about algebraizable schemes is already held in the formal scheme. Our one example of $\operatorname{Spf} \mathbb{Z}_p[x_1, \ldots, x_n]/(f)$ is algebraizable. We must wait until after a discussion of abelian varieties to find non-algebraizable formal schemes.

3 Infinitesimal deformations

The question at hand is the following: Let us say we have an object X (like a scheme) over some field k, and an artin local ring A with residue field k, can we find an other object X' such that $X' \otimes_A k = X$? If we can, we call this X' a deformation of X to A. These will be related to formal schemes and the last section, in that a limit of infinitesimal deformations of a scheme will be a formal scheme.

Of course, the object being deformed need not be a scheme, but could be a map between schemes, a vector bundle on a scheme, a map between vector bundles on a scheme, a group. Following [3] we state some theorems about the existence of such deformations and discuss briefly their use. We begin with schemes. By induction it is often necessary to consider "thickenings" with one of the two following properties either the ideal in question squares to zero, or when working over a local ring we could take the ideal to be zero when multiplied against the maximal ideal.

To make specific what we mean we give the following definitions:

Definition 3.1. A **thickening** is a map $X_0 \to X$ defined by a quasi-coherent ideal $\mathscr{I}$ such that $\mathscr{I}^n = 0$ for some n.

Given a thickening $i : S_0 \to S$ with defining ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, a **deformation** of a flat scheme X_0 over S_0 is a scheme X flat over S and map $j : X_0 \to X$ such that $X_0 = X \times S_0$.

We point out that the definition for a deformation includes the requirement of flatness. The reason for this is that flatness guarantees that many properties of the original scheme are preserved after deforming. For instance, we can check smoothness of X/S by considering X_0/S_0.

Furthermore, if we require flatness properties of S-morphisms $f : X \to Y$ can be checked by looking at S_0-morphisms $f_0 : X_0 \to Y_0$. This will prove useful in the forthcoming cohomological obstructions to deformations. We make this precise:

Proposition 3.2 ([13] 3.3)**.** *Let A be a ring and J is an ideal of A such that $J^n = 0$ and $u : M \to N$ a homomorphism of A modules with N flat over A, then if $\overline{u} : M/JM \to N/JN$ is an isomorphism then u is an isomorphism.*

Proof. Let K be the cokernel of u and K' the kernel, then we consider:

$$0 \to K' \to M \to N \to K \to 0,$$

and we tensor this with A/J to find that $K/JK = 0$ and as J is nilpotent this implies that $K = 0$. Therefore, since N is flat tensoring $0 \to K' \to M \to N \to 0$ with A/J gives

$$0 \to K'/JK' \to M/JM \to N/JN \to 0,$$

and similarly we are able to conclude that $K'/JK' = 0$ so $K' = 0$, and therefore u is an isomorphism. $\square$

We begin with deformations of schemes and consider the following theorem. We let for a morphism of schemes $f : X \to Y$, $T_{X/Y}$ denote the relative tangent space. The following is ([3] Theorem 8.5.9):

Theorem 3.3.

(a) Let X and Y be schemes over some scheme S and let Y be smooth over S. Let $j : X_0 \to X$ be a closed immersion defined by some ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $g : X_0 \to Y$ be an S-morphism. Then there is an obstruction

$$\mathfrak{o}(g, j) \in H^1(X_0, \mathscr{I} \otimes_{\mathscr{O}_{X_0}} g^* T_{Y/S})$$

whose vanishing is equivalent to the existence of an S-morphism $h : X \to Y$ extending g. In the case when $\mathfrak{o}(g, j) = 0$, the set of extensions h of g is an affine space under $H^0(X_0, \mathscr{I} \otimes_{\mathscr{O}_{X_0}} g^* T_{Y/S})$

(b) Let X_0 be a separated smooth S_0 scheme with structure morphism f_0. Let $i : S_0 \to S$ be a thickening of order one defined by an ideal $\mathscr{I}$ of square zero, then we have an obstruction

$$\mathfrak{o}(X_0, i) \in H^1(X_0, f_0^* \mathscr{I} \otimes_{\mathscr{O}_{X_0}} T_{X_0/S_0})$$

whose vanishing is equivalent to the existence of a (flat) deformation X/S such that $X \otimes_S S_0 = X_0$. In the case when the obstruction vanishes the set of isomorphism classes of such deformations is an affine space under $H^1(X_0, f_0^* \mathscr{I} \otimes_{\mathscr{O}_X} T_{X_0/S_0})$ and the set of automorphisms of a given deformation is an affine space under $H^0(X_0, f_0^* \mathscr{I} \otimes_{\mathscr{O}_X} T_{X_0/S_0})$

Proof. To begin these two questions, we consider the situation for affine schemes. First consider (a). Let $\operatorname{Spec} B = V \subseteq S$ be an open set. By smoothness given any $y \in Y$ we can find an open set U containing y such that there are integers n, r and equations $g_1, \ldots, g_r \in B[t_1, \ldots, t_n]$ with $\operatorname{rank}(\partial g_j/\partial t_j)_{ij} = r$ (the rank of $A/\mathfrak{m}$ for the unique maximal ideal $\mathfrak{m}$) and $U \cong \operatorname{Spec} B[t_1, \ldots, t_n]/(g_1, \ldots, g_r)$. Let $W_0 = \operatorname{Spec} A_0$ be an affine open subset of $g^{-1}(U_0)$ and let $W = \operatorname{Spec} A$ be its thickening in X. Extending $g : W_0 \to U$ to $W \to U$ is the same as extending $B_0[t_1, \ldots, t_n]/(g_1, \ldots, g_r) \to A_0$ to a map to a map $B[t_1, \ldots, t_n]/(g_1, \ldots, g_r) \to A$. Taking a lift of $t_i \to \overline{a_i}$ to $t_i \to a_i$, we note that $g_j(a_i)$ may not be zero. However, the condition on rank and the fact that modifying by a_i by something in $\mathscr{I}$ will only affect the first order part as $\mathscr{I}^2 = 0$ means that we can modify the lifts so that indeed $g_j(a_i) = 0$. This means that we indeed can always lift affine-locally.

Now we choose open $V = \operatorname{Spec} B \subseteq S$ and $U = \operatorname{Spec} C \subseteq Y_V$ and $W = \operatorname{Spec} A \subseteq g^{-1}(U)$. Let us have two S-morphisms $W \rightrightarrows U$ extending g. On rings, these are two B-algebra morphism $h, h' : C \to B$ extending a given one $h_0 : C \to A_0 = A/J$ where J is an ideal. Let $\phi = h - h'$. Note this is a map $C \to J$ as h, h' agree to A/J.

We will verify by hand that ϕ gives a derivation of C into J. $\phi(aa') = (h - h')(aa') = h(aa') - h'(aa') = h(a)h(a') - h'(a)h'(a') = h(a)h(a') + (\phi(a) - h(a))h'(a') = \phi(a)h'(a') - h(a)(h(a') - h'(a')) = \phi(a)h'(a') - h(a)(h(a') - h'(a') = \phi(a)h'(a) - h(a')\phi'(a')$. Since $\phi(a), \phi(a') \in J$, and changing $h'(a), h(a')$ by anything in J in does not affect$\phi(a)h'(a) - h(a')\phi'(a')$. This means ϕ actually amounts to a derivation A into J over S. This says that locally on Y the extensions are defined up to an S-derivation of $\mathscr{O}_Y$ into g_*J: an element of $\operatorname{Hom}(\Omega_{Y/S}, g_*J)$ locally on Y. This is the same as an element of $\operatorname{Hom}(g^*\Omega_{Y/S}, J)$ locally on X_0, which is a local section of $g^*T_{Y/S} \otimes J$.

Having done the case locally, we cover X with affines $\{U_\alpha\}_\alpha$ such that g extends for all α. Then we note for these to piece together to a morphism, it is necessary and sufficient that they agree on overlaps. We will denote intersections $U_\alpha \cap U_\beta$ by $U_{\alpha\beta}$. On $U_{\alpha\beta}$ the two morphisms, one from U_α one from U_β, thus define a local section, $c_{\alpha\beta}$, of $g^*T_{Y/S} \otimes J$. That $c_{\alpha\beta}$ defines a cocycle is clearly seen. Thus we get an element $[c]$ of $H^1(X_0, g^*T_{Y/S} \otimes J)$. The condition for g_0 being extendable to $X \to Y$ is precisely if the local extensions can be changed into local extensions that agree on overlap. This is precisely the condition that $[c]$ is a coboundary. Thus $[c]$ is the necessary obstruction. Once we have a extension, note that another one differs by a section of $H^0(X_0, g^*T_{Y/S} \otimes J)$, and thus we have that the extensions for an affine space under $H^0(X_0, g^*T_{Y/S} \otimes J)$. We note that the obstruction in H^1 does not depend on the chosen extensions $U_\alpha \to X$, as changes the extensions just changes the obstruction by a coboundary. This means that if we take a refinement of $\{U_\alpha\}$, inheriting these morphisms, we get that the same element in H^2 as the obstruction. From this we conclude that the obstruction did not depend on the initial open cover.

For part (b), we again consider the problem first locally. First choose open $\operatorname{Spec} B_0 = V_0 \subseteq S_0$ and let $\operatorname{Spec} A = V \subseteq S$ be its thickening. As in part (a) we look at an open set of X_0 above V_0 of the form $\operatorname{Spec} B_0[t_1, \ldots, t_n]/(g_1, \ldots, g_r)$, where $g_i \in B_0[t_1, \ldots, t_n]$ and $\operatorname{rank} (\partial g_j / \partial t_j)_{ij} = r$. We can take lifts of the g_i to get a lift of V_0. Since we are working with affines, H^1s vanish, and thus we get a morphism between extensions by part (a), and since they agree over S_0, that morphism must be an isomorphism by flatness, and again by part (a) maps are in bijection sections of the open set of the sheaf $f_0^* \mathscr{I} \otimes T_{X_0/S_0}$. Any morphism of deformations must be isomorphisms by Nakayama's lemma and flatness.

Now we work on building a global extension. We let $\{U_\alpha\}$ be a cover on which we can find extensions. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (and similarly for more indices). We let $\xi_{\beta\alpha}$ be the map taking the deformation of U_α restricted to $U_{\alpha\beta}$ ot the deformation of U_β restricted to $U_{\alpha\beta}$. We let $\xi_{\beta\alpha}^\gamma$ denote $\xi_{\beta\alpha}|_{U_{\alpha\beta\gamma}}$. Scheme gluing says these deformations glue if and only if all $c_{\alpha\beta\gamma} = (\xi_{\gamma\alpha}^\beta)^{-1} \xi_{\gamma\beta}^\alpha \xi_{\beta\alpha}^\gamma$ are the identity. Giving a $c_{\alpha\beta\gamma}$ in this form, however, is the same as giving a local section of $f_0^* \mathscr{I} \otimes T_{X_0/S_0}$. Thinking of the $c_{\alpha\beta\gamma}$ this way, the scheme glues if and only if $c_{\alpha\beta\gamma}$ vanishes. First we claim that $c_{\alpha\beta\gamma}$ is a Cech 2-cocycle. We have that $(\partial c)_{\alpha\beta\gamma\delta}$ is $c_{\beta\gamma\delta} c_{\alpha\gamma\delta}^{-1} c_{\alpha\beta\delta} c_{\alpha\beta\gamma}^{-1}$. Since all the c_{***} can be viewed as local sections of $f_0^* I \otimes T_{X_0/S_0}$, they commute. Therefore, we look at

$$c_{\beta\gamma\delta} c_{\alpha\gamma\delta}^{-1} c_{\alpha\beta\delta} c_{\alpha\beta\gamma}^{-1} = c_{\alpha\gamma\delta}^{-1} c_{\alpha\beta\delta} c_{\beta\gamma\delta} c_{\alpha\beta\gamma}^{-1},$$

and expanding the factors we note everything cancels, so we do have a cocycle. Of course, being a coboundary means that the $\xi_{\alpha\beta}$ can be modified so that gluing is achieved, and therefore,the vanishing of this class $[c]$ is precisely what is needed to ensure an extension. Furthermore, in the case we can extend to X, any other extension can be found modifying $\xi_{\alpha\beta}$ so that it still agrees on overlap, and therefore, we get extensions form precisely elements of $H^1(X_0, f_0^* \mathscr{I} \otimes T_{X_0/S_0})$. We also note that the above considerations gives that an automorphisms of an extension are a gluing of local automorphisms, ie a gluing of local sections of $f_0^* \mathscr{I} \otimes T_{X_0/S_0}$, so therefore a global section of $f_0^* \mathscr{I} \otimes T_{X_0/S_0}$.

Again we note that the obstructions do not depend on the choices of $\xi_{\alpha\beta}$, as a change in this choice just modifies the obstruction by a coboundary. Again the obstruction remains the same under refinement of the open sets and hence is independent of initial open cover.

□

The obstruction behaves well, satisfying the following functorial property:

Proposition 3.4. *Let X_0 and Y_0 be smooth S_0-schemes with the same properties as before with structure morphisms f_0 and g_0 respectively. Let $h_0 : X_0 \to Y_0$ be an S_0-morphism. Then there exist canonical maps under which the obstructions $\mathfrak{o}(X_0, i)$ and $\mathfrak{o}(Y_0, i)$ have the same image in $H^2(X_0, f_0^*\mathscr{I} \otimes h_0^* T_{Y_0/S_0})$:*

$$H^2(X_0, f_0^*\mathscr{I} \otimes T_{X_0/S_0}) \xrightarrow{df_0} H^2(X_0, f_0^*\mathscr{I} \otimes h_0^* T_{Y_0/S_0}) \xleftarrow{f_0^*} H^2(Y_0, g_0^*\mathscr{I} \otimes T_{Y_0/S_0})$$

Proof. On open sets U of Y_0, for every element of $\Gamma(h_0^{-1}(U), f_0^*\mathscr{I} \otimes T_{X_0/S_0})$ we have a map $\Omega_{Y_0/S_0} \to (h_0)_*(f_0^*\mathscr{I} \otimes T_{X_0/S_0})$ found by viewing an element of $f_0^*\mathscr{I} \otimes T_{X_0/S_0}$ as a $\mathscr{O}_{X_0}$ map from Ω_{X_0/S_0} to $f_0^*\mathscr{I}$. Using the adjoint property, this gives us maps for $V \subseteq X_0$ from $\Gamma(V, f_0^* I \otimes T_{X_0/S_0}) \to \Gamma(V, f_0^*\mathscr{I} \otimes h_0^* T_{Y_0/S_0})$, and this induces the first map (where we use smoothness to guarantee local freeness of the cotangent sheaf to guarantees that pullback behaves well with taking duals).

