Ph.D. Qualifying Exam, Real Analysis

Fall 2012, part I

Do all five problems. Write your solution for each problem in a separate blue book.

1. Two short problems.
   a. Suppose $u$ is a distribution on $\mathbb{R}$ and $x^k u = 0$ for some $k \in \mathbb{N}, k \geq 1$. Show that there exists $a_j \in \mathbb{C}$ such that $u(\phi) = \sum_{j=0}^{k-1} a_j \phi^{(j)}(0)$.
   b. Suppose that $(X, \mu)$ is a measure space, $1 < p < \infty$, $u_n \in L^p(X, d\mu)$ for $n \in \mathbb{N}$, and for all $\phi \in L^q(X, d\mu)$, $q^{-1} + p^{-1} = 1$, $\lim_{n \to \infty} \int_X |u_n \phi| d\mu$ exists. Show that there exists $C \geq 0$ such that $\int_X |u_n|^p d\mu \leq C$ for all $n$.

2. Let $D_N, N \geq 1$ integer, be the Dirichlet kernel
   $$D_N(\theta) = \frac{1}{2\pi} \frac{\sin(N + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta}.$$  
   Let $L_N = \int_0^{2\pi} |D_N(\theta)| d\theta$. Prove that there exist $C_1, C_2 > 0$ such that for all $N \geq 2$,
   $$C_1 \log N \leq L_N \leq C_2 \log N.$$

3. Two short problems.
   a. Show that there is a closed subset $E$ of $[0, 1]$ with positive Lebesgue measure and with empty interior.
   b. Show that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and $A \subset [0, 1]$ is Lebesgue measurable with measure 0 then $f(A)$ is measurable with measure 0.

4. Suppose that $X, Y$ are Banach spaces, and let $T_s$ denote the norm topology on $X$, $U_s$ the norm topology on $Y$. Let $T_w$ denote the weak topology on $X$, and $U_w$ denote the weak topology on $Y$.
   a. Show that $(X, T_s)$ has the following property, sometimes called (T3$\frac{1}{2}$) or completely regular: if $x \in X$ then $\{x\}$ is $T_w$-closed, and given any $x \in X$ and $C \subset X$ $T_w$-closed with $x \notin C$, there is a continuous function $f : X \to [0, 1]$ with $f(x) = 1$ and $f$ identically 0 on $C$.
   b. Show that a linear map $T : X \to Y$ is continuous as a map from $(X, T_s)$ to $(Y, U_s)$ if and only if it is continuous as a map from $(X, T_w)$ to $(Y, U_w)$.

5. Suppose that $X, Y$ are Hilbert spaces. An operator $A \in \mathcal{L}(X, Y)$ is Fredholm if $A$ has closed range, and $\text{Ker} A$ as well as $Y/\text{Ran} A$ are finite dimensional.
   a. Show that $A \in \mathcal{L}(X, Y)$ is Fredholm if and only if there are finite dimensional vector spaces $V, W$ and a finite rank operator $P \in \mathcal{L}(W \oplus X, V \oplus Y)$ such that $A + P$ is invertible, where $\tilde{A} \in \mathcal{L}(W \oplus X, V \oplus Y)$ is defined by $\tilde{A}(w, x) = (0, Ax), x \in X, w \in W$.
   b. Suppose $A \in \mathcal{L}(X, Y)$ is Fredholm. Show that there exists $\delta > 0$ such that if $R \in \mathcal{L}(X, Y)$ with $\|R\|_{\mathcal{L}(X,Y)} < \delta$ then $A + R$ is Fredholm.
Ph.D. Qualifying Exam, Real Analysis
Fall 2012, part II

Do all five problems. Write your solution for each problem in a separate blue book.

1 Two short problems.
   a. Show that the spectrum of a bounded linear operator $A$ on a Banach space $X$ is non-empty.
   b. Show that if $1 < p < \infty$ then for $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $p^{-1} + q^{-1} = 1$, $f * g$ defined by $(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy$ is a bounded continuous function with sup norm $\leq \|f\|_{L^p}\|g\|_{L^q}$.

2 Two short problems.
   a. Let $P$ denote the space of continuous piecewise affine functions on $[0, 1]$. Show that any $f \in C([0, 1])$ is a uniform limit of elements of $P$.
   b. Show that the inclusion map $i : C([0, 1]) \to L^2([0, 1])$ is not compact.

3 Let $\ell^2(\mathbb{Z})$ denote the space of square summable bi-infinite sequences. Let $L$ denote the operator of multiplication by $n$, and let $\mathcal{H} = \{\{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) : \{na_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})\} \subseteq \ell^2(\mathbb{Z})$ with $\|\{a_n\}_{n=-\infty}^{\infty}\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)|a_n|^2$. Let $R \in \mathcal{L}(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}))$.
   a. Show that there is a discrete set $D \subseteq \mathbb{C}$ such that $L + R - \lambda I : \mathcal{H} \to \ell^2$ is invertible for $\lambda \notin D$.
   b. With $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, show that if $V \in C(\mathbb{T})$ then the set of $\lambda \in \mathbb{C}$ for which there exists $f \in C^1(\mathbb{T})$, not identically 0, for which $f' + Vf = \lambda f$, is discrete.

4 In this problem, let $\|\cdot\|_p$ be the $L^p(\mathbb{R}^n)$ norm, $1 \leq p \leq \infty$, and let $C^\infty_0(\mathbb{R})$ be the set of compactly supported $C^\infty$ functions.
   a. Show, including the explicit constant, that for $\phi \in C^\infty_0(\mathbb{R})$, $\|\phi\|_\infty \leq \frac{1}{2} \int_{-\infty}^{\infty} |\phi'(t)| \, dt$.
   b. Suppose that there is $C > 0$ such that for all functions $\phi \in C^\infty_0(\mathbb{R}^n)$ there is an inequality of the form $\|\phi\|_q \leq C\|
abla \phi\|_p$. Show that one would necessarily then have the relationship $q^{-1} = p^{-1} - n^{-1}$. (Hint: consider the functions $\phi_t$ defined by $\phi_t(x) = \phi(tx)$.)
   c. When $n = 2$, part b) suggests that one might have an inequality of the form $\|\phi\|_\infty \leq C\|\nabla \phi\|_2$. Show that there is no $C > 0$ such that this inequality holds for all $\phi \in C^\infty_0(\mathbb{R}^n)$.

5 Let $\Omega$ be a compact polygonal domain in $\mathbb{R}^2$, i.e. $\Omega$ is an open set with compact closure such that each $x \in \partial \Omega$ has a neighborhood $O_x$ such that $\Omega \cap O_x$ is given by the intersection of one or two half-planes with $O_x$.
   Show that there is $C > 0$ such that the Fourier transform of the characteristic function $\chi_\Omega$ of $\Omega$ satisfies $|\mathcal{F}(\chi_\Omega)(\xi)| \leq C(1 + |\xi|)^{-1}, \xi \in \mathbb{R}^2$. (Hint: reduce to the case of the Fourier transform of a product of a cutoff function with one or two characteristic functions of half-planes.)
   Are there non-trivial open cones $\Sigma$ in $\mathbb{R}^2 \setminus \{0\}$ such that a faster decay estimate holds as $\Sigma \ni \xi \to \infty$?