A1. (10 pts) Suppose $G$ is a group of odd order and suppose $p$ is the smallest prime dividing $|G|$. If the $p$-Sylow subgroup $S \subset G$ is normal and has order $p^2$ or $p$, prove that $S$ is contained in the center of $G$.

A2.
(a) (6 pts) Find the Galois groups of $x^6 - 2$ and $x^6 + 3$ over $\mathbb{Q}$.
(b) (4 pts) Find the Galois group of $x^6 - 2$ over $\mathbb{F}_5$ and $\mathbb{F}_7$.

A3. Let $A$ be a commutative ring. For $f \in A$, take the definition of $A_f$ to be via equivalence classes of fractions.

(a) (4 pts) Prove that $A_f$ and $A[X]/(1 - Xf)$ are uniquely isomorphic as $A$-algebras.

(b) (6 pts) Let $g = f + n$ for nilpotent $n \in A$. Prove that $A_f$ and $A_g$ are uniquely isomorphic as $A$-algebras. (You do not need to use (a) for this.)

A4. Let $n$, $m$ be integers $0 < m < n$ and with $m$ dividing $n$. Let $R = \mathbb{Z}/n\mathbb{Z}$ and $M = R/mR$.

(a) (3 pts) Show that $R$ is injective as a module over itself. (Hint: Zorn.)

(b) (7 pts) Compute $\text{Tor}_i^R(M, M)$ and $\text{Ext}_i^R(M, M)$ for all $i \geq 0$.

A5. Let $k$ be a field with characteristic $p > 0$ and $F$ a finitely generated extension field. A transcendence basis $\{x_1, \ldots, x_d\}$ of $F$ over $k$ is separating if the finite extension $F/k(x_1, \ldots, x_d)$ is separable. This problem proves such transcendence bases exist when $k$ is perfect. Fix a transcendence basis $\{x_1, \ldots, x_d\}$.

(a) (4 pts) Let $n := [F : k(x_1, \ldots, x_d)]_i$ be the inseparable degree. Assume $n > 1$. Show there is $a \in F$ not separable over $k(x_1, \ldots, x_d)$ such that its $(T$-monic) minimal polynomial $f \in k(x_1, \ldots, x_d)[T]$ lies in $k[x_1, \ldots, x_d, T]$, and prove $f$ is irreducible in $k[x_1, \ldots, x_d, T]$, and irreducible in $k(x_1, \ldots, x_{d-1}, T)[x_d]$ if $f$ involves $x_d$.

(b) (4 pts) Assume $k$ is perfect. For $a$ and $f$ as in (a), prove $\partial_{s_j}f \neq 0$ for some $j$; relabel so $j = d$. Prove that $\{x_1, \ldots, x_{d-1}, a\}$ is a transcendence basis of $F$ over $k$, and use multiplicativity of inseparable degree to show $[F : k(x_1, \ldots, x_{d-1}, a)]_i < n$.

(c) (2 pts) Deduce the existence of a separating transcendence basis when $k$ is perfect.
Fall 2014 Qualifying Exam, Algebra

Afternoon

B1. (10 pts) Find all abelian groups $G$ for which there exists an exact sequence

$$0 \to \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}) \to G \to \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}) \to 0.$$ 

B2. Let $K = \mathbb{Q}(\alpha)$ with $\alpha^3 = 2$. The following proves that the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in $K$ is $\mathbb{Z}[\alpha]$. (In contrast, $L = \mathbb{Q}(\theta)$ with $\theta^3 = 10$ has $(1 + \theta + \theta^2)/3 \in \mathcal{O}_L$, so $\mathcal{O}_L \neq \mathbb{Z}[\theta]$. You needn’t show this.)

(a) (3 pts) Show $\mathbb{Z}[\alpha]$ has discriminant $-2^2 \cdot 3^3$, and deduce that $|\mathcal{O}_K : \mathbb{Z}[\alpha]|$ divides 6.