The second map is clearer as it is just the map induced by pullbacks, noting that $h_0^* g_0^* \mathscr{I} = f_0^*\mathscr{I}$ as $g_0 h_0 = f_0$.

We must show that the images of the obstructions under these two maps is the same. We first pick opens of Y_0, $\{U_\alpha\}$, on which we have flat deformations, and $V_{\alpha,i}$ opens on X_0 in $h_0^{-1}(U_\alpha)$ where we have deformations of X_0 and deformations of h_0. Now we view maps from a thickening of an open V in X_0 to a thickening of Y_0 as an element of the affine space under $\Gamma(U, f_0^*\mathscr{I} \otimes g^* T_{Y_0/S_0})$ in part (a) of the deformation of schemes, Theorem 3.3.

We let f be some morphism of $V_{\alpha\beta,ij} \to U_{\alpha\beta}$ deforming h_0. Then on triple intersections $V_{\alpha\beta\gamma,ijk} \to U_{\alpha\beta\gamma}$ the image of obstruction of X_0 will give the section of $f_0^*\mathscr{I} \otimes g^* T_{Y_0/S_0}$ representing $c_{X_0,\alpha\beta\gamma,ijk} f$ and the image of the obstruction of Y_0 will give the map $f c_{Y_0,\alpha\beta\gamma}$ (where the c are the maps on the triple overlaps like in the proof of Theorem 3.3). The fact these are the same has to do with if we change a cocycle f by its conjugate $c_{\alpha\beta\gamma}^{-1} f c_{\alpha\beta\gamma,ijk}$ the two differ by a coboundary. This is seen if to every $V_{\delta,\ell}$ we associate a map from the deformation to the deformation of U_δ. If we view the local sections of $f_0^*\mathscr{I} \otimes h_0^* T_{Y_0/S_0}$ as the difference on $\mathscr{I}$ from some fixed map V to an open containing $f(V)$, then we get that conjugating changes the base map and therefore is the difference on $V_{\delta\epsilon}$ of the basemap from V_δ and that on V_ϵ. Therefore, conjugating by $c_{\alpha\beta\gamma}$ changes everything by a coboundary, and therefore we have indeed that images of the two obstructions are the same. □

We also note that if we have a product $X_0 \times_{S_0} Y_0$, we maps on cohomology i_j corresponding to the components, and the obstruction to lifting $X_0 \times_{S_0} Y_0$ is the same of images of the obstructions to lifting the two individual components. This can be seen using the functoriality, but more easily it can be seen covering $X_0 \times Y_0$

by opens of the form $\operatorname{Spec} A_i \otimes B_\alpha$, where X_0 is covered by opens of the form $\operatorname{Spec} A_i$ and Y_0 opens of the form $\operatorname{Spec} B_\alpha$.

We have the following corollary to Theorem 3.3:

Corollary 3.5 ([3], 8.5.19). *Let A be a complete local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k. Let X_0 be a separated smooth scheme over $\operatorname{Spec} k$ with $H^2(X_0, T_{X_0/k}) = 0$, then there exists a smooth formal scheme over $\widehat{S}$ lifting X_0.*

Proof. We let $S_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$

We assume we already have deformed X_0 to X_n/S_n. The obstruction to deforming X_n to S_{n+1} lives in $H^2(X_0, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes T_{X_n/S_n})$. However $\mathfrak{m}$ since $A/\mathfrak{m}^{n+1} \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} = \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes A/\mathfrak{m}$, we get

$$H^2(X_0, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes T_{X_n/S_n}) = H^2(X_0, T_{X_0/k}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2},$$

and this vanishes, so by Theorem 3.3 X_n/S_n lifts to X_{n+1}/S_{n+1}. $\square$

Example 3.6. Since $H^2(C_0, T_{C_0/k})$ vanishes for smooth curves C_0, curves supply us already with a large class of objects that can be deformed. We will see later than the deformations of proper curves can even always be algebraized.

We also give the analogous statement for schemes for vector bundles:

Theorem 3.7 ([3] Theorem 8.5.3). *Let $i : X_0 \to X$ be a closed immersion defined by an ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$*

(a) If E and F are vector bundles on X and $i^*E = E_0$ and $i^*F = F_0$, and we have $u_0 : E_0 \to F_0$ be a map of vector bundles on $\mathscr{O}_{X_0}$, then there is an obstruction

$$\mathfrak{o}(u_0, i) \in H^1(X_0, \mathscr{I} \otimes \mathcal{H}om(E_0, F_0))$$

whose vanishing is necessary and sufficient for the existence of a extension $u : E \to F$ over X. When the obstruction vanishes, the extensions form an affine space under $H^0(X_0, \mathscr{I} \otimes \mathcal{H}om(E_0, F_0))$

(b) If E_0 is a vector bundle over X_0, there is an obstruction:

$$\mathfrak{o}(E_0, i) \in H^1(X_0, \mathscr{I} \otimes \mathcal{E}nd(E_0, F_0))$$

whose vanishing is necessary and sufficient for the existence of a vector bundle E on X such that $i^*E = E_0$. When the obstruction vanishes the set of extension is an affine space under $H^1(X_0, \mathscr{I} \otimes \mathcal{E}nd(E_0, F_0))$

We do not provide the proof for this statement, but only remark that it proceeds along very similar lines to the proof for extensions of schemes.

Corollary 3.8 ([3] 8.5.5)**.** *Let A is a complete local noetherian ring with maximal ideal $\mathfrak{m}$. Let $S_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$ and $\widehat{S} = \varinjlim S_n$. We let $\mathcal{X}$ be a flat adic locally noetherian formal scheme over $\widehat{S}$ with $X_n = S_n \times_{\widehat{S}} \mathcal{X}$. If $H^2(X_0, \mathscr{O}_{X_0}) = 0$ then given a line bundle L_0 on X_0, the line bundle can be lifted to a line bundle L on $\mathcal{X}$. If $H^1(X_0, \mathscr{O}_{X_0}) = 0$ such liftings are unique.*

Proof. We assume we can have lifted L_0 to X_n. We note that the obstruction to lifting it to X_{n+1} lives in $H^2(X_n, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes_{\mathscr{O}_{X_0}} \mathcal{E}nd(L_0))$. Now since we are working a line bundle, this is just $H^2(X_n, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes \mathscr{O}_{X_n}) = H^2(X_0, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes_{\mathscr{O}_{X_0}} \mathscr{O}_{X_0}) = H^2(X_0, \mathscr{O}_{X_0}) \otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$ as $\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$ is killed by $\mathfrak{m}$. Now by (b) of the above theorem we have the uniqueness of lifting to each X_n and hence to $\mathcal{X}$.

We get that the lack of automorphisms by showing that at every level there are no automorphism. Completely analogously, $H^1(X_n, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes \mathscr{O}_{X_n} = H^1(X_0, \mathscr{O}_{X_0}) \otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Therefore, if the $H^1(X_0, \mathscr{O}_{X_0})$ vanishes, there will be no automorphisms. □

We will not give the details, but will state one particular case when a formal scheme can be concluded to algebraizable.

Theorem 3.9 ([3], 8.4.10)**.** *Let $\mathcal{X} = \varinjlim X_n$ be a proper, adic $\widehat{Y}$-formal scheme, where $X_n = \mathcal{X} \times_{\widehat{Y}} Y_n$. If L is an invertible $O_{\mathcal{X}}$ module such that $L_0 = L \otimes \mathscr{O}_{X_0}$ is ample. $\mathcal{X}$ is algebraize, and if X is a proper Y-scheme such that $\widehat{X} = \mathcal{X}$, then there exists a unique line bundle M on X such that $L = \widehat{M}$ and M is ample.*

Remark 3.10. We return the discussion of C is a proper curve over a field. We recall that 3.5 implied that if A is a local noetherian ring with residue field k, there existed a formal deformation of C to Spf A. Now, as proper curves are always projective we can pick an ample line bundle $\mathscr{L}$ on C. Since $H^2(C, \mathscr{O}_C)$ for curves, Corollary 3.8 ensures that $\mathscr{L}$ can be lifted to a line bundle on formal deformation and Theorem 3.9, give that the formal deformations must in fact by algebraizable. Therefore, for this one set of examples, in fact all deformations to local noetherian rings are algebraizable. This is not always the case, but we must wait until after a discussion of abelian varieties to give an example.

4 Variety with no deformation to characteristic zero

In this section we give Serre's example from [3] of variety over a field of characteristic p with no deformation to characteristic zero.

The general idea of Serre's example is to construct a representation of a group G over a field of characteristic $p > 0$, which does not lift to characteristic zero. From this, we construct an object such that if the deformation was algebraizable we could also get a lift of the representation to characteristic zero.

We begin by building the object to be deformed.

The setup is as followed: We let k be an algebraically closed field of characteristic p, G a finite group, $\rho : G \to \mathrm{PGL}_{n+1}(k)$ a representation. We set $P = \mathbb{P}^n_k$ and note that G acts on p via ρ. We then set Q to be the union over $g \in G$ of the fixed points of P by g.

A key proposition in the construct is the following.

Proposition 4.1. *If $r + \dim Q < n$, then there exists $d_0 > 0$ such that for any $d | d_0$ such that there is a smooth intersection, $Y = V(h_1, \ldots, h_{n-r})$, of dimension r in P with the property that $\deg(h_i) = d$ for all i, and this Y is stable under the G action and G acts freely on it.*

Proof. Since P is projective, and G acts freely, we get that we can form the quotient $Z = P/G$, and furthermore we have $(f_*\mathscr{O}_P)^G = \mathscr{O}_Z$ and since G is finite the map $f : P \to Z$ is finite ([10] Theorem 2.1)). Furthermore, we can conclude that Z is projective.

Since Z is projective, we can pick an embedding $i : Z \to \mathbb{P}^n_k$, and hence $(if)^*\mathscr{O}_{\mathbb{P}^n_k}(1) = \mathscr{O}_P(d_0)$ for some $d_0 > 0$. Now, denote i_m to be the composition of i with the mth Veronese embedding and let $c(m) = \binom{n+m}{m}$, then we get that $(i_m f)^*\mathscr{O}_{P_k^{c(m)-1}}(1) = \mathscr{O}_p(md_0)$.

Now we have that f is finite, hence a closed map, so $f(Q)$ is closed in Z, and since f is finite, we have too that $\dim(f(Q)) = \dim(Q)$. By Bertini like theorem ([7] Corollary 6.11) we may find L, the intersection of $n - r$ hyperplanes in $P_k^{c(m)-1}$, such that $L \cap Z$ does not meet $f(Q)$ and L is transversal to $Z - f(Q)$.

Let $U = Z - f(Q)$. It is clear $f^{-1}(U)$ is stable under the G action and as it does meet Q G acts freely on it. Thus $f|_{f^{-1}U}$ is étale as it is the quotient of a free action by the same theorem of [10]. Therefore, if we look at local rings of generic points of irreducible components of $f^{-1}(U)$ we can conclude that the pull back of the hyperplanes, ℓ_i, given by Bertini's theorem give a complete intersection. Y satisfies the desired properties, so we are done. $\square$

Next we take $d \geq 2$ and Y as given in the last proposition. Let $X = Y/G$ be the quotient of Y by G, and again by we have that X is smooth over k, projective, and because G is finite $\dim X = \dim Y = r$, and $f : Y \to X$ is an étale cover.

The lack of deformation will follow from the following proposition:

Proposition 4.2. *If $r \geq 3$ or $(p, n+1)$ and $p|d$ and A is a complete local noetherian ring with residue field k, then if $\mathcal{X}$ is a flat, formal scheme lifting X, then $\mathcal{X}$ is algebraizable with algebraization X_1 over A. Furthermore, in the above case X_1 is projective and smooth over A and the representation ρ_0 lifts to a representation $\rho : G \to \mathrm{PGL}_{n+1}(A)$*

Given this proposition, if we can find a representation that does not lift to characteristic zero, we will have an example of a scheme that deform to characteristic zero.

Before we begin we will need a lemma:

Lemma 4.3. *Let C in $\mathbb{P}^n_k$ be a complete intersection of positive dimension cut out by equations $f_1, \ldots, f_r$. Then for all $1 \leq i < n - r$ and all $d \geq 0$, $H^i(C, \mathscr{O}_C(-d)) = 0$, for $i = 0$ $H^0(C, \mathscr{O}_C) = k$ and $H^0(C, \mathscr{O}_C(-d)) = 0$ for $d \geq 1$. If the degree of each f_i is greater than equal to 2 we also have that $H^0(C, \mathscr{O}_C(1)) = k^{n+1}$ and $H^i(C, \mathscr{O}_C(1)) = 0$ for all $i \geq 1$*

Proof. The base case is just a fact known from the cohomology of projective space.

We assume we have the statement for $C = V(f_1, \ldots, f_r)$, and we consider $C' = V(f_1, \ldots, f_r, f_{r+1})$ a complete intersection of one dimension lower. We let $d_{r+1} \geq 1$ be the degree of f_{r+1}. Let i be the inclusion of C' into C. Consider:

$$0 \to \mathscr{O}_C(-d_{r+1}) \to \mathscr{O}_C \to i_* O_{C'} \to 0,$$

or more generally consider the twisting for $d \geq 0$:

$$0 \to \mathscr{O}_C(-d_{r+1} - d) \to \mathscr{O}_C(-d) \to i_* O_{C'}(-d) \to 0.$$

We look at the piece of the long exact sequence for $i \geq 1$:

$$H^i(C, \mathscr{O}_C(-d)) \to H^i(C', O_{C'}(-d)) \to H^{i+1}(C, \mathscr{O}_C(-d - d_{r+1})).$$

By the induction hypothesis, for $i < n - r - 1$ the first and last terms vanish, and therefore the middle term also vanishes.

For $i = 0$, we look at the segment of the long exact sequence:

$$0 \to H^0(C, \mathscr{O}_C(-d)) \to H^0(C', \mathscr{O}_{C'}(-d)) \to H^0(C, \mathscr{O}_C(-d - d_{r+1})).$$

By induction the last term always vanishes and the first term vanishes if $d \geq 1$ and is k if $d = 0$.

If all the degrees of f_i are greater than or equal to 2, we consider the following piece of the associated long exact sequence:

$$0 \to H^0(C, \mathscr{O}_C(1)) \to H^0(C', \mathscr{O}_{C'}(1)) \to H^0(C, \mathscr{O}_C(1 - d_{r+1})).$$

As $d_{r+1} \geq 2$, the last term is zero so the first two terms are isomorphic, and hence $H^0(C', \mathscr{O}_{C'}(1)) = k^{n+1}$.

