1(a). In both the dihedral group of order 8 and the quaternion group of order 8, determine how many subgroups there are of order 4. In both these groups, $G = D_8$ or $Q_8$, determine the number of $\text{Aut}(G)$ orbits in the set of subgroups of order 4.

[The action is the obvious one, $H \mapsto \phi H$, for $H \subset G$ and $\phi \in \text{Aut}(G).$]

1(b). Determine the isomorphism classes of groups of order 2008 that have a non-abelian Sylow-2 subgroup.

[The odd factor of 2008 is prime.]

2. Suppose $k$ is a field and suppose $E \supset k$ is an integral domain that is finite dimensional as a vector space over $k$.

(a). Explain why $E$ is a field.

(b). If $a \in E$, consider the $k$-linear transformation $\alpha : E \to E$ defined by $\alpha(x) = ax$. In terms of the minimal polynomial $f(T) \in k[T]$ of $a$ over $k$, describe the rational canonical form of $\alpha$.

(c). If $a \in E$ is purely inseparable over $k$, that is, if its minimal polynomial has form $f(T) = T^{(p^r)} - r$, where $r \in k$, $p = \text{char}(k)$, describe the Jordan canonical form of $\alpha$ over the algebraic closure of $k$.

3(a). Let $A$ be a commutative Noetherian ring and suppose $I \subset A$ is an ideal so that every prime ideal $P \supset I$ is maximal. Explain why $A/I$ is isomorphic to a direct product $A_1 \times \cdots \times A_n$, where each ring $A_j$ has a unique prime ideal.

3(b). Suppose $(0) = \bigcap_{1 \leq i \leq r} Q_i$ is an irredundant primary decomposition of the zero ideal in a commutative ring $B$. Let $P_i = \sqrt{Q_i}$. Show that the set of zero divisors in $B$ is exactly $\bigcup_{1 \leq i \leq r} P_i$.

[A zero divisor is an element $b \in B$ so that $ab = 0$ for some $a \neq 0$. Irredundant includes the fact that $\bigcap_{j \neq i} Q_j \not\subset Q_i$.]
4. Let $A$ be a Dedekind domain with field of fractions $K$. Let $E \supset K$ be a finite Galois extension, $G = \text{Gal}(E/K)$. Let $B \supset A$ be the integral closure of $A$ in $E$. 

(a). Explain why the map from the set of ideals in $A$ to the set of ideals in $B$ given by $I \mapsto IB$ is injective. 

[Hint: You should use facts about factorization of ideals in Dedekind domains.]

Define an operation, $N$, on ideals $J \subset B$ by 

$$N(J) = \prod_{\sigma \in G} \sigma J \subset B.$$ 

(b). If $I, J \subset B$ are ideals, show $N(IJ) = N(I)N(J)$. If $b \in B$ and $n = N_{E/K}(b) \in A$ is the field norm of $b$, show that $N(bB) = nB$, where $bB$ and $nB$ denote principal ideals.

(c). If $Q \subset B$ is a prime ideal and $P = Q \cap A$, let $f = |B/Q : A/P|$. Show that $N(Q) = (PB)^f$.

[Hint: Make use of the factorization of the ideal $PB$ in $B$.]

(d). If $J \subset B$ is an ideal so that $N(J) = PB$ for some prime ideal $P \subset A$, then $J$ is a prime ideal of $B$.

5(a). Let $\zeta_7, \zeta_9 \in \mathbb{C}$ denote primitive $7^{th}$ and $9^{th}$ roots of unity, respectively. Find the Galois group of $E = \mathbb{Q}[\zeta_7, \zeta_9]$ over $\mathbb{Q}$.

5(b). Let $\rho_7, \rho_9 \in \mathbb{R}$ denote the real parts of $\zeta_7$ and $\zeta_9$, respectively. Explain why $L = \mathbb{Q}[\rho_7, \rho_9]$ is normal over $\mathbb{Q}$, and determine the Galois group of $L$ over $\mathbb{Q}$.
1(a). Let $H$ be a non-abelian group of order 20. Determine all possible pairs $\{(n_5, n_2)\}$, where $n_5$ and $n_2$ denote the number of Sylow-5 and Sylow-2 subgroups in such an $H$.

1(b). Let $G$ be a group of order 60 that contains a non-abelian subgroup of order 20. Determine all possible pairs $\{(m_5, m_3)\}$, where $m_5$ and $m_3$ denote the number of Sylow-5 and Sylow-3 subgroups in such a $G$.

[Hint: Normalizers. Use part (a).]

2. Let $G$ be a group. Let $Aut(G)$ denote the group of automorphisms of $G$ and let $Inn(G)$ denote the group of inner automorphisms of $G$.

(a). Show that $Inn(G)$ is a normal subgroup of $Aut(G)$.

(b). Determine the group $Inn(S_4)$, where $S_4$ is the symmetric group on four letters.

(c). Show that all automorphisms of $S_4$ are inner.

[Hint: Conjugacy classes. Show that an automorphism must map a 2-cycle to a 2-cycle. Then do some counting.]

3(a). Suppose $A \subset B = A[x_1, \ldots, x_n]$ are integral domains so that as a ring $B$ is finitely generated over $A$, with generators $x_i$ algebraic over $A$. Explain why there exists a non-zero element $a \in A$ so that every element of $B[1/a]$ is integral over $A[1/a]$.

3(b). Suppose $A \subset B$ is an integral extension of commutative rings and suppose $K$ is an algebraically closed field. Explain why every homomorphism $\phi: A \to K$ extends to a homomorphism $\Phi: B \to K$.

4(a). Determine the Galois group of $X^5 - 2$ over the fields $\mathbb{R}, \mathbb{F}_7,$ and $\mathbb{F}_{11}$.

4(b). If $E$ is the splitting field of $X^5 - 2$ over $\mathbb{Q}$, describe the fixed fields $L_i$ of all Sylow-2 subgroups $H_i \subset Gal(E/\mathbb{Q})$.

5(a). If $p$ is a prime integer, how many irreducible monic polynomials of degree 3 are there over the field $\mathbb{F}_p$?

5(b). How many conjugacy classes are there in the group $GL(3, \mathbb{F}_p)$?

[Hint: Use linear algebra, specifically invariant factors.]