Do all five problems. Write your solution for each problem in a separate blue book.

1. Let \( \mu \) denote the Lebesgue measure on \([0, 1]\). Suppose that \( f_k, k \in \mathbb{N} \), are Lebesgue measurable (real-valued) functions on \([0, 1]\), and let \( m_{kn} = \mu(\{x \in [0, 1] : |f_k| \leq 2^{n-1}, 2^n \}) \) for \( n \in \mathbb{N} \).
   a. Suppose that \( \sum_{n=1}^{\infty} n2^nm_{kn} \leq 1 \) for all \( k \) and suppose that \( f_k \to 0 \) a.e. Show that \( \int f_k \, d\mu \to 0 \).
   b. Give an example of \( f_k \) as above such that \( \sum_{n=1}^{\infty} 2^nm_{kn} \leq 1 \) for all \( k \), \( f_k \to 0 \) a.e, but \( \int f_k \, d\mu \) does not tend to 0.

2. Let \( S'(\mathbb{R}^n) \) denote the space of tempered distributions on \( \mathbb{R}^n \). A locally \( L^1 \) function \( f \) on \( \mathbb{R}^n \) (i.e. the restriction to compact sets is \( L^1 \)) is said to lie in \( S'(\mathbb{R}^n) \) if the map \( C^\infty_c(\mathbb{R}^n) \ni \phi \mapsto \int_{\mathbb{R}^n} f\phi \in \mathbb{C} \) has a (necessarily unique) continuous extension to an element of \( S'(\mathbb{R}^n) \). Here \( C^\infty_c(\mathbb{R}^n) \) is the set of compactly supported \( C^\infty \) functions on \( \mathbb{R}^n \).
   a. Show that the function \( |x|^{-(n-\beta)} \) lies in the space \( S'(\mathbb{R}^n) \) if \( 0 < \beta < n \). For what values of \( \beta \) is it a sum of an \( L^1 \) and a \( L^2 \) function?
   b. Show that the function \( e^x \cos(e^x) \) on \( \mathbb{R} \) lies in \( S'(\mathbb{R}) \).
   c. Prove that there is a locally \( L^1(\mathbb{R}^n) \) function that does not lie in \( S'(\mathbb{R}^n) \).

3. Suppose that \((X, ||.||_X)\) is a normed vector space, \( M \) and \( N \) are (not necessarily closed) subspaces equipped with norms \( ||.||_M \), resp. \( ||.||_N \) such that the identity maps \((M, ||.||_M) \to (M, ||.||_X)\), resp. \((N, ||.||_N) \to (N, ||.||_X)\) are continuous. Let \( M + N \) be the algebraic sum: \( M + N = \{m + n \in X : m \in M, n \in N\} \). For \( x \in M + N \), let
   \[
   ||x||_{M+N} = \inf\{||m||_M + ||n||_N : m \in M, n \in N, x = m + n\}.
   
   a. Show that \( ||.||_{M+N} \) is a norm on \( M + N \).
   b. Show that if \((M, ||.||_M) \) and \((N, ||.||_N) \) are complete then \((M + N, ||.||_{M+N}) \) is complete.

4. Suppose that \( f, g \) are holomorphic functions on a non-empty open connected set \( \Omega \subset \mathbb{C} \), and suppose that \(||f||^2 + ||g||^2\) is constant on \( \Omega \). Show that \( f \) and \( g \) are constant on \( \Omega \).

5. Suppose that \( X, Y \) are reflexive separable Banach spaces, \( X^*, Y^* \) the duals, \( P \in \mathcal{L}(X,Y) \), and suppose that its adjoint \( P' \in \mathcal{L}(Y^*,X^*) \) satisfies the following property: There is a Banach space \( Z \) and a compact map \( \iota : Y^* \to Z \) and \( C > 0 \) such that for all \( \phi \in Y^* \),
   \[
   ||\phi||_{Y^*} \leq C(||P'\phi||_{X^*} + ||\iota\phi||_Z).
   