Similarly looking at:

$$H^1(C, \mathscr{O}_C(1)) \to H^1(C', \mathscr{O}_{C'}(1)) \to H^1(C, \mathscr{O}_C(1 - d_{r+1})),$$

we see by induction the first and last terms are zero as $d_{r+1} \geq 2$, and hence the middle term is zero.

This completes the proof of the lemma. □

We will only prove the case when $r \geq 3$ here. We will give the whole proof with the exception of details relating to quotients by group actions and facts about the norm of an ample line bundle.

Proof of Proposition 4.2. Let us assume that X deforms to a formal scheme $\mathcal{X}$. Let X_n be the successive thickenings of X be the defining ideal. Assume that Y_0 extends to finite étale Y_n over X_n and any two such deformations over X_n agree up to unique isomorphism as deformations. We seek to extend Y_n to Y_{n+1}. We apply Theorem 3 to the morphism $Y_n \to X_{n+1}$ given by the composite $Y_n \to X_n \to X_{n+1}$. However, since Y_n is finite étale over X_n, the T_{Y_n/X_n} vanish and all the cohomology groups in question vanish, so indeed we can find a deformation Y_{n+1} and since even H^0 vanishes, it is unique up to unique isomorphism by Theorem 3.3.

Now by the uniqueness of lifting at every step and the fact we can lift after a g action for $g \in G$, we find the action of G on Y_0 extends uniquely to an action on $\mathcal{Y}$. Furthermore, since $Y_0 \to X_0$ is G-Galois, it is not hard to see that $Y_n \to X_n$ remains G-Galois, as we get maps $gY_n \to X_n$ and X_n is equivariant under the G action, so we get a map $Y_n/G \to X_n$, but it is an isomorphism over k so must be an isomorphism by flatness.

From the lemma we get that $H^0(Y_0, \mathscr{O}_{Y_0}) = k$ and that $H^1(Y_0, \mathscr{O}_{Y_0}) = 0$. Then by cohomology and base change, since H^1 vanishes, we get that $H^0(Y_m, \mathscr{O}_{Y_m}) = A_m$ for all m.

Set $L_0 = \mathscr{O}_{Y_0}(1)$. We claim L_0 lifts to a line bundle $\mathscr{L}$ on $\mathcal{Y}$. In the case that $r \geq 3$, can apply the lemma again to ensure $H^2(Y_0, \mathscr{O}_{Y_0}) = 0$, so the line bundle does lift so by Theorem 3.8 the line bundle does lift and since $H^1(Y_0, \mathscr{O}_{Y_0}) = 0$ the lift is unique.

Then since L_0 is ample, there exists a projective, flat scheme $\mathcal{Y}$ is algebraizable. By ([4]2:6.6.1) we have that $E_0 = N_{Y_0/X_0}L_0$ is an ample line bundle on X_0. Then we can let $E_m = N_{Y_m/X_m}L_m$, and thus $\mathcal{E} = \lim E_n$ lifts E_0, and since E_0 was ample, we get that $\mathcal{X}$ is algebraizable and its algebraization X is projective by Theorem 3.9.

The action of G on $\mathcal{Y} \to \mathcal{X}$ gives an action on $Y \to X$ by Theorem 2.9, and we get that $E = N_{Y/X}L$ by the analogous theorem for line bundles ([3] Theorem 8.4.2).

By Lemma 4.3, we have $H^0(Y_0, L_0) = k^{n+1}$ and $H^1(Y_0, L_0) = 0$. Again the vanishing of H^1 and cohomology and basechange allow us to conclude that $H^0(Y_m, L_m) = A_m^{n+1}$. Therefore $H^0(Y, L) = A^{n+1}$. Now, again by the discussion after Theorem 2, since we have $H^1(Y_0, \mathscr{O}_{Y_0}) = 0$ there is an isomorphism $a(g)$ of $L \to L$ above $g : Y \to Y$ and it is unique up to isomorphism automorphism of L_0. Locally at global sections, thus we get a representation $\rho : G \to \mathrm{PGL}(H^0(Y, L)) = \mathrm{PGL}_{n+1}(A)$.

We note that this restricts to an action $G \to \mathrm{PGL}_{n+1}(k)$ on Y_0. Since the global sections of L_0 are spanned by i^*x_j where x_j are the homogeneous variables on P, and G acts on $H^0(Y_0, P_0)$ be ρ_0, so ρ lifts ρ_0. □

The next step is to present a characteristic p representation that cannot be lifted to characteristic zero. We let r and n be such that $1 \leq r < n$ and $s \geq n+1$ and $G = \mathbb{F}_p^s$. We assume that $p \geq n+1$ so that we can take exponential of matrices. We choose an injective homomorphism of additive groups, $h : G \to k$. We then let N be the $(n+1) \times (n+1)$ matrix with 1s above the diagonal and zeros everywhere else.

We then define $\widetilde{\rho}_0(g) = \exp(h(g)N)$ and $\rho_0(g)$ the image of $\widetilde{\rho}_0(g)$ in $\mathrm{PGL}_{n+1}(k)$. The exponential is makes sense as powers of N move the diagonal of 1s up and to the right until $N^{n+1} = 0$ This is clearly faithful as $h(g)$ is injective.

Next we claim:

Proposition 4.4. *In the above situation, if A is a noetherian local ring with residue field k, and K is field of fractions of characteristics zero, then there is no lifting $\rho : G \to \mathrm{PGL}_{n+1}(A)$ of ρ_0*

Proof. We suppose by contradiction that such a ρ exists. Clearly ρ is injective since ρ_0 is. Furthermore, we can view $\rho : G \to \mathrm{PGL}_{n+1}(\overline{K})$. Let $V = \overline{K}^{n+1}$

Next we claim ρ can be lifted to a representation $\rho' : G \to \mathrm{GL}_{n+1}(\overline{K})$. Let $g_1, \ldots, g_p$ generate G, we can clearly pick lifts $\rho'(g_i)$ such that $\det \rho'(g_i) = 1$. Note then that $(\rho'(g_i))^p = c$ such that $c^{n+1} = 1$ and since $p > n+1$ raising to the pth power is a bijection on the $n+1$ roots of unit, so we can notify $\rho'(g_i)$ again to additionally assure that $\rho'(g_i)^p = 1$. The ρ' induced by these choices is clearly well defined, so we have $\rho' : G \to \mathrm{SL}(V)$. It is clear that this is still faithful.

As G is commutative and $\overline{K}$ has characteristic zero, we have a decomposition into one dimensional $\overline{K}[G]$ modules $V = \oplus_{i=1}^{n+1} V_i$. Let the associated characters be χ_i. As the image of ρ' is in $\mathrm{SL}(V)$, we have that $\prod_i \chi_i = 1$.

Let H_i be the kernel of the χ_i. Since ρ' is faithful, $\cap_i H_i = \{1\}$. However, each χ_i is a map $G \to \mu_p(\overline{K})$, and therefore is a map from an s-dimensional vector space to a 1 dimensional vector space. Since $\prod_i \chi_i = 1$ there is a linear dependence between the maps of $G \to \mu_p(\bar{(K)})$ viewed as maps over $\mathbb{F}_p$. Then as $s \geq n+1$, we get that χ_i as viewed as $\mathbb{F}_p$ vector space maps have nontrivial common kernel. Therefore, we get $\cap_i H_i$ is nontrivial, and therefore ρ' is actually not faithful. This is a contradiction, so ρ_0 in fact cannot lift. □

Therefore, if a formal scheme X deformed to a formal scheme over characteristic zero, we would get that the repreatnation lifts. Since the representation does not lift, no such deformation to characteristic zero is possible.

5 Schlessinger's Criteria

This section follows [13].

Letting Λ be a complete noetherian local ring, $\mathfrak{n}$ its maximal ideal, and $k = \Lambda/\mathfrak{n}$ its residue field. We let C be the category of artinian local Λ-algebra. We say a covariant functor F be a covariant functor from C to **Set** is *pro-representable* if it isomorphic to $\mathrm{Hom}(R, A)$ where $A \in C$ and the Hom is taken as local Λ-algebras.

It is often useful and interesting to ask when a given functor is representable. In particular having the representing ring will allow us to get all deformations by base-changing from one object. Futhermore, the deformation ring, will tells us the formal completion of a point on a moduli scheme, and thus can help us glean local information about moduli problems.

The main theorem of the paper is copied below ([13] Theorem 2.11). A *small extension* of local artinian Λ-algebras is a is a map of local artinian Λ-algebras $A \to B$ such that the kernel I is killed by the maximal ideal of A

Theorem 5.1 (Schlessinger). *If F is a functor from C to* **Set** *such that $F(k)$ is one point, Consider the following map*

$$F(A' \times A'') \to F(A') \times_{F(A)} F(A'')$$

Then F is pro-representable if the following hold:

(a) The above is a surjection whenever $A'' \to A$ is a small extension.

(b) The above map is a bijection whenever $A = k$ and $A'' = k[\epsilon]$

(c) If $t_F \cong \mathrm{Hom}_\Lambda(R, k[\epsilon])$ and $\dim(t_F) < \infty$

(d) If for any small extension $A' \to A$ the following is a bijection

$$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$$

More can be said when the functor F is smooth:

Definition 5.2. A functor F is smooth if for any surjective map of artin local rings $A \to B$ in C, $F(B)$ surjects onto $F(A)$.

In particular we have the following:

Proposition 5.3. *If R is a local noetherian Λ algebra with residue field k, then the functor from C to* **Set***, $h_R(A) := \mathrm{Hom}_\Lambda(R, A)$, is formally smooth if and only if R is a formal power series ring over Λ.*

Proof. First we assume that h_R is formally smooth. Let $\mathfrak{m}$ be the maximal ideal of R, $\mathfrak{n}$ be the maximal ideal of Λ, and let us consider $\mathfrak{m}/(\mathfrak{n}, \mathfrak{m}^2)$. This is a k vector space and hence has a basis. By the noetherian property, we have the basis is finite. We choose $x_1, \ldots, x_n$ to be elements of R in $\mathfrak{m}$ form in the basis in $\mathfrak{m}/(\mathfrak{n}, \mathfrak{m}^2)$. Note that $x_1, \ldots, x_n$ will then generate the maximal ideal of $R/\mathfrak{n}$.

I claim that R is isomorphic to $\Lambda[[t_1, \ldots, t_n]]$. Let the maximal ideal of $\Lambda[[t_1, \ldots, t_n]]$ be M. Note that $M = (\mathfrak{n}, t_1, \ldots, t_n)$. We do this in the following way. Note by the above the we get a map $R \to \Lambda[[t_1, \ldots, t_n]]/M$. We assume by induction we have a surjective map $R \to \Lambda[[t_1, \ldots, t_n]]/M^m$. By smoothness, we get a map $R \to \Lambda[[t_1, \ldots, t_n]]/M^{m+1}$. Note since this is a map of Λ algebras, we must have the Λ action, $\Lambda \to R \to \Lambda/\mathfrak{n}^{m+1} \subseteq \Lambda[[t_1, \ldots, t_n]]/M^{m+1}$ is surjective. Therefore, as this is a surjection on $\Lambda[[t_1, \ldots, t_n]]/M^m$ and have in the image elements that reduce to t_i mod M^2, we get that this map too is surjective. We therefore, get a compatible set of surjections $R \to \Lambda[[t_1, \ldots, t_n]]/M^m$. It is clear looking at where the x_i live that for each of these, $\mathfrak{m}^m$ must be killed. Therefore, we have a surjective map $R/\mathfrak{m}^m t \to \Lambda[[t_1, \ldots, t_n]]/M^m$. The right hand side has fewer less relations than the left hand side, so now surjectivity implies all these are isomorphisms, and we have built an isomorphism $R \to \Lambda[[t_1, \ldots, t_n]]$.

Now we consider the case when $R = \Lambda[[t_1, \ldots, t_n]]$ is already known. Note that any Λ algebra surjection $A \to B$ and map $R \to B$ can be lifted to $R \to A$ just by lifting the images of t_i, so in this case h_R is smooth. $\square$

Thus in the case of smooth prorepresentable functors, we know even that R is power series ring

Schlessinger then considers three examples of this in use. The one most of interest to us is the discussion of the representability of the deformation functor. We begin with a scheme over X/k where k is still and field and we let and we let for which for local artinian A algebras gives:

$$\mathbf{D}(A) = \{\text{isomorphism classes of flat deformations of } X/k \text{ to } A\}$$

The key result relating to these deformations is the following

Theorem 5.4 (Schlessinger)**.** *If X is either*

1. *proper over k or*
2. *affine with at most isolated singularities*

then $\mathbf{D}$ is prorepresented iff for each small extension $A' \to A$ and each deformation Y' of X/k to A', every automorphism of the deformation $Y \otimes_{A'} A$ is induced by an automorphism of Y'

Schlessinger proves that this last condition is equivalent to condition (d) of Theorem 5.1 in this context of deformations of schemes. We note that for schemes that are guaranteed to having liftings, like for instance for curves as their H^2 vanishes, we have deformation ring is a power series ring over Λ. This will come up in the next section.

With this we can define the following:

Definition 5.5. If X is a k-scheme, a *universal deformation* of X with respect to artin local Λ algebras, is a (formal) scheme X_u over the ring R that represents the deformation functor, such that deformations to Λ local rings B correspond to maps $R \to B$ via $(\gamma : R \to B) \mapsto X_u \otimes_{R,\gamma} B$.

If the deformation functor of a scheme is representable, we can recover the universal representation as a formal scheme simply by taking formal limits over deformations to local artin local rings found by taking quotients. Base change from U captures all the deformation properties to local Λ algebras. Note a universal deformation exists as a formal scheme.

6 Two examples

We will compute the universal deformations of two particular curves. Before we begin to do that we will need a few propositions

Proposition 6.1. *Let R and R' be two abstractly isomorphic complete local Noetherian rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ respectively and residue field k, $\phi : R \to R'$ be a local map.*

If for every linear form, ℓ on $\mathfrak{m}'/(\mathfrak{m}')^2$ the associated map $R \to k[\epsilon]$ is not a map only into k, then ϕ is an isomorphism.

Proof. Note that it is immediate that $R/\mathfrak{m} \xrightarrow{\phi} R'/\mathfrak{m}'$ by localness of ϕ.

