Let $5$ $4$

Consider the Hilbert space $X$.

Recall that $S'(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$, and $S'(\mathbb{R})$ the space of tempered distributions. Show that there exists no $u \in S'(\mathbb{R})$ such that for $\phi \in S(\mathbb{R})$ with $\text{supp } \phi \subset (0, \infty)$, $u(\phi) = \int e^{1/x} \phi(x) \, dx$. (Hint: if such a distribution $u$ existed, it would satisfy an estimate!)

Let $\mu$ be a non-negative Borel measure on $\mathbb{R}^n$ such that $\mu(A) < \infty$ for each bounded Borel subset $A \subset \mathbb{R}^n$.

Suppose $\rho > 0$, $x \mapsto \mu(\overline{B}_\rho(x))$ is an upper semi-continuous function on $\mathbb{R}^n$. (A real-valued function $\theta$ on $\mathbb{R}^n$ is upper semi-continuous if for all $x \in \mathbb{R}^n$, $\theta(x) \geq \limsup_{y \to x} \theta(y)$.)

Give an example of a Borel measure $\mu$ as above and $\rho > 0$ such that $x \mapsto \mu(\overline{B}_\rho(x))$ is not continuous.

Let $\mathcal{H} = L^2(\mathbb{R}, e^{-x^2} \, dx)$, and let $\phi_n(x) = x^n$ for $n \geq 0$ integer, so $\phi_n \in \mathcal{H}$.

Let $e_{\xi}(x) = e^{i\xi x}$ for $x \in \mathbb{R}$. Prove that $\sum_{n=0}^{k} \frac{(i\xi)^n}{n!} \phi_n$ converges to $e_{\xi} \in \mathcal{H}$ in the norm topology as $k \to \infty$.

Using (a) or otherwise show that if $\phi \in \mathcal{H}$ and $\langle \phi_n, \phi \rangle_\mathcal{H} = 0$ for all $n$ then $\phi = 0$. (Hint: show that the Fourier transform of $e^{-x^2} \phi$ vanishes!)

Show that there is an orthonormal basis $\{\psi_n\}_{n=0}^{\infty}$ of $\mathcal{H}$ such that for all $n$

$$\text{span}\{\phi_0, \phi_1, \ldots, \phi_n\} = \text{span}\{\psi_0, \psi_1, \ldots, \psi_n\}.$$

Let $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$ be the unit circle, and for $m > 0$ let $H^m(\mathbb{T})$ be the Sobolev space consisting of $L^2(\mathbb{T})$-functions $f$ whose Fourier coefficients $\hat{f}(n)$ satisfy $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 (1 + n^2)^m < \infty$. Suppose $A : H^m(\mathbb{T}) \to L^2(\mathbb{T})$ is a continuous linear map, and there is $B \in \mathcal{L}(L^2(\mathbb{T}), H^m(\mathbb{T}))$ such that $AB - I = E$ and $BA - I = F$ are compact on $L^2(\mathbb{T})$, resp. $H^m(\mathbb{T})$, [and in fact $E^* \in \mathcal{L}(L^2, H^m)]$. Suppose also that $\langle Af, g \rangle_{L^2} = \langle f, Ag \rangle_{L^2}$ when $f, g \in H^m(\mathbb{T})$, [and $\langle Bf, g \rangle_{L^2} = \langle f,Bg \rangle_{L^2}$ for $f, g \in L^2(\mathbb{T})$]. Show that $A - \lambda I : H^m \to L^2$ is invertible when $\lambda \notin \mathbb{R}$.
Do all five problems. Write your solution for each problem in a separate blue book.

1. Show that if $\mu$ is a $\sigma$-finite measure on a measurable space $(X, A)$ (i.e. on $A$, where $A$ is a $\sigma$-algebra of subsets of $X$), then there is a finite measure $\nu$ on $(X, A)$ with $\nu << \mu$ and $\mu << \nu$.

2. Suppose $K \in L^2(\mathbb{R}^n)$, and define $(Tf)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, dy$. Show that $T \in L(L^2(\mathbb{R}^n))$, and $T$ is compact.

3. Recall that $S(\mathbb{R})$ is the space of Schwartz functions on $\mathbb{R}$, and $S'(\mathbb{R})$ the space of tempered distributions. Let $\phi_n \in S(\mathbb{R})$, $n \geq 1$. Suppose that $u \in S'(\mathbb{R})$, and the distributions corresponding to $\phi_n$, namely $u_n(\psi) = \int \phi_n \psi$ for $\psi \in S(\mathbb{R})$, converge to $u$ in $S'(\mathbb{R})$.

   a. Suppose that there exists $C > 0$ such that $\|\phi_n\|_{L^2} < C$ for all $n$. Show that there exists $\phi \in L^2$ such that $u(\psi) = \int \phi \psi$, $\psi \in S(\mathbb{R})$ and $\phi_n$ converge weakly to $\phi$ in $L^2$.

   b. Show that the analogous statement is not true if $L^2$ is replaced by $L^1$, namely show that there exist $\phi_n$ and $u$ with $\|\phi_n\|_{L^1} < C$ such that $u$ is not a distribution given by an $L^1$ function $\phi$.

4. Suppose $X \subset Z$, $Y \subset V$, with $X, Y, Z, V$ Banach spaces, and with both inclusions continuous with respect to their respective norms. Suppose that $P : Z \to V$ is continuous and linear and has the property that $u \in Z$, $Pu \in Y$ implies $u \in X$. Show that there exists $C > 0$ such that for all $u \in Z$ satisfying $Pu \in Y$, one has

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_Z).$$

5. Let $S^1 = \mathbb{R}/\mathbb{Z}$.

   a. Assume that for a given function $\phi \in L^1([0,1])$ there exists an irrational number $\alpha$ such that $\phi(x) = \phi(x + \alpha)$ for almost all $x \in [0,1]$, where $+$ is addition modulo $\mathbb{Z}$. Show that $\phi(x)$ equals to a constant for almost all $x \in [0,1]$.

   b. Given an irrational number $\alpha$, consider the equation

$$g(x + \alpha) - g(x) = p(x), \quad x \in S^1,$$

for an unknown function $g(x)$, with a given function $p \in C^\infty(S^1)$, such that

$$\int_{S^1} p(x) \, dx = 0.$$

Give a condition on $\alpha$ that would guarantee that $g \in C^1(S^1)$ for any such function $p$. 

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Ph.D. Qualifying Exam, Real Analysis
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