   a. Show that \( \text{Ker}P' \) is finite dimensional.
   b. Let \( V \) be a closed subspace of \( Y^* \) with \( V \oplus \text{Ker}P' = Y^* \). Show that there is \( C' > 0 \) such that for \( \phi \in V \), \( ||\phi||_{Y^*} \leq C'||P'\phi||_{X^*} \).
   c. Show that for \( f \in Y \) such that \( \ell(f) = 0 \) for all \( \ell \in \text{Ker}P' \), there is \( u \in X \) such that \( Pu = f \).
Suppose \( X \) is a compact metric space, and \( \ell \) is a positive linear functional on \( C(X) \), the Banach space of real valued continuous functions, i.e. \( f \geq 0 \) implies \( \ell(f) \geq 0 \). Suppose \( K \subset X \) compact. Let \( O_j, j \in \mathbb{N} \), be open sets satisfying \( O_j \supset \overline{O}_{j+1} \) and \( \cap_{j \in \mathbb{N}} O_j = K \), and let \( f_j \in C(X; [0, 1]) \) be such that \( f_j = 1 \) on \( K \), \( f_j = 0 \) on the complement of \( O_j \). Show directly from first principles (without quoting a major theorem) that \( \lim_{j \to \infty} \ell(f_j) \) exists, and is independent of the choice of \( O_j \) and \( f_j \) satisfying these conditions.

Suppose \( p \in (0, 1), 0 < a < p \), and suppose \( N \in \mathbb{N}, A_1, \ldots, A_N \) are Lebesgue measurable subsets of \([0, 1]\) with average measure \( \frac{1}{N} \sum_{i=1}^{N} \mu(A_i) \geq p \). Let \( E = \{x \in [0, 1] : x \in A_i \text{ for at least } aN \text{ values of } i \} \).

a. Show that \( \mu(E) \geq \frac{p}{1-a} \).

b. Show that if \( c > \frac{p}{1-a} \) then there exist \( N \in \mathbb{N} \) and sets \( A_1, \ldots, A_N \) with \( \mu(A_i) \geq p \) for all \( i \) such that \( \mu(E) < c \).

Let \( S \) denote the set of Schwartz functions on \( \mathbb{R}^n \). The Fourier transform on \( S \), and more generally on \( L^1 \), is given by \( (F\phi)(\xi) = \int e^{-ix \cdot \xi} \phi(x) \, dx \), while on tempered distributions \( \phi \in S' \) it is given by \( (F\phi)(v) = v(F\phi), \phi \in S \).

a. Show that there exists a compactly supported \( C^\infty \) function \( \phi \) on \( \mathbb{R} \) such that \( \phi \geq 0, \phi(0) > 0 \), and \( F\phi \) is non-negative, \( (F\phi)(0) > 0 \). (Hint: when is the Fourier transform of a function real?)

b. Show that if \( u \) is a compactly supported distribution then the Fourier transform (as defined above) is \( C^\infty \), and there exists \( N \in \mathbb{R} \) such that for all \( \alpha \in \mathbb{N}^n \) there exist \( C_\alpha > 0 \) with \( |(D^\alpha F u)(\xi)| \leq C_\alpha (1 + |\xi|)^N \).

Suppose \( X \) is a vector space over \( \mathbb{C} \) and \( F \) is a collection of linear maps \( X \rightarrow \mathbb{C} \). Equip \( X \) with the \( F \)-weak topology, i.e. the weakest topology in which all elements of \( F \) are continuous.

a. Show that the vector space operations \( + : X \times X \rightarrow X \) and \( \cdot : \mathbb{C} \times X \rightarrow X \) are continuous in this topology (where \( X \times X \) and \( \mathbb{C} \times X \) are equipped with the product topology).

b. Suppose that \( \rho : X \rightarrow [0, \infty) \) is continuous and is a seminorm. Show that there exist \( k \in \mathbb{N}, \ell_1, \ldots, \ell_k \in F \) and \( C > 0 \) such that \( \rho(x) \leq C \sum_{j=1}^{k} |\ell_j(x)| \) for all \( x \in X \).

For \( s \geq 0 \), let \( H^s(\mathbb{T}) \) be the space of \( L^2 \) functions \( f \) on the circle \( \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}) \) whose Fourier coefficients \( \hat{f}_n = \int e^{-inx} f(x) \, dx \) satisfy \( \sum (1 + n^2)^s |\hat{f}_n|^2 < \infty \), with norm \( \|f\|^2_s = (2\pi)^{-1} \sum (1 + n^2)^s |\hat{f}_n|^2 \).

a. Show that for \( r > s \geq 0 \), the inclusion map \( \iota : H^r(\mathbb{T}) \rightarrow H^s(\mathbb{T}) \) is compact.

b. Show that if \( s > 1/2 \), then \( H^s(\mathbb{T}) \) includes continuously into \( C(\mathbb{T}) \), and indeed the inclusion map is compact.