We note that the condition above guarantees that $(\mathfrak{m}'/\mathfrak{m}'^2)^\vee \to (\mathfrak{m}/\mathfrak{m}^2)^\vee$ is injective, which then implies that $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/\mathfrak{m}'^2$ is a surjection. Since these are abstractly isomorphic k-vector spaces, this implies this is an isomorphism. Therefore, we have that $\mathfrak{m}' = \phi(\mathfrak{m}) + \mathfrak{m}'^2$. By induction we assume that for a given $n > 0$, $\mathfrak{m}'^n = \phi(\mathfrak{m}^2) + \mathfrak{m}'^{n+1}$. Multiplying by $\mathfrak{m}'$ we get $\mathfrak{m}'^{n+1} = \phi(\mathfrak{m}^n)\mathfrak{m}'^{n+1} + \mathfrak{m}'^{n+2}$, and substituting $\mathfrak{m}' = \phi(\mathfrak{m}) + \mathfrak{m}'^2$, we get $\mathfrak{m}'^{n+1} = \phi(\mathfrak{m}^{n+1}) + \mathfrak{m}'^{n+2}$. Therefore, for all $n > 0$ we have $\mathfrak{m}'^n = \phi(\mathfrak{m}^n) + \mathfrak{m}'^{n+1}$. This implies that ϕ is surjective as a map from $\mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{m}'^n/\mathfrak{m}'^{n+1}$, and again by dimension considerations this implies it is an isomorphism. Note that this enough, by dimension considerations to conclude that $R/\mathfrak{m}^n \to R'/\mathfrak{m}'^n$ is an isomorphism. We can therefore conclude that ϕ is an isomorphism from R to R'

□

The following proposition will be useful to argue that given we know the automorphism of curves in question come from those of the ambient $\mathbb{P}^2_k$ that in order to have identity on those curves we must have identity on the $\mathbb{P}^2_k$.

Proposition 6.2. *Let X be a closed subscheme of $\mathbb{P}^n_k$ such that $H^0(X, \mathscr{O}_X) = k$ and such that $H^0(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(1)) \to H^0(X, \mathscr{O}_X(1))$ is an injection, and let f be an automorphism of $\mathbb{P}^n_k$ that restricts to the identity on X. Then f is the identity on $\mathbb{P}^n_k$.*

Proof. First we note that we have a map $H^0(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(1)) \to H^0(\mathbb{P}^n_k, f^*\mathscr{O}_{\mathbb{P}^n_k}(1))$ sending $x_i \mapsto f^*x_i$.

We then pick a map $f^*\mathscr{O}_{\mathbb{P}^n_k}(1) \to \mathscr{O}_{\mathbb{P}^n_k}(1)$ restricting to an automorphism of $\mathscr{O}_X(1)$. Note that since the $H^0(X, \mathscr{O}_X) = k$ this automorphism on $\mathscr{O}_X(1)$ corresponds to scaling by an element of $c \in k^*$ as locally $\mathscr{O}_X(1)$ is free of rank 1 so locally it must be a scaling, and gluing these local scalings we get an invertible element of $\mathscr{O}(X)$.

Now we have the composite map:

$$H^0(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(1)) \to H^0(\mathbb{P}^n_k, f^*\mathscr{O}_{\mathbb{P}^n_k}(1)) \xrightarrow{\phi} H^0(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n_k}(1))$$

Identifying the last two terms in as subgroups of $H^0(X, \mathscr{O}_X)$, we see that the composite map is multiplication by c.

We must translate this back to tell us what this means about f. Note that a choice of isomorphism ϕ encodes the change of coordinates, or the change map of $f_*x_0, \dots, f_*x_n$ to linearly independent sums of $x_0, \dots, x_n$ giving $n+1$ sections of $\mathscr{O}_{\mathbb{P}^n_k}(1)$. This is a change of coordinates. This says the associated change of coordinate matrix is a scaling, so its effect on $\mathbb{P}^n_k$ the identity, so $f = \mathrm{id}$ $\square$

We also prove a commutative algebra proposition that will be useful later:

Proposition 6.3. *Let $\phi : A \to B$ be a local map of local Noetherian rings with residue fields k_A and k_B respectively and maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. Set $M_0 := M \otimes_A k_A$ is $B_0 := B \otimes_A k_A$. If M is A-flat and finitely generated over B, then M is B-free iff M_0 is B_0 free.*

Proof. Note that forward implication is trivial, so we assume that M_0 is B_0-free.

Note first that M is finitely generated over B, so M is $\mathfrak{m}_B$-adically separated. Note that since ϕ is a local map that this implies that M is $\mathfrak{m}_A$-adically separated.

Let us have sequence $0 \to K \to \oplus_n B \to M \to 0$. By Matsumura Theorem 22.3 [9] $\mathrm{Tor}_1^A(k_A, M)$ vanishes as M is a A-flat and M is $\mathfrak{m}_A$-adically separated. Therefore tensoring, we get $0 \to K \otimes k_A \to \oplus_n B_0 \to B_0 \to M_0 \to 0$. Note that since M_0 is B_0 free, it is B_0 flat, so again by Matsumura, we get that $\mathrm{Tor}_1^{B_0}(k_B, M_0) = 0$, so $0 \to K \otimes k_B \to \oplus_n k_B \to M_0 \otimes k_B \to 0$. However, the exactness of this sequence computes that $\mathrm{Tor}_1^B(k_B, \oplus_n B)$ surjects onto $\mathrm{Tor}_1^B(k_B, M)$, but the former is zero so the latter is zero. Therefore, by Matsumura again M is B-flat and since B is local, M is B-free, and we are done. $\square$

6.1 Example One; an elliptic curve

For this section we will let k be a field of characteristic neither 2 nor 3, and we let Λ be a complete noetherian local ring with residue field k. Note that the characteristic of Λ be 0 or that of k. For the remainder of this section C_0 we be the curve defined by $y^2 = x^3 + x$ in $\mathbb{P}^2_k$. We will consider deformations of this curve.

The point of this section is to prove:

Theorem 6.4. *The universal deformation of the curve C_0 is the curve $y^2 = x^3 + x + t$ in $\mathbb{P}^k_{\Lambda[[t]]}$.*

By the rigidity theorem for elliptic curves([8] Theorem 2.4.1) gives us that there are no automorphisms of deformations of X to artinian local Λ-algebra A which fixes a chosen section and reduces to the identity on C_0. Therefore, the only automorphisms that reduce to the identity on C_0 are those that come from translation. In particular, Schlessinger's criteria are satisfied.

Schlessinger [13] gives that get the tangent space to deformations is given by $H^1(C_0, T_{C_0/k}) \cong H^0(C_0, \Omega^1_{C_0/k})$, which is one dimensional as C_0 is a genus one curve. By smoothness, we then have that the deformation ring is abstractly isomorphic to $\Lambda[[t]]$.

Now by the Proposition 6.1 to prove that the candidate $y^2 = x^3 + x + t$ in $\mathbb{P}^2_{\Lambda[[t]]}$ is the universal deformation, it suffices to check that $y^2 = x^3 + x + t$ in $\mathbb{P}^2_{k[\varepsilon]/\varepsilon^2}$ is nontrivial. We call this curve D. This is to say not isomorphic to $y^2 = x^3 + x$ in $\mathbb{P}^2_{k[\varepsilon]/\varepsilon^2}$. We call this curve C_{trv} to denote it is the trivial deformation. To do this we will argue that if they are isomorphic the isomorphism comes from an automorphism on the ambient projective space.

Let e be the section at infinity. We first prove the following lemma:

Lemma 6.5. *For any curve E over a noetherian base S with geometrically connected genus one fibers and proper smooth structure map $f : E \to S$, for $n > 0$, $f_*\mathscr{O}(ne)$ is locally free of rank n. Furthermore,*

$$0 \to f_*\mathscr{O}(ne) \to f_*\mathscr{O}((n+1)e) \to f_*(\mathscr{O}((n+1)e)/\mathscr{O}(ne)) \to 0$$

is exact, and finally $\mathscr{O}_S \to f_*\mathscr{O}_E$ is an isomorphism

Proof. First we note that f is proper and smooth, hence flat. Then we consider the ideal sheaf, $\mathscr{I}$, defining the section e . We note that from the sequence

$$0 \to \mathscr{I} \to \mathscr{O}_S \to \mathscr{O}_E \to 0,$$

and the fact that the last two terms are flat that the first term must be flat. We see that the formation of $\mathscr{I}$ commutes with base change.

We then wish to claim that $\mathscr{I}$ is an invertible sheaf. By the fact that its formation commutes with base change and by Proposition 6.3, we can check this residually, where it is the case as e restricts to sections of elliptic curves over a field.

This we have that $\mathscr{O}(ne) = \mathscr{I}^{-n}$ and the formation of this commutes with base change. If $n > 0$, at every point $q \in S$, $h^1(E_q, \mathscr{O}(ne)_q) = 0$ by Serre duality for proper smooth curves over a field. By cohomology and base change we then get that for any $q \in S$, there is an open set U containing q such that $f_*\mathscr{O}(ne)$ is free on U

and furthermore ϕ_q^0 the cohomology and base change morphism is an isomorphism, and therefore it is free on U of rank $h^0(E_q, \mathscr{O}(ne)_q) = n$.

We can conclude the exact sequence

$$0 \to f_*\mathscr{O}(ne) \to f_*\mathscr{O}((n+1)e) \to f_*(\mathscr{O}((n+1)e)/\mathscr{O}(ne))$$

if we show that $R^1 f_*\mathscr{O}(ne) = 0$. We can conclude by noting locally this is true by cohomology and base change and the fact that $H^1(E_q, \mathscr{O}(ne)_q) = 0$ for $q \in S$.

It remains to show that the natural map $\mathscr{O}_S \to f_*\mathscr{O}_E$ is an isomorphism. Again, we do this by cohomology and basechange. We note first that for any $q \in S$, $H^0(E_q, \mathscr{O}_{E_q}) = k_q$. To see that ϕ_q^0 is surjective we examine:

$$\mathscr{O}_S \otimes k_q \to (f_*\mathscr{O}_E) \otimes k_q \xrightarrow{\phi_q^0} H^0(E_q, \mathscr{O}_{E_q})$$

takes 1 to 1 and hence by dimension considerations is surjective. Since ϕ_q^0 is surjective, it is an isomorphism. Since ϕ_q^{-1} is trivially surjective, $f_*\mathscr{O}_E$ is locally free of rank 1 and hence a line bundle. Thus to check that $\mathscr{O}_S \to f_*\mathscr{O}_E$ is an isomorphism it suffices to check residually, where it becomes the calculation that $\mathscr{O}_S \otimes k_q \to (f_*\mathscr{O}_E) \otimes k_q$ is an isomorphism which follows the what has been proved about the last exact sequence above. This completes the proof of the lemma. □

From the lemma we are able to conclude that the global sections $\mathscr{O}(2e)$ and $\mathscr{O}(3e)$ are free $k[\epsilon]$ algebras of rank 2 and 3 respectively, and that the formation of these global sections behaves well with respect to basechange to the closed point.

Next we will argue that x and y are global sections of $\mathscr{O}(2e)$ and $\mathscr{O}(3e)$ respectively. We begin by noting that $D - e$ is affine as $D - e$ is a closed subscheme of an affine of $\mathbb{P}^2$. To do this we will prove the following lemma:

Lemma 6.6. *Let X be a scheme and $\mathscr{J}$ an invertible ideal sheaf and $j : X - Z(\mathscr{J}) \to X$ the open immersion. Then we have that $\varinjlim_n \mathscr{J}^{-n} \to j_*(\mathscr{O}_U)$ is an isomorphism of quasi-coherent sheaves*

Proof. Zariski locally on opens contained in U the claim is clear. On opens intersecting $\mathscr{O}_U$ $\mathscr{J}$ admits a free generator and the limit includes all functions on the complement of $Z(J)$. □

For our particular purposes, this gives us that $H^0(D-e, \mathscr{O}_D) = \varinjlim_n H^0(D, \mathscr{O}(ne))$ and we can thereby identity x and y with global sections of $\mathscr{O}(ne)$ for large n. We want that x is a global section of $\mathscr{O}(2e)$ and y one of $\mathscr{O}(3e)$. The question is local near e as away from e they are sections of the correct degree. Let V containing e be such that $\mathscr{I}$ free on V and let T be the local generator of $\mathscr{I}$ on V.

Let T be a local generator of $\mathscr{I}$ on V. Let $R = k[\epsilon]$. Therefore, we get $H^0(V - e, \mathscr{O}_D)/H^0(V, \mathscr{O}_C)$ is isomorphic to $R[1/T]/R$. Note that any element of $R[1/T]/R$ can be represented by a polynomial in $1/T$ over R with no $(1/T)^0$ coefficient. We wish to identify x with such a polynomial of degree 2 and y one of degree 3. On fibers this is known, so it must be the case that any higher degree terms on x and y are nilpotent. If R' is a reduced ring with R as quotient, if we lift the Weierstrass cubic over R' we see that there can be no higher order nilpotent terms and we are done.

Then we have that $\mathscr{O}_D(1) \to j_*(\mathscr{O}(D-e))$ lands inside of $\mathscr{O}(3e)$ and we can check this is an isomorphism on the level of residues where it is a result over fields.

Next we consider any elliptic curve (E, e) over a noetherian local base ring R. By the above discussion we saw that for any Weierstrass cubic inclusion $\mathbb{P}^2_R$ we have that x is a section of $\mathscr{O}(2e)$ and y is a section of $\mathscr{O}(3e)$, and now we can check residually that $\{1, x\}$ is a basis of $\mathscr{O}(2e)$ and $\{1, x, y\}$ is basis $\{1, x, y\}$, which is true. Therefore, we have that any transformation in $\mathbb{P}^2_R$ of Weierstrass cubic must preserve such bases of $\mathscr{O}(2e)$ and $\mathscr{O}(3e)$. That is to say that x is a basis of $\mathscr{O}(2e)/\mathscr{O}(e)$ and y a basis element of $\mathscr{O}(3e)/\mathscr{O}(2e)$. Therefore, we must have that $x \mapsto wx + r$ for a unit w and $r, w \in R$ and $y \mapsto vy + sx + t$ where v is a unit and $v, s, t \in R$.

Since the map carries one Weierstrass cubic to another we must have that $(v^2 - u^3)x^3$ must lie in $\mathscr{O}(5e)$ as $y^2 = x^3 \bmod \mathscr{O}(5e)$ and there are the only elements. Rephrasing this we have that $(v^2 - u^3)x$ vanishes in $\mathscr{O}(6e)/\mathscr{O}(5e)$, but we know by the calculations above that x^3 is a basis for $\mathscr{O}(6e)/\mathscr{O}(5e)$ and hence we must in fact have that $v^2 - u^3 = 0$ Thus lettering $u = v/w$, we have that $u^3 = v$ and $u^2 = w$.

