1. Let \( A = \mathbb{C}[X, Y]/(X^2, XY) \).
   (a) (3 pts) Describe all prime ideals of \( A \).
   (b) (3 pts) Show that the localization \( A_Y \) is an integral domain.
   (c) (4 pts) For which prime ideals \( p \) of \( A \) is \( A_p \) an integral domain?

2. Let \( k \) be a field.
   (a) (5 pts) Show that if \( f \in k[x] \) is monic of degree \( n > 0 \) with no repeated irreducible factors and \( f(0) \neq 0 \), then there is exactly one conjugacy class of elements \( M \in \text{GL}_n(k) \) with characteristic polynomial \( f \). \textbf{Hint:} one approach is to use the rational canonical form over \( k \).
   (b) (5 pts) Find a representative of each conjugacy class of \( g \in \text{GL}_3(R) \) such that \( g^5 = 1 \).

3. (10 pts) Let \( G = \text{SL}_2(F_p) \) with \( p \) an odd prime. Prove \( |G| = p(p^2 - 1) \), and that the Sylow \( \ell \)-subgroups of \( G \) are cyclic for every odd prime \( \ell \) dividing \( |G| \). \textbf{Hint:} \( F_p^\times \subset GL_2(F_p) \).

4. (a) (4 pts) Explain why \( \mathbb{Z}[\sqrt{19}] \) is the integral closure of \( \mathbb{Z} \) in \( \mathbb{Q}(\sqrt{19}) \).
   (b) (3 pts) For an odd prime \( p \neq 19 \), explain why \( p\mathbb{Z}[\sqrt{19}] \) is prime if and only if 19 is not a square mod \( p \).
   (c) (3 pts) Using that \( 19 \equiv 6^2 \mod 17 \), factor \( 17\mathbb{Z}[\sqrt{19}] \) into a product \( pq \) of two prime ideals.

5. For \( n \geq 1 \), let \( S_n \) act on \( \mathbb{C}^n \) by permutation of the standard basis, and let \( V_n := \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum x_i = 0\} \) be the natural \( S_n \)-stable hyperplane in \( \mathbb{C}^n \).
   (a) (2 points) For \( n \geq 2 \) and the inclusion \( S_{n-1} \hookrightarrow S_n \) onto the stabilizer of a choice of \( i \in \{1, \ldots, n\} \), show \( V_n \simeq V_{n-1} \oplus \mathbb{C} \) as \( S_{n-1} \)-representations.
   (b) (3 points) For \( n \geq 1 \), show \( V_n \) is an irreducible \( S_n \)-representation. \textbf{(Possible approach: work by induction on \( n \), and use (a).)}
   (c) (3 points) For \( n \geq 2 \), show \( \wedge^2 V_n \simeq (\wedge^2 V_{n-1}) \oplus V_{n-1} \) as \( S_{n-1} \)-representations.
   (d) (2 points) For \( n \geq 2 \), show \( \wedge^2 V_n \) is an irreducible \( S_n \)-representation. \textbf{(Hint:} induction on \( n \).)
1. (10 pts) Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, and let $B : V \times V \to \mathbb{C}$ be a non-degenerate symmetric bilinear form. Let $T : V \to V$ be a nilpotent linear transformation that is skew-symmetric with respect to $B$: for all $x, y \in V$ we have

$$B(Tx, y) = -B(x, Ty).$$

If dim($\ker T$) = 1, deduce dim $V$ is odd.

2. (a) (3 points) Prove $x^7 - 15$ and $(x^7 - 1)/(x - 1) = x^6 + x^5 + \cdots + x + 1$ are irreducible over $\mathbb{Q}$.

(b) (7 points) Describe (with proof) the splitting field $K/\mathbb{Q}$ of $x^7 - 15$, and describe the group $\text{Gal}(K/\mathbb{Q})$, and describe its action on $K$.

3. Let $K/k$ be a finitely generated extension of fields with characteristic 0.
   (a) (5 pts) If $k'/k$ is an algebraic extension inside $K$ then show $[k' : k]$ is finite.
   (b) (5 pts) Suppose $k$ is algebraically closed in $K$; i.e., anything in $K$ algebraic over $k$ belongs to $k$. Show that if $\ell/k$ is a finite extension then $K \otimes_k \ell$ is a field in which $\ell$ is algebraically closed. (Hint: use the primitive element theorem.)

4. (a) (5 pts) If $A$ is an integrally closed noetherian domain with fraction field $F$ and $F'/F$ is a finite separable extension, explain why the integral closure of $A$ in $F'$ is finitely generated as an $A$-module. Identify where the hypotheses on $A$ are used in your argument. (Hint: use the trace to show that the integral closure lies in a finitely-generated $A$-module.)

   (b) (2 pts) State the Noether normalization theorem for integral domains finitely generated over $\mathbb{C}$, and its geometric interpretation.

   (c) (3 pts) If $R$ is an integral domain finitely generated over $\mathbb{C}$ with fraction field $K$, explain why the integral closure of $R$ in $K$ is finitely generated as an $R$-module. (Hint: use (a) and (b).)

5. Let $A$ be a commutative ring, and $M$ an $A$-module. Recall that $M$ is called injective if for any injection of $A$-modules $N \to N'$ and $A$-linear $f : N \to M$ there is an $A$-linear $f' : N' \to M$ satisfying $f'|_N = f$.

   (a) (2 pts) Show that every $A$-module with one generator is isomorphic to $A/J$ for some ideal $J$ of $A$.

   (b) (2 pts) Show that every $A$-module with one generator is isomorphic to $A/J$ for a unique ideal $J$ of $A$.

   (c) (6 pts) Assume that $\text{Ext}_A^1(A/J, M) = 0$ for every ideal $J$ of $A$. Show that $M$ is injective. (Hint: Zorn’s Lemma and (a).)