(b) (2 pts) For $x = c_0 + c_1 \alpha + c_2 \alpha^2$ with $c_i \in \mathbb{Q}$, compute the matrix for multiplication by $x$ on $K$ with respect to the ordered $\mathbb{Q}$-basis $\{1, \alpha, \alpha^2\}$ and obtain $N_{K/\mathbb{Q}}(x) = c_0^2 + 2c_1^3 + 4c_2^3 - 6c_0c_1c_2$, so $N_{K/\mathbb{Q}}(x) \equiv c_0 \pmod{2}$ if $x \in \mathbb{Z}[\alpha]$.

(c) (2 pts) If $x = c_0 + c_1(\alpha - 2) + c_2(\alpha - 2)^2$ with $c_i \in \mathbb{Z}$ then show $N_{K/\mathbb{Q}}(x) \equiv c_0 \pmod{3}$.

(Hint: $x = (c_0 - 2c_1 + 4c_2) + (c_1 - 4c_2)\alpha + c_2\alpha^2$.) Also show that $N_{K/\mathbb{Q}}(\alpha - 2) = -6$.

(d) (3 pts) Prove that $\mathcal{O}_K \cap (1/2)\mathbb{Z}[\alpha] = \mathbb{Z}[\alpha]$ and $\mathcal{O}_K \cap (1/3)\mathbb{Z}[\alpha] = \mathbb{Z}[\alpha]$, and deduce that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. (Hint: $N_{K/\mathbb{Q}}(\alpha) = 2$ and $N_{K/\mathbb{Q}}(\alpha - 2) = -6$)

B3. Let $G$ be a finite group, and let $C_1, C_2, C_3$ be three conjugacy classes in $G$.

(a) (6 pts) For any irreducible character $\chi$ of $G$, show that

$$\sum_{x \in C_1, y \in C_2, z \in C_3} \chi(xyz) = \frac{#C_1#C_2#C_3 \cdot \chi(C_1)\chi(C_2)\chi(C_3)}{\chi(1)^2}.$$ 

(Hint: The left side is the trace of $\sum \rho(xyz)$, where $\rho : G \to \text{GL}(V)$ has character $\chi$.)

(b) (4 pts) Show that the set $\{(x, y, z) \in C_1 \times C_2 \times C_3 | xyz = 1\}$ has cardinality

$$\frac{#C_1#C_2#C_3}{#G} \sum_{\chi} \frac{\chi(C_1)\chi(C_2)\chi(C_3)}{\chi(1)}.$$ 

Here the sum runs over all irreducible characters of $G$.

B4. Let $V$ be a nonzero finite-dimensional vector space over a field $F$. A linear endomorphism $T$ of $V$ is *semi-simple* if every $T$-stable subspace admits a $T$-stable linear complement.
(a) (4 pts) State the theorem on rational canonical form for \( T \), and deduce that \( T \) is semi-simple if and only if its minimal polynomial has no repeated monic irreducible factors, and that if \( F \) is algebraically closed then this is equivalent to diagonalizability of \( T \).

(b) (3 pts) Assume \( F \) is algebraically closed. State the theorem of Jordan canonical form for \( T \), and use it to prove that \( T = S + N \) for commuting \( F \)-linear endomorphism \( S \) and \( N \) of \( V \) that are respectively semi-simple and nilpotent.

(c) (3 pts) Prove that \( S \) and \( N \) as in (b) are unique.

**B5.** Let \( k \) be an algebraically closed field, and \( A \) and \( B \) domains finitely generated over \( k \).

(a) (3 pts) Let \( X \) be the space of maximal ideals of \( B \) (with the Zariski topology), and for \( x \in X \) let \( m_x \) be the corresponding maximal ideal in \( B \). For nonzero \( f \in A \otimes_k B \) show there is a dense open \( U \subseteq X \) so that the element \( f \mod m_x \in A \) is nonzero for all \( x \in U \). (Hint: write \( f \) as a finite sum of elementary tensors.)

(b) (4 pts) Prove that \( A \otimes_k R \) is a domain for any \( k \)-algebra domain \( R \). (Hint: reduce to the case where \( R \) is finitely generated over \( k \).)

(c) (3 pts) For any extension field \( K/k \), use Noether normalization to prove \( A \otimes_k K \) has the same dimension as \( A \).