We return to the case when $R = k[\epsilon]$. We must only check that there are no change of coordinates of the form $x \mapsto u^2x + r$, $y \mapsto u^3y + sx + t$ that sends the Weierstrass cubic for the trivial deformation to that of D

Next we note that the ideal that cuts out C_0/k from $\mathbb{P}^2_k$ is isomorphic to $\mathscr{O}(-3)$, we have $0 \to \mathscr{O}(-3) \to \mathscr{O}(\mathbb{P}^2_k) \to \mathscr{O}(C_0) \to 0$. Twisting by $\mathscr{O}(1)$ and noting that $\mathscr{O}(-2)$ has no global sections, we get that $H^0(\mathbb{P}^2_k, \mathscr{O}(1)) \to H^0(C_0, \mathscr{O}(1))$ is injective. Furthermore, we know from that $H^0(C_0, \mathscr{O}_{C_0}) = k$. We can therefore apply Proposition 6.2.

Since any isomorphism of C_{trv} to D should reduce to the identity on C_0/k and therefore as maps of $\mathbb{P}^2_k$ by Proposition 6.2, we must have x, x', y and y' must be related by $x' = u^2x + r$, $y' = u^3y + sx + t$, where u is a unit and r, s, t must be multiples of ϵ. The only possibility for from a t factor in x^3 is from $3u^4rx^3$, so $r = 0$. Similarly for the x^2 term we have $3u^4rx^2$, so $r = 0$. Then the y terms are force $2su^3xy$ and $2u^3yt$ force $s = t = 0$. But then the equation for C_{trv} cannot be taken to that of D, which can be seen looking at the constant term, so D is a nontrivial deformation and hence gives the universal deformation.

6.2 Example two; Klein quartic

Our second example is the same Klein quartic cut out by $x^3y + y^3z + z^3y$ in $\mathbb{P}^2_k$ where for this we take $\operatorname{char} k \neq 2, 7$ so that the Klein quartic is smooth. Again we let C_0 represent the Klein quartic over k and C_{trv} be the trivial deformation cut out by $x^3y + y^3z + z^3y$ in the appropriate projective space.I claim that the universal deformation is $x^3y + y^3z + z^3x + tx^4 + sx^3y + rx^3z + 3sy^3z + sx^2y^2 + ty^2z^2 + rx^2z^2$ in $\mathbb{P}^2_{\Lambda[[t]]}$. We call this last curve C.

To apply Schlessinger's to this case, it is sufficient to show that for any deformation to an Artin local Λ-algebra with residue field k there are no automorphisms that restrict to the identity automorphism on C_0. We will do this by induction on the length of Artin rings. Let A be a local Artin Λ-algebra with residue field k on which it has been established that no deformation of C_0 to A has nontrivial automorphisms fixing the C_0/k. Let $B \to A$ be a small extension with kernel I. Assume that the deformation C'/B has a nontrivial automorphism fixing C_0/k, and call this a. Note that this must fix $C' \otimes A/A$ by hypothesis. Let us look at the map on sections sending s to $a(s) - s$. Let U be an affine open. Note for any s, $a(s) = s + i_s$ where i_s is in the kernel $\mathscr{O}_{C'}(U) \to \mathscr{O}_{C' \otimes A}(U)$. Call this map d. Note that $i_s^2 = 0$, as $i_s \in I\mathscr{O}_{C'}(U)$. Note that $d(st) = (s + i_s)(t + i_t) - st = ti_s + si_t = tds + sdt$, so this is a derivation, therefore it factors as a map through $\Omega^1_{C'/B}$, but therefore, it represents an element of $(\Omega^1_{C'/B})^\vee$. Note as the formation of Ω^1 commutes with base change, and Ω^1 is locally free, we have that the formation of $(\Omega^1_{C'})^\vee$ commutes with basechange. Since $(\Omega^1_{C_0/k})^\vee$ has no nontrivial global sections as the genus is greater than 1 we may apply cohomology and base change while noting that ϕ^0_q and ϕ^{-1}_q are trivially surjective to get that $(\Omega^1_{C'})^\vee$ has no nontrivial global sections, and d represents to zero map, so a was the identity, and we have no nontrivial automorphisms of C'/B fixing which are the identity on C_0/k.

Again, by the same reasons as in the last example but different genus and this time we have that there are no deformations of the identity automorphism, we get that the deformation ring is $\Lambda[[r, s, t]] = A$ (and it prorepresents the deformation functor). This time, we must check this is the nontrivial deformation over $k[\epsilon]/\epsilon^2$ where r, s, t are non zero multiples of ϵ as again we get a map from the deformation ring to $\Lambda[[r, s, t]]$ which is abstractly isomorphic and to get isomorphism we must check non-triviality over all these $k[\epsilon]$ where $\epsilon^2 = 0$

To do this, first let us be in $B = k[\epsilon]$ and let r, s, t be multiples of ϵ where at least one is nonzero. We want so show that the embedding of C in $\mathbb{P}^2_B$ comes from its Ω^1

sheaf.

Let $\pi : C \to \Lambda[[w]]$ be the structure morphism of the deformation at hand. We check that $h^0(C_q, \Omega^1_C|_{C_q})$ is locally constant, which amounts to constant by the nature of $\operatorname{Spec}\Lambda[[w]]$. Over the closed point, it is clear that $h^0(C_q, \Omega^1_C|_{C_q}) = 3$ by genus considerations. Over the other point this will be clear too, if we can show that equation in question is actually smooth, but this holds as the smooth locus is open. Therefore, $h^0(C_q, \Omega^1_C|C_q) = 3$ constantly, and we may apply Grauert's theorem. Therefore, we have that $\pi_*\Omega^1$ is locally free, and its formation commutes with all $Z \to k[t]]$. In particular for all Λ-algebra maps $\Lambda[[w]] \to k[\epsilon]$ the global sections of $\Omega^1_{C\otimes k[\epsilon]/k[\epsilon]}$ are free of rank 3, and the formation of the Ω^1 commutes with base change to the one point.

In the following proof, much is adapted from Hartschorne's discussion over a field [5]. Now let f be a degree d polynomial with at least one unit coefficient over a ring, R. We note that multiplication by f map $\mathscr{O}(-d) \to \mathscr{O}$ is an injection fiber wise, and therefore as $\mathscr{O}(-d), \mathscr{O}$ are invertible sheaves, it is an injection. This makes $\mathscr{O}(-d)$ into an invertible ideal sheaf $\mathscr{I}$ and it is clear over the standard affine open cover that ideal it cuts out is precisely f on that affine open, and therefore it cuts out the closed subscheme defined by $f = 0$. Let Z be this closed subscheme Thereby, computing $\mathscr{I}/\mathscr{I}^2$, by tensoring the exact sequence $0 \to \mathscr{I} \to \mathscr{O}_{\mathbb{P}^2_R} \to \mathscr{O}_Z \to 0$ with $\mathscr{I}$ and using the fact that it is flat, we get that $\mathscr{I}/\mathscr{I}^2$ is $\mathscr{O}_Z \otimes \mathscr{I} \cong \mathscr{O}_Z \otimes \mathscr{O}(-d) = \mathscr{O}_Z(-d)$.

We may therefore conclude for the example at hand of the Klein quartic that $\mathscr{I}/\mathscr{I}^2$ is locally free and this commutes with base change.

By abuse of notation we denote over $C \otimes \operatorname{Spec} B$ by C as a deformation of C_0 to $\operatorname{Spec} B$. Then we consider the following exact sequence of vector bundles:

$$\mathscr{I}/\mathscr{I}^2 \to \Omega_{\mathbb{P}^2_B/\operatorname{Spec} B} \otimes \mathscr{O}_C \to \Omega^1_{C/\operatorname{Spec} B} \to 0$$

We can check that the above sequence is exact at the residue level by Nakayama's lemma, where the statement holds, because it holds on C_0.

Taking the second order piece of the associated graded short exact sequence for exterior algebras, we get $0 \to \Omega_C \otimes \mathscr{I}/\mathscr{I}^2 \to \omega_{\mathbb{P}^2_B} \otimes \mathscr{O}_C \to 0$, so $\Omega_C \otimes \mathscr{I}/\mathscr{I}^2 \to \omega_{\mathbb{P}^2_B} \otimes \mathscr{O}_C$ is an isomorphism. As C is cut out by a monic quartic polynomial we have, $\mathscr{I}/\mathscr{I}^2 \cong \mathscr{O}_C(-4)$, as done by the calculation above.

We also have the exact sequence (Hartshorne's theorem 8.13. I will not prove it)

$$0 \to \Omega^1_{\mathbb{P}^2_B} \to \mathscr{O}_{\mathbb{P}^2_B}(-1)^3 \to \mathscr{O}_{\mathbb{P}^2_B} \to 0$$

Taking the second piece of the associated exterior power exact sequence, we get that $\Omega_{\mathbb{P}^2_B} \cong \mathscr{O}_{\mathbb{P}^2_B}(-3)$. Combining the last two results and using the fact that $\mathscr{L}$ has degree 4, we get that $\Omega^1_C \cong \mathscr{O}_C(4-3) = \mathscr{O}_C(1)$. Next note that x, y, z sections Ω^1_C from the map $\mathscr{O}_{\mathbb{P}^2_{k[\epsilon]}} \to \mathscr{O}_C(1)$ and these give a basis because they do so residually, as the fiber of the closed point in the dual numbers is a smooth genus 3 curve, and therefore the embedding came from the canonical sheaf.

Having done all this if C was a trivial deformation, we must be able to get from C_{trv} to C by a projective change of coordinates corresponding to a different global sections of Ω^1_C. Furthermore, this change must reduce to the identity, because we have that $H^0(\mathbb{P}^2_k, \mathscr{O}(1)) \to H^0(C_0, \mathscr{O}(1))$ is an injection as $\mathscr{O}_{\mathbb{P}^2_k}(-3)$ has no global sections, $H^0(C_0, \mathscr{O}_{C_0}) = k$, and Proposition 6.2. Therefore we can assume it is of the form:

$$\begin{aligned} x &\mapsto x + u_x x + u_y y + u_z z \\ y &\mapsto y + v_x x + v_y y + v_z z \\ z &\mapsto z + w_x x + w_y y + w_z z \end{aligned}$$

where u_i, v_i, w_i are all k linear combinations of the r, s, t.

In the following we implicitly use the fact that r, s, t are multiples of ϵ and hence multiply to give 0, and thus the v_i, u_i, w_i pairwise multiply to zero. In change of coordinates x^3y goes to $(x^3 + 3u_x x^3 + 3u_y x^2 y + 3u_z x^2 z)(y + v_x x + v_y y + v_z z)$. This is the only term that can generate x^4 terms or x^3y, x^3z terms. From this we get that $v_x = t$, $v_z = r$, and $3u_x + v_y = s$. From similar arguments on the other terms we get $3v_y + r_z = 3s$, and $3w_z + u_x = 0$, which gives that $v_y = s$, $u_x = 0 = w_z$ in all characteristics but 2 and 7, as the linear system is not redundant away from those characteristics. Similar considerations as those above give that $u_i = 0 = r_i$ for $i = x, y, z$.

Note now that this should make all the x^2y^2, y^2z^2, x^2z^2 terms be zero, but they are s, t, r. Therefore, no matter what multiples of ϵ s, t, r are, the deformation is nontrivial. Therefore, we conclude the deformation in question is universal.

7 Moduli spaces

An important use of deformation theory is that it gives information about moduli schemes. In particular, infinitesimal deformations of objects within given contexts can give information about infinitesimal nature of the moduli scheme near a point. Because of moduli spaces being a use for infinitesimal deformation theory, we give discuss important initials about moduli spaces, and prove facts about Picard schemes to illustrate methods for working with them.

One important class of moduli spaces are those representing functors. Yoneda's lemma says that a scheme is determined up to unique isomorphism by the functor **Schemes** $\to$ **Set** given by $\mathrm{Hom}(X, -)$. Given an arbitrary functor, **Schemes** $\to$ **Set**, one asks if it is isomorphic to a functor of the above form given by a scheme, or in other works is the functor representable by a scheme. If a functor is representable, many properties of the scheme can be read easily from the functor using their functorial definitions. For instance a functorial definition of smoothness was given in Definition 5.2.

Two important examples of moduli schemes are the Hilbert schemes and Picard schemes. We will only briefly discuss the former and will discuss the later at more length.

Following [3] we define $\mathfrak{H}ilb_{\mathbb{P}^n}$ to be the functor from the category of noetherian schemes to the category of sets such that

$$\mathfrak{H}ilb_{\mathbb{P}^n}(S) = \{Y \subseteq \mathbb{P}^n_S : Y \text{ is flat over } S\}$$

The question then arises of if this functor is representable, and it is theorem due to Grothendieck that the answer is yes and this representing scheme is normally given by denoted $\mathrm{Hilb}_{\mathbb{P}^n}$ and is called the Hilbert scheme. The Hilbert scheme is especially useful in the construction of representing schemes for other functors. For instance proof of the existence of the Picard scheme uses the existence of Hilbert schemes.

Before stating the existence theorem for Picard schemes we will include a discussion of two Picard functors. Here we are mostly following [3] but also following [2].

To begin with, we define the *absolute Picard functor* to be the functor the category of locally noetherian S-schemes to the category of abelian groups by having $\mathrm{Pic}_X(T)$ be $\mathrm{Pic}(X_T)$, ie letting $\mathrm{Pic}_X(T)$ be the abelian group of isomorphism classes of line bundles on the scheme $X_T = X \times_S T$. However, it is relatively easy to see that this functor is never representable ([3] page 252 for details).

Instead, there is another candidate functor, labelled $\mathrm{Pic}_{X/S}$ called the *relative Picard functor* given by $\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$. In good situations this functor will turn out to be representable. In particular:

Theorem 7.1 (Grothendieck, [3] Theorem 9.4.8)**.** *Assume that $f : X \to S$ is projective Zariski locally over S and is flat with integral geometric fibers, and that for any S scheme T, $\mathscr{O}_T \to f_{T*}\mathscr{O}_{X_T}$ is an isomorphism, and that f has a section locally Zariski locally.*

Then the scheme $\mathrm{Pic}_{X/S}$ exists representing the relative Picard functor, and it is separated and locally of finite type over S. Furthermore if S is Noetherian and X/S

is projective, then $\mathrm{Pic}_{X/S}$ *is a disjoint union of open subschemes, each an increasing union of open quasi-projective* S*-schemes.*

One consequence of the representability of the Picard functor is that we get the Poincaré sheaf, $\mathscr{P}$, on $X \times \mathrm{Pic}_{X/S}$ with the property that S-maps $h :\to Pic_{X/S}$ correspond to any line bundles $\mathscr{L}$ on $X_T = X \times_S T$ of the form $\mathscr{L} \cong (1 \times h)^* \mathscr{P} \otimes f_T^* \mathscr{N}$ up to choice of line bundle $\mathscr{N}$ on T.

We denote $\mathrm{Pic}^0_{X/k}$ be the connected component of the identity.

In various cases the Picard scheme can be proved to have nice properties. Firstly:

Proposition 7.2. *If* X/k *is smooth and projective and geometrically integral, then* $\mathrm{Pic}^0_{X/k}$ *is proper.*

Proof. It is already known from the proof of the existence of the Picard scheme that it is separated.

We apply the valuative criterion for properness. Let A be a DVR with fraction field K. Let p be the closed point of $\mathrm{Spec}\, A$ and let q be the generic point. We assume we have a map $\mathrm{Spec}(K)$ to $\mathrm{Pic}_{X/k}$. As $\mathrm{Spec}\, K$ has no nontrivial line bundles, this is the same thing as a line bundle on X_K. Since we already have that it is separated, we must only show that this line bundle comes from a line bundle on X_A. This is to say we have to we must prove that $\mathrm{Pic}(X_A) \to \mathrm{Pic}(X_K)$ is a surjection. Note since $X_A \to A$ is smooth, it is regular and line bundles are all given by divisors. Because of this we discuss briefly irreducible codimension 1 closed subschemes of X_A

Note that since X is geometrically integral, X_p is irreducible. Therefore, $(X_A)_p$ is an irreducible codimension one irreducible closed subscheme of X_A, but any closed subscheme of it cannot be. Next we note that any closed subscheme of $(X_A)_q$ cannot be closed in X_A, as $X_A \to \mathrm{Spec}\, A$ will be proper as $X \to \mathrm{Spec}\, k$ is projective hence proper.

Thus let us take a divisor D_q of $(X_A)_q$. Let D be its closure. $\mathscr{O}_{X_A}(D)$ restricts to $\mathscr{O}_{(X_A)_q}(D_q)$ on $(X_A)_q$, and hence the map is surjective so $\mathrm{Pic}_{X/k}$ satisfies the valuative criterion for properness, and hence $\mathrm{Pic}^0_{X/k}$ is proper for such X. $\square$

One particular use of the Picard scheme is in the study of curves for these we have the following:

Proposition 7.3. *If* C *is a smooth curve over a field* k*, then* $\mathrm{Pic}_{C/k}$ *is smooth.*

Proof. To show that it is smooth, we will apply the infinitesimal smoothness criterion. Let $A \to B$ be a surjective morphism of local artinian k-algebras. We must show that any map $\mathrm{Spec}\, B \to \mathrm{Pic}_{C/k}$ can be factored through $\mathrm{Spec}\, A \to \mathrm{Pic}_{C/k}$. Rephrasing this using the fact that such $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ have nontrivial line bundles, we get that $\mathrm{Pic}(C_A) \to \mathrm{Pic}(C_B)$ is a surjection. By induction we can assume that $\ker(A \to B) = I$ is killed by the maximal ideal of A, ie is a so called *small extension.* Let us call the kernel I.

To do this we will examine the sequence:

$$0 \to I \otimes \mathscr{O}_C \to \mathscr{O}_{C_A}^\times \to \mathscr{O}_{C_B}^\times \to 1$$

with the second arrow given by $s \otimes x \mapsto 1 + sx$. The well-definedness of this arrow follows from the *small extension* proper of I as A acts on I only through the quotient by its maximal ideal. The injectivity of the second arrow is clear. The exactness of the second arrow is also clear, as is the surjectivity of the third arrow.

Taking cohomology we get

$$H^1(C_A, \mathscr{O}_{C_A}^\times) \to H^1(C_B, \mathscr{O}_{C_B}^\times) \to H^2(C, I \otimes \mathscr{O}_C)$$

However as C is a curve the last term vanishes. Furthermore, we recognize $H^1(C_A, \mathscr{O}_{C_A}^\times)$ as $\mathrm{Pic}(C_A)$ and $H^1(C_B, \mathscr{O}_{C_B}^\times)$ as $\mathrm{Pic}(C_B)$, so our result follows and $\mathrm{Pic}_{X/k}$ is smooth for curves. □

We note here, we managed to learn about the scheme representing the relative Picard functor, just using information about the functor itself.

Knowing now that that $\mathrm{Pic}^0_{X/k}$ is smooth and proper for curves, we may to also know its dimension, we can state and prove this in slightly more generality.

Proposition 7.4. *If X is a proper, geometrically integral k-scheme, then there is a k-linear isomorphism $H^1(X, \mathscr{O}_X) \cong T_0(\mathrm{Pic}^0_{X_k})$.*

Proof. To do this we recall that we can calculate $T_0(\mathrm{Pic}^0_{X_k})$ as maps $\mathrm{Spec}\, k[\epsilon] \to \mathrm{Pic}^0_{X_k}$ such that the induced a morphism $\mathrm{Spec} \to \mathrm{Pic}^0_{X_k}$ goes to the identity point. Rephrasing this, this says our tangent space is precisely $\ker \left[\mathrm{Pic}_{X/k}(\mathrm{Spec}(k[\epsilon])) \to \mathrm{Pic}_{X/k}(\mathrm{Spec}\, k)\right]$ as $\mathrm{Spec}\, k[\epsilon]$ and $\mathrm{Spec}\, k$ have no nontrivial line bundles. Like in the last proof we take:

$$0 \to \epsilon \otimes_k X \to \mathscr{O}_{X_{k[\epsilon]}}^\times \to \mathscr{O}_X^\times \to 1,$$

where again the second arrow is given by $\epsilon \otimes x \mapsto 1 + \epsilon \otimes x$. We then look at a piece of the associated long exact sequence.

$$H^0(X, \mathscr{O}_{X_{k[\epsilon]}}^\times) \to H^0(X, \mathscr{O}_X^\times) \to H^1(X, \epsilon \otimes_k X) \to H^1(X, \mathscr{O}_{X_{k[\epsilon]}}^\times) \to H^1(X, \mathscr{O}_X^\times)$$

The first arrow is a surjection. Therefore the following is exact:

$$0 \to H^1(X, \epsilon \otimes_k X) \to H^1(X, \mathscr{O}_{X_{k[\epsilon]}}^\times) \to H^1(X, \mathscr{O}_X^\times)$$

Now we note that $H^1(X, \epsilon \otimes_k \mathscr{O}_X)$ is isomorphic to $H^1(X, \mathscr{O}_X)$, and again $H^1(X, \mathscr{O}_{X_{k[\epsilon]}}^\times)$ is isomorphic to $\mathrm{Pic}(X_{k[\epsilon]})$ and $H^1(X, \mathscr{O}_X^\times)$ to $\mathrm{Pic}(X_k)$. Therefore, $H^1(X, \mathscr{O}_X)$ is isomorphic to $T_0(\mathrm{Pic}_{X/k})$. □

Therefore we have that for smooth, proper curves, C, over a field, $\mathrm{Pic}^0_{C/k}$ is a proper, smooth k-scheme with dimension equal to $\dim_k H^1(X, \mathscr{O}_X)$, which is the genus of the curve. We now wish to understand a little better the identity component of $\mathrm{Pic}_{X/k}$, at least in the case of curves.

Proposition 7.5. *If $k = \overline{k}$ is a field and C/k is a smooth curve (geometrically connected of dimension 1) such that $p \in C(k) \neq \emptyset$ then components of $\mathrm{Pic}_{C/k}$ are indexed $\mathbb{Z}$ and correspond to the degree of the line bundle.*

Proof. We assume first we have already proved the result for algebraically closed fields. By [3] formation of Pic^0 commutes with basechange. Translating by $\mathscr{O}(dp)$ have that the inverse image of the degree d component of $\mathrm{Pic}_{X \otimes \overline{k}/\overline{k}}$ is the translation of Pic^0 by $\mathscr{O}(dp)$. Therefore, we have in fact that this inverse image must be connected, and the connected components correspond to degrees of the line bundle, with the rational points corresponding to degree d line bundles.

We now prove the statement over an algebraically closed field. By translation and fact $\mathrm{Pic}^0_{C/k}$ is locally of finite type it suffices to show that $\mathscr{L} \in \mathrm{Pic}^0_{C/k}(k)$ if and only if x is $\mathscr{L}$ is a degree zero line bundle. The only if follows immediately from the local constancy of degrees of line bundles. We get the other direction as follows. Let L be a degree zero line bundle with $L \cong \mathscr{O}(D)$ where $D = \sum_{i \in I} n_i p_i$ for points p_i. Let $m = |I|$. Let us look at X^m (that is $X \times_k \cdots \times_k X$ with m X's). We take the map $(x_i)_{i \in I} \mapsto \prod_i \mathscr{O}(n_i x_i)$ (defined on S valued points x_i). This gives a map from X^m, which is connected, to Pic. Note that when all x_i equal a fixed x, this maps to the identity, when $x_i = p_i$ it maps to $\mathscr{L}$. Therefore, $\mathscr{L}$ is in $\mathrm{Pic}^0_{X/k}$ and we are done. □

Next we discuss an example that if we remove properness $\mathrm{Pic}^0_{C/k}$ is no longer proper.

Proposition 7.6. *Let k be a field and let C be the (non-smooth) curve cut out by $y^2 = x^3 - xy$ in $\mathbb{P}^2_k$. As group schemes $\mathrm{Pic}^0_{C/k} \cong \mathbb{G}_{m/k}$.*

Proof. We prove the result first over $k = \overline{k}$ and then reduce the general case to this case.

Let x_0 be the point at infinity. First let us define a map $\mathbb{G}_m \to C - \{0\}$ given by $t \mapsto ((1-t)t, t, (1-t)^3)$. Then we have the map $C - \{0\} \to \mathrm{Pic}^0_{C/k}$ given on R valued points by $x \mapsto \mathscr{O}(x) \otimes \mathscr{O}((x_0)_R)^{-1}$.

We check first that the composite map $\mathbb{G}_m \to \mathrm{Pic}_{X/k}$ is a map of group schemes. It suffices to check this on k valued points. Here we note that if p, q are k-valued points of $C - \{0\}$ that the line in $\mathbb{P}^2$ joining p and q must meet another point, r. As 0 is a double point on C r cannot be zero. Note that $-\mathscr{O}(r - x_0)$ is the image of $\mathscr{O}(p - x_0) \otimes \mathscr{O}(q - x_0)$ in $\mathrm{Pic}^0_{X/k}$ as $\mathscr{O}(3x_0) = \mathscr{O}$. The map from $\mathbb{G}_m$ is precisely the one so that the third point on the line of the image of s and the image of t is $1/st$. This proves we have a map of group schemes.

We establish the kernel of the morphism is trivial. Let p be an R valued point for k-algebra R of $C - \{0\}$, we note by genus considerations $\mathscr{O}(p \otimes (x_0)_R^{-1})$ is trivial if and only if $p = (x_0)_R$. Therefore, the scheme theoretic image of the kernel is trivial.

To establish surjectivity, let us begin with a line bundle $\mathscr{L}$. Note that in the case we are working the Cartier class group is still isomorphic to $\mathrm{Pic}(X)$. Multiplying a principle Cartier divisor we can assure that $\mathscr{L}$ comes from a Cartier divisor that has neither zeros nor poles at 0. This means line bundles are completely determined by their pullback to $C - \{0\}$, but this the smooth locus of C and line bundles are given by Weil divisors. We note we get in the image all line bundles given by twists of points such that the sum of twists is zero. These can be the only things in the $\mathrm{Pic}^0_{C/k}$ as if $\widetilde{C}$ is the normalization, $\mathrm{Pic}_{C/k} \to \mathrm{Pic}_{\widetilde{C}/k}$ preserves degree in this sense. Therefore, the map is a surjection, hence an isomorphism as the kernel is trivial.

For general k we note that to check that $\mathbb{G}_m \to \mathrm{Pic}^0_{C/k}$ is an isomorphism of group schemes, it suffices to check it over $\overline{k}$ where it holds. □

8 Abelian Varieties

Definition 8.1. An **abelian variety** is a proper, smooth, geometrically integral group scheme over field with a rational identity section.

The first thing we notice is that the connected components of the identity of the Picard scheme of curve with a rational point is an abelian variety. Therefore, the study of abelian variety is fact useful in the study of curves.

We will list some of their properties listed in [10] and [2], and then will pass to a more detailed discussion of their intersection theory.

Among the important properties of abelian varieties are that they are always commutative and projective. Furthermore, morphisms as k-schemes that carry the identity of one abelian variety to the identity of another are automatically homomorphism.

Another important aspect in the study of abelian varieties is the dual abelian abelian variety, which is taken to be $\mathrm{Pic}^0_{A/k}$. Once it is known it is abelian variety, it is denoted $\hat{A}$ and called the dual abelian variety. We have already established that it is a proper group scheme over k, so it remains to check that it is smooth. This is done via cohomological considerations to compute the tangent space at the identity and using maps with finite kernels from A to $\mathrm{Pic}^0_{A/k}$.

As $\widehat{A}$ arises form a piece $\mathrm{Pic}_{A/k}$, $A \times \widehat{A}$ comes with the Poincaré line bundle, $\mathscr{P}_A$ which we can and do pick to be trivial along $\{0\} \times \widehat{A}$ and along $A \times \{\widehat{0}\}$. These choices then completely characterize $\mathscr{P}_A$ on $A \times \widehat{A}$. By the universal property of the Picard scheme, we then get that this gives us a map from $A \to \mathrm{Pic}_{\widehat{A}/k}$ and the trivilazation assumptions then give us that 0 is taken to 0, and hence we have a map

$$i_A : A \to \hat{\hat{A}}$$

and this is in fact an isomorphism ([10], Last Corollary of 3.13)

Maps $A \to \hat{A}$ are important in the study of abelian varieties. One important class of these maps are of defined as follows. Given a line bundle $\mathscr{L}$ on A, we define $\phi_{\mathscr{L}}$ to be the map $A \to \widehat{A}$ given on $x \in X(R)$ by $x \mapsto t_x^*(\mathscr{L}_R) \otimes \mathscr{L}_R^{-1}$, where t_x is translation by x. For ample $\mathscr{L}$, in fact these maps turns out to have finite kernel and surjects onto $\widehat{A}$. These morphisms in fact have other nice properties. A map $f : A \to \widehat{A}$ gives a dual map $\widehat{f} : \hat{\hat{A}} \to \widehat{A}$, and identifying $\hat{\hat{A}}$ with A via the map $i_A : A \to \hat{\hat{A}}$ described above, we get a map $\widehat{f} \circ i_A : A \to \widehat{A}$. For $f = \phi_{\mathscr{L}}$ for ample $\mathscr{L}$, these maps agree. This leads us to the following definitions.

Definition 8.2. A map $f : A \to B$ between two abelian varieties over a field k is called an *isogeny* if it is surjective and has kernel finite as a group scheme. A map $f : A \to \widehat{A}$ is called *symmetric* if $f = \widehat{f} \circ i_A$. Furthermore if $f : A \to \widehat{A}$ is a symmetric isogeny we call f a *polarization.*

We can rephrase the above discussion then to saying that for ample $\mathscr{L}$ on A, $\phi_{\mathscr{L}}$ is a polarization. If A/k is an abelian variety over an alebraically closed field, all polarizations turn out to be of the $\phi_{\mathscr{L}}$ for some ample $\mathscr{L}$. However, this is not the case for abelian varieties over general fields.

Remark 8.3. We note that ([10] Theorem 3.16.1 in fact gives us that the $\deg \phi_{\mathscr{L}} = \chi(\mathscr{L})^2$ and hence the degree of every polarization is a square, as these the degree is not affected by base change, and therefore, we can base change to a field for which the polarization is of the form $\phi_{\mathscr{L}}$.

We now begin to discuss deformations of abelian varieties.

Definition 8.4. By **abelian scheme** over a noetherian base S, we mean a group scheme $\pi : X \to S$ such that π is smooth and proper with geometrically connected fibers.

We also give the following example definition in the context of abelian schemes:

Definition 8.5. For an abelian scheme A/S, a *polarization* is a map $A \to \mathrm{Pic}_{A/S}$ that corresponds to a polarization on all fibers.

In the context of abelian schemes [11] (Proposition 3.1) proves a rigidity lemma which allows one to conclude among other things that abelian schemes are always commutative and that maps taking an abelian scheme X to another group scheme over S taking the identity to the identity is actually a homomorphism.

Deformations of abelian varieties behave particularly nicely. We give one example to illustrate this point:

Proposition 8.6. *If A/S is an abelian scheme and $\mathscr{L}$ a line bundle on A flat over S then if for one field valued* $\mathrm{Spec}\, k = s \in S$*, if A_s is the fiber over s and $\mathscr{L}|_{A_s}$ over s is ample over k then $\mathscr{L}$ is relatively ample over S.*

Proof. The ample locus for locally noetherian morphisms is always open.([4] 3.1:4.7.1)

Therefore to check the assertion, we must only check that they are closed. In fact since we know that they are open, the ample locus is constructible, so we can reduce to showing that A/S is an abelian scheme over the spectrum discrete valuation ring $S = \mathrm{Spec}\, D$, if η is the generic point of p the closed point, we may assume that $\mathscr{L}_\eta$ is ample, and we must check that $\mathscr{L}_p$ is ample. We can assume that $\mathscr{L}_\eta$ is a line bundle from an effective divisor, D_η. Taking closure we may assume that $\mathscr{L}$ comes from effective D divisor.

At this point we use Mumford's criterion for ampleness, which says if A is an abelian variety over an algebraically closed field that $\mathscr{L}$ being ample is equivalent to the kernel of $\phi_{\mathscr{L}}$ defined above being finite. Both of these conditions are preserved by basechange.

However, we note that we may define $\phi_{\mathscr{L}}$ be the same formula on A/S. As the kernel is closed, the closure of the kernel on the generic fiber must contain the kernel on the special fiber. As the latter is finite, the former must also be finite.

∎

Example 8.7. We note briefly that this property that the ampole locus of line bundles must be closed is indeed special. Let us take x, y be the coordinates of $\mathbb{P}^2_k$ and t be the coordinate of $\mathbb{P}^1_k$. Then we then look at the surface S cut out by $x^2+y^2+txy+1=0$. Let D be the divisor corresponding to $x=0, y=0$. Over most fibers, for instance $t=1$ this divisor is ample. However, the divisor is not ample over the point at infinity in $\mathbb{P}^1$ as the divisor becomes $(0,0)$ in the curve cut out by xy.

Note that $S/\mathbb{P}^1$ is flat and proper but not smooth over $\mathbb{P}^1$, though it is smooth over k. We wish to make a better example with smoothness over the base. α be the coordinate corresponding to another $\mathbb{P}^1$ We look at $x^2+\alpha tx+y^2-\alpha ty+txy+1$. Here the divisor $x=y=0$ is ample over $\alpha=0$ over every fiber but $\alpha=0$. Over $\alpha=0$ we have the previous example where divisor is not ample.

8.1 Infinitesimal deformations of abelian varieties

Abelian varieties give a particular good class of examples for deformation theory, because their deformations behave particularly well. For instance, Grothendieck proved the following theorem important to the study of deformations of abelian varieties. We give the proof following [11]

Theorem 8.8. *Consider $S=\operatorname{Spec} A$ where A is an Artin local ring and I is an ideal of A such that its $A \to A/I$ is a small extension. If $\pi : X \to \operatorname{Spec} A$ is a smooth proper morphism and $e : S \to X$ a section, if $X \times \operatorname{Spec}(A/I)$ is known to be an abelian scheme over $\operatorname{Spec}(A/I)$ with identity $e_{\operatorname{Spec}(A/I)}$, then X is an abelian scheme over S with identity e*

The proof uses a fact above left invariant differential forms on abelian varieties. Before give the proof will give a discussion of these differential forms (following [10]). We let $\Omega^{1,\ell}_{A/k}$ denote the space of global left invariant 1-forms, and $\Omega^1_{A/k}$ the space of global 1-forms. We let Ω^0_e the cotangent space of A at e. First we state, but will not prove, some important tools we wish to use:

Proposition 8.9 ([1], Proposition 4.2.1)**.** *The map $\Omega^{1,\ell}_{A/k} \to \Omega^0_e$ given by evaluating at e is an isomorphism.*

Thus we have:

Proposition 8.10. *The natural map*

$$\Omega_0 \otimes_k \mathscr{O}_A \to \Omega^1_A$$

is an isomorphism

Proof. Both sides are locally free sheaves. By Nakayama's lemma, it suffices to check this residually. By translation it suffices to check this at the origin, where it holds by the last proposition. □

Now we can give the proof

Proof of Theorem 8.8. We denote $S_0 = \operatorname{Spec}(A/I)$ and $X_{S_0} = X \times S_0$. We let k be the residue field A and we let $\mathfrak{m}$ be its maximal ideal. We let X_k be $X \times_S \operatorname{Spec} k$.

We consider the map $\mu_{S_0} : X_{S_0} \times_{S_0} X_{S_0} \to X_0$ given by $\mu_{S_0}(x, y) = x - y$. We let μ_k be the restriction to X_k.We work to extend μ_{S_0} to $\mu_S : X \times_S X \to X$. By our discussion in the first sectionwe have that the obstruction to this is some $\mathfrak{o} \in H^1(X_k \times X_k, I \otimes \mu_k^*(T_{X/k}))$.

Let $g_1, g_2 : X_{S_0} \times X_{S_0} \to X_{S_0}$ defined respectively by $x \mapsto (x, e)$ and $x \mapsto (x, x)$. It is clear that that both $\mu_0 g_1$ and $\mu_0 g_2$ extend to morphisms from X to X. Therefore, their corresponding obstructions, $\mathfrak{o}_1$ and $\mathfrak{o}_2$, in $H^1(X_k, I \otimes (\mu \circ g_i)_k^*(T_{X/k}))$ do vanish.

However, by functoriality of the obstructions, we have that $o_i = g_i * o$. Therefore, we have that $g_i^* \mathfrak{o} = 0$.

Since X_k is an abelian variety, its tangent bundle must be trivial, this is discussed in [10]. In fact we have that $T_{X_k} = \mathscr{O}_{X_k} \otimes H^0(X_k, T_{X_k})$.

Now we have that

$$H^1(X_k \times X_k, I \otimes \mu_k^*(T_{X_k/k}) \cong H^1(X_k \times X_k, \mathscr{O}_{X_k \times X_k}) \otimes H^0(X_k, T_{X_k,k}) \otimes I.$$

At this point we use the Künneth formula to figure that

$$H^1(X_k \times X_k, \mathscr{O}_{X_k \times X_k}) \cong p_1^* H^1(X_k, \mathscr{O}_{X_k}) \oplus p_2^* H^1(X_k, \mathscr{O}_{X_k})$$

Let $o = \operatorname{pr}_1^* x + \operatorname{pr}_2^* y$. Then we have that $g_1^* \mathfrak{o} = x$ and that $g_2^* \mathfrak{o} = x + y$. Since both are zero, x and y are zero, so $\mathfrak{o}$ is zero, which means that μ_0 actually lifts! Let us call its lift $\mu : X \to X \to X$.

We next claim that there is exactly one lift that sends $(e, e) \to e$. The lifts of μ is an affine space under $H^0(X_k \times X_k, I \otimes \mu_k^*(T_{X_k/k}))$. This is isomorphic to $H^0(X_k \times X_k, I \otimes \mathscr{O}_{X_k \times X_k} \otimes H^0(X_k, T_{X_k/k}) \cong H^0(X_k \times X_k, \mathscr{O}_{X_k \times x_k}) \otimes I \otimes H^0(X_k, T_{X_k/k}) \cong I \otimes H^0(X_k, T_{X_k/k})$. The $H^0(X_k, T_{X_k/k})$ is isomorphic to the tangent space at e_0 via left translation (the content of dualizing the above proposition). The extension is then completely determined by what happens to $e \times e$. Note that the extensions of $e_0 \to e_0$ is a homogenous space given by $H^0(e_0 \times e_0, I \otimes \mu_k^*(T_{X_k/k})) \cong I \otimes H^0(X_k, T_{X_k/k})$. Therefore, we have a one to one correspondence between maps $e \times e \to X$ extending $e_0 \times e_0 \mapsto e_0$ and maps extending μ_0 of $X \times X \to X$. Therefore, we pick the unique extension of μ that sends $e \times e$ to e.

From this we can define i to be the composition of $X \hookrightarrow \{e\} \times X \xrightarrow{\mu} X$ and m to be the composition $X \times X \xrightarrow{1 \times i} X \times X \to X$.

Now we must check that with e, m, and i make certain diagrams commutative and take the identity sections to identity sections. Rigidity ([11] Corollary 6.2) implies the this must be checked only on X_{S_0}, where it is true.

□

In [11], Mumford actually states, but does not fully prove the following theorem. We include it just to highlight how the extent to which moduli of abelian varieties are indeed special.

Theorem 8.11. *Let S is a connected, locally noetherian scheme, and $\pi : X \to S$ is a smooth, projective morphism, and $e : S \to X$ a section of π. If for one geometric point $s \in S$, the fiber X_s of π is an abelian variety with identity $e(s)$, then in fact X is an abelian scheme over S with identity e*

In addition to this we have that any abelian variety can be deformed to an artin local ring. Here we follow [12].

Proposition 8.12. *Let $R \to R'$ be a small surjection of artin local ring with residue field k. Let A_0 be an abelian variety over k and A' a lift to R'. Then the obstruction to lifting A' to R vanishes.*

Proof. Let $P' = A' \times_{R'} A'$. The lifting of P' is obstructed by

$$\mathfrak{o}(P') \in H^2(P_0, I \otimes T_{P_0/k}) \cong H^2(P_0, T_{P_0/k}) \otimes I.$$

By the Künneth formula and the fact that $T_{P_0/k} = \mathrm{pr}_1^* T_{A_0/k} + \mathrm{pr}_2^* T_{A_0/k}$, the two projections of P_0 give injections $H^2(A_0, T_{A_0/k}) \to H^2(P_0, T_{P_0/k})$.

By what we proved about products, we get that $\mathfrak{o}(P') = i_1(\mathfrak{o}(A')) + i_2(\mathfrak{o}(A'))$.

Now similar to the last proof, we define a' the automorphism of P' taking $(x, y) \mapsto (x+y, y)$, and we have $\mathfrak{o}(P') = (a^*)^{-1}\,\mathrm{da}\,\mathfrak{o}(P')$. Looking at the projections onto both factors and looking and here working directly with the obstructions and using the fact that we know this must be $2i_1(\mathfrak{o}(A')) + i_2(\mathfrak{o}(A'))$. This makes $i_1\mathfrak{o}(A') = 0$, but since i_1 is an injection, $\mathfrak{o}(A') = 0$, which is the result we wanted.

□

The above proposition is one of the main steps of the following theorem.

Theorem 8.13 (Grothendieck, [12] Theorem 2.3.1)**.** *Let A_0 be an abelian variety over a field k, and let M be the functor of deformations of A_0 to local artin Λ algebras (equivalently as abelian varieties or just as schemes), then this functor is prorepresentable by $R \cong \Lambda[[t_{1,1}, \ldots, t_{g,g}]]$*

The next representablity result has to do with polarizations. We have discussed polarizations on abelian varieties. We now give a definition on an arbitrary abelian scheme.

Definition 8.14. Let A/S be an abelian scheme. A **polarization** is a map

$$\psi : A \to \mathrm{Pic}_{A/S}$$

such that on geometric fibers it is a map $\phi_{\mathscr{L}}$ of the form described above.

We ask when can an abelian variety be lifted with a polarization. We have the following result:

Theorem 8.15 (Mumford, [12] Theorem 2.3.3)**.** *Let P be the functor taking local artin Λ algebras to flat deformations of $(A_0, \phi_0)/k$ where A_0 is an abelian variety and ϕ_0 is a polarization. Let $d = \dim_k H^2(X_0, \mathscr{O}_{X_0})$. The functor P is a subfunctor of functor M (defined in the statement of the last theorem), and there exists d elements $a_1, \ldots, a_d \in \Lambda[[t_{1,1}, \ldots, t_{g,g}]]$ and such that if $\mathfrak{a} = (a_1, \ldots, a_d)$, then $\Lambda[[t_{1,1}, \ldots, t_{g,g}]]/\mathfrak{a}$ prorepresents P.*

The following, which we will prove also holds:

Proposition 8.16. *Furthermore, if the polarization is separable, the deformation functor is smooth.*

To prove this we will the following fact about cohomology

Proposition 8.17. *For an abelian variety A, $H^p(A, \mathscr{O}_A) = \bigwedge^p H^1(A, \mathscr{O}_A)$*

Now we take a polarization $\phi : A \to \widehat{A}$ of an abelian variety A. As it must be an isogeny, its kernel must be a finite group scheme, K. In his book, [10], Mumford includes a discussion of commutative finite group schemes and their duals. Important for our purposes is if $A \to B$ is an isogeny of abelian varieties, the kernel of the dual map $\widehat{B} \to \widehat{A}$ is isomorphic to the dual of the kernel ([10] Theorem 3.15.1). For polarizations, because of the symmetric condition, this implies that $K \cong \widehat{K}$, where the hat here denotes the dual commutative finite group scheme. Mumford's discussion of finite group schemes in ([10] 3.14.2) implies that if this is the case then, after passing to the algebraic closure, K must contain a copy of $\mu_{p^n} = \mathrm{Spec}\, k[T, T^{-1}]/((T-1)^{p^n})$ and then the morphism cannot be separable. Therefore, if p divides the order of the kernel of a polarization, the polarization is not separable, and furthermore, if a polarization is separable, it is étale.

Proof of 8.16. We let B, B' be local Λ-algebras, and $B \to B'$ a surjection with kernel I. We assume that A' is an abelian scheme over B, and $\phi' : A' \to \hat{A}'$ be a polarization. We will show that A' can lifted to $A/\operatorname{Spec} B$ and ϕ' lifted to $\phi : A \to \widehat{A}$ a polarization over B. Let $A_0 = A' \otimes k$.

The proof for this will proceed in two steps. First we will establish the result for polarizations ϕ' of the form $\phi_{\mathscr{L}'}$ for some L' first. Then we will reduce to this case.

For convenience we will refer to $T_{A_0,k}$ as T and $T_{0,\widehat{A_0}/k}$ as $\widehat{T}$.

Proposition 8.10 gives us that $H^1(A, T_{A_0/k}) \cong H^1(A, O_{A_0}) \otimes T$. Proposition 7.4 gives us that $H^1(A, \mathscr{O}_A) \cong T_{0,\widehat{A_0}/k}$. We also have that $H^2(A, \mathscr{O}_{A_0}) = \widehat{T} \wedge \widehat{T}$ by the preceding proposition.

We will give a map $f : H^1(A, T_{A_0/k}) \to H^2(A, \mathscr{O}_{A_0})$ by giving a map $\widehat{T} \otimes T \to \widehat{T} \wedge \widehat{T}$, where the map is $\mathrm{id} \wedge d\lambda_0$ where $d\lambda_0$ is the natural map on tangent spaces.

We let Y be a fixed deformation of A to B. Let $\mathfrak{o}(Y, \mathscr{L}')$ be the obstruction to lifting $\mathscr{L}'$ to Y.

Now we fix a deformation of A' to B, Z. We claim

$$\mathfrak{o}(Y, \mathscr{L}') = \mathfrak{o}(Z, \mathscr{L}') + (f \otimes \mathrm{id}_I)(\iota_X(Y))$$

where $\iota_Z(Y)$ is the element $H^1(X, \mathscr{O}_Z) \otimes I$ correponding to the deformation of Y relative to Z.

One can imagine constructing the obstruction of lifting $\mathscr{L}'$ with respec to Y in the following way. Cover A_0 with affines, $\{U_\alpha\}$, over which $\mathscr{L}'$ trivializes we can lift $\mathscr{L}'$ over Y and Z uniquely and such that the deformations Y and Z are isomorphic on that overlap. We set an isomorphism of Y and Z locally, we note this a derivation into I, which we will call D_α. Then we specify a generator of $\mathscr{L}'$ on each open $\{U_\alpha\}$ with respect to Z and therefore with respect to Y. Let transition function $\mathscr{L}'$ on $U_\alpha|_{U_{\alpha\beta}}$ to $U_\beta|_{U_{\alpha\beta}}$ on A_0 be given by multiplication by $u_{\alpha\beta}$ and on Z and Y by $v_{\alpha\beta}$ (identifying Y with Z via the chosen isomorphisms). We will compute the obstruction with respect Y. Passing from $U_\alpha|_{U_{\alpha\beta\gamma}}$ to $U_\beta|_{U_{\alpha\beta\gamma}}$ on Y we take the generator, s, to $v_{\alpha\beta s}$. Passing to $U_\beta|_{U_{\alpha\beta\gamma}}$ to $U_\gamma|_{U_{\alpha\beta\gamma}}$ we get $v_{\alpha\beta}s$ to $(v_{\alpha\beta}v_{\beta\gamma} + (D_\gamma - D_\beta)v_{\alpha\beta})$, and finally passing back to $U_\alpha|_{U_{\alpha\beta\gamma}}$, this becomes

$$(v_{\alpha\beta}v_{\beta\gamma}v_{\gamma\alpha} + v_{\gamma\alpha}(D_\gamma - D_\beta)v_{\alpha\beta} + (D_\gamma - D_\alpha)v_{\alpha\beta}v_{\beta\gamma}).$$

Recall that U_α and U_β are defined as deformations to Y relative to a deformation to Z as a differential. The difference between the differentials on $U_{\beta\gamma}$ is the sum of things of the form $f_{\beta\gamma} \otimes t_{\beta\gamma} \otimes i_{\beta\gamma}$ where f is a local section of $\mathscr{O}_{A_0}$, $t \in T$, and $i \in I$. Derivations of the transition functions, $u_{\alpha\beta}$, by derivations of the form of the form $D_\gamma - D_\beta$ will then be $f_{\beta\gamma}t_{\beta\gamma}(u_{\alpha\beta}) \otimes i_{\beta\gamma}$ where $t_{\beta\gamma}$ acts by the left vector field given by $t_{\beta\gamma}$ at the identity. The $t_{\beta\gamma}(u_{\alpha\beta})$ gives the $d\lambda_0$ term. Note that difference between the obstruction is precisely $v_{\gamma\alpha}(D_\gamma - D_\beta)v_{\alpha\beta} + (D_\gamma - D_\alpha)(v_{\alpha\beta}v_{\beta\gamma})$. Therefore, the above computation shows that the appropriate cup product shows the differences in the obstruction is precisely

Now we use the fact that λ_0 is separable to guarantee it is étale, to guarantee $d\lambda_0$ is an isomorphism, we get that we may pick $\iota_Z(Y) \in H^1(X, \mathscr{O}_{X_0}) \otimes I$, $\delta_Y(\mathscr{L}')$ to vanish. Doing this lifts the abelian scheme and the polarization.

Now we consider the case when the polarization does not come from a line bundle. Let k' be large enough so that on it the polarization actually does come form a line bundle. We consider $C \to B$ over $C' \to B'$ as local artin rings, assuming we have a deformation to B', where C, C' have residue field k'.

Lemma 2.3.2 of Oort ([12]) gives that the polarization over C', since it comes from a line bundle on the residue field, also comes from a line bundle on C'.

We consider $\mathscr{N}$ on $A' \times A'$ giving the polarization $A' \to \widehat{A}'$. We must show we can lift $\mathscr{N}$ to a line bundle on $A \times \widehat{A}$ for some deformation of A' to A/B. Again we get an equation $\delta_{Y\times Y}(\mathscr{N}) = \delta_{Z\times Z} + ((f \oplus f) \otimes \mathrm{id}_I)(\iota_{Z\times Z}(Y \times Y))$. Now it not clear that we can pick a deformation of Y to get $\delta_{Y\times Y}(\mathscr{N})$ to vanish. However, tensoring to k' via base changing to C shows us that we can infact do this over k', and hence, by linear algebra of field extensions, we could actually do this over k, and hence in this case too we lift the field extension.

$\square$

Remark 8.18. This fact about separable polarizations is especially important, because it means that abelian varieties with separable polarizations over fields of characteristic $p > 0$ can always be lifted to characteristic zero.

By Proposition 8.16 any variety with a separable polarization can be lifted to a formal scheme over a local noetherian ring in characteristic zero, like the ring of Witt vectors. Then we note that if $A/\operatorname{Spec} B$ is an abelian scheme over an artin local ring with a separable polarization ψ, we have by Mumford's discussion of polarizations [11] $(1 \times \psi)^* \mathscr{P}_A$ will be ample. This gives a compatible system of ample line bundles, and hence by Theorem 3.9 the formal scheme was actually algebraizable.

We have the following example that there exist abelian varieties which indeed have no polarizations. What is shown is actually stronger: there are abelian varieties with such that p divides the order of the kernel of every polarization.

Example 8.19 (Taken from and following closely [6]). Let k be a field. We begin with E and E' two non-isogenous elliptic curves over k, and we assume that both contain a copy of a group scheme over of prime order p over k. We fix an embedding of G into both E and E' and consider G to be the diagonal embedding of G into the product $E \times E'$. We then consider the abelian surface $A = (E \times E')/G$. We will let $\pi : E \times E' \to A$ be the quotient map which is an isogeny of degree p.

We claim that any polarization of A over k kernel whose order is divisible by p. We assume by sake of contradiction that there were a polarization $\phi : A \to \widehat{A}$ such that the kernel had order over k coprime with p.

We consider $j : E \to A$ to the inclusion of E into A, and we let $\widehat{j} : \widehat{A} \to \widehat{E}$ be the dual map. Consider:

$$f : E \xrightarrow{j} A \xrightarrow{\phi} \widehat{A} \xrightarrow{\widehat{j}} \widehat{E}$$

That this is a polarization can be seen by dualizing and noticing that identifications of the double-dual maps commute. We let f' be the analogous map for $j' : E' \to A$.

We now consider

$$E \times E' \xrightarrow{\pi} A \xrightarrow{\phi} \widehat{A} \xrightarrow{\widehat{\pi}} \widehat{E} \times \widehat{E'}. \tag{8.20}$$

If we add to the beginning an inclusion of E (resp E') and to the end a projection onto $\widehat{E}$ (resp $\widehat{E'}$), we see that the composite is f (resp f').

As E and E' are not isogenous any homomorphism from one to the other must be the zero map as they are both one-dimensional. Therefore, the composite (8.20) must be the direct product of $E \to \widehat{E}$ and $E' \to \widehat{E'}$, as composing by inclusion of E (resp. E') then projection onto E' (resp. E) must be zero, and therefore must be the map $f \times f'$.

As the degree of ϕ is coprime to p and the degree of π and $\widehat{\pi}$ are both p, we have that the p^2 must divide $f \times f'$ but not p^3. Therefore, since degrees of polarizations are p^2 divides one $\deg f$, and p does not divide the other.

Polarizations of elliptic curves, viewing them as self dual, are simply the multiplication by $[n]$ maps. Applying this to f, f' one must have kernel which contains none of the p-torsion, and the other must have kernel that contains all of it. Without loss of generality we will assume that the kernel of f contains all of the p-torsion.

Returning to the composition (8.20) we note that the p-parts of the kernels of f and f' must occur at where meets the kernel of $\widehat{\pi}$, as the π bit is an isomorphism, and ϕ has degree prime to p. We note that the kernel of f must contain G. This implies the image of G is in the kernel of $\widehat{\pi}$. However, this then implies that the kernel of f' must contain G.

Therefore, the degree of both f and f' are divisible by p which is a contradiction so the degree of ϕ must be divisible by p, and in particular A has no separable isogenies.

We thus note that the case for abelian surfaces is already quite different than the case for elliptic curves which not only always had separable polarizations, but always had polarizations which were isomorphisms.

8.2 Non-algebraizable formal scheme

We now can discuss the existence of non-algebraizable formal schemes. It will arise as the deformation of an abelian variety.

We will need the following lemma.

Lemma 8.21. *Let k be an uncountable field, and let V be an n dimensional k vector space. Let $\{H_i\}$ be a countable number of affine subspaces of codimension at least 1. Then $\bigcup_n H_n$ is not equal to V.*

Proof. First it suffices to prove the assertion for H_n of codimension 1, and we do this by induction on the dimension of V. Notice it is clearly true if $\dim_k V = 1$. We assume it is true for $\dim_k V = n$.

Note that V has uncountably many hyperplanes. First instance, let $e_1, \ldots, e_n$ be a basis of V, and we take the hyperplanes spanned by $e_1, e_2, \ldots, e_{n-2}, e_{n-1} + \alpha e_n$ for $\alpha \in k$ gives uncountably many different hyperplaces no two of which are parallel.. Therefore, we may find a hyperplane H which is not parallel to any o the H_n. Note that $H \cap H_n$ will then be an affine subspace of H of codimension at least 1, and by induction, we may find an element of H which is not contained in any of the H_n. □

We also need the following:

Lemma 8.22. *An abelian scheme over a DVR is projective.*

Proof. We know the generic fiber is an abelian variety, hence projective. Take an ample effective divisor D_η on the generic fiber and take its closure D on the whole abelian scheme. Since $\mathscr{O}(D_\eta)$ is ample and ampleness on abelian schemes can be checked on any point, $\mathscr{O}(D)$ is ample on all fibers. We call $\mathscr{O}(D)$, $\mathscr{L}$. By Proposition 8.6 ampleness on the special fiber is enough to conclude ampleness of the line bundle, and therefore we have an ample line bundle, so we have finished.

□

Therefore, we can construct a non-algebraizable formal scheme in the following way. Taking an abelian variety, A, over an uncountable field of characteristic zero of dimension $g \geq 2$, we see that the tangent space of deformations of this abelian variety is k^{g^2} and deformations of a given polarization have tangent space $k^{g(g+1)/2}$. If $g \geq 2$ the former's dimension is greater than the latter's. Using the fact that there are only countably may polarization types of any abelian variety, we immediately get by the above lemma that there exists x in the tangent space to the deformation functor that is not in the tangent space of the deformation of the polarization of any of the abelian variety. This is to say that there is a deformation of A to $k[\epsilon]$ for which there does not exist a polarization. Since we know polarizations always lift, we can find a formal deformation $\mathcal{A}$ of A to $k[[x]]$ such that $\mathcal{A} \otimes \operatorname{Spec} k[\epsilon]$. If $\mathcal{A}$ was algebraizable, by the last lemma we could find an ample line bundle on, and then using the $\phi_{\mathscr{L}}$ construction could get a polarization on the algebraization, and hence on $\mathcal{A} \otimes \operatorname{Spec} k[\epsilon]$. Since we know this cannot happen, we have shown the existence of non-algebraizble formal deformations.

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