1. MORNING (FALL, 2012)

(1) Let \(A\) be an \(n \times m\) matrix of integers, thus defining a linear map
\[ f_A : \mathbb{Z}^m \rightarrow \mathbb{Z}^n. \]

Similarly, the transpose \(A^t\) of \(A\) defines a linear map
\[ f_{A^t} : \mathbb{Z}^n \rightarrow \mathbb{Z}^m. \]

Prove that the cokernel of \(f_A\) and the cokernel of \(f_{A^t}\) have isomorphic torsion subgroups.

(2) Let \(G = \text{SL}_n(F_p)\) for a prime \(p\) and an integer \(n > 1\).

(i) Find a Sylow \(p\)-subgroup \(P\) of \(G\) and compute its order.

(ii) Give an explicit sequence of subgroups \(1 = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_m = P\) such that for all \(0 \leq i < m\), \(P_i\) is normal in \(P_{i+1}\) and the quotient \(P_{i+1}/P_i\) is abelian.

(3) Let \(R\) be a commutative ring, and \(M\) a free \(R\)-module of finite rank \(n > 0\).

(i) (3 pts) Let \(M^*\) be the dual module. For all \(i > 0\) prove that \(\text{Sym}^i(M)\) is free, and show that there is a unique bilinear pairing
\[ \text{Sym}^i(M) \times \text{Sym}^i(M^*) \rightarrow R \]

 such that \((m_1 \ldots m_i, \ell_1 \ldots \ell_i) \mapsto \sum_{1 \leq j \leq i} s_{ij} \ell_j(m_{s(j)})\) for all \(m_j \in M\) and \(\ell_j \in M^*\), where \(S_i\) is the symmetric group on \([1, \ldots, i]\).

(ii) (2 pts) Prove that the linear map \(\text{Sym}^i(M^*) \rightarrow \text{Sym}^i(M)^*\) induced by the pairing in (i) is an isomorphism when \(i! \in R^\times\) (hint: use a basis of \(M\) to make bases of these symmetric powers that are “dual” to each other up to \(R^\times\)-multipliers), and for any prime \(p\) and \(M = R = F_p\) show that this map vanishes when \(i = p\).

(iii) (5 pts) Let \(0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0\) be a short exact sequence of finitely generated free \(R\)-modules, with \(M\) of rank \(n > 0\). For any integer \(0 \leq a \leq n\), let \(X_a \subset \wedge^n(M)\) be the R-submodule generated by elementary wedge products involving at least \(a\) factors from \(M'\) (and define \(X_{n+1} = 0\)). Construct a natural isomorphism \(\wedge^a(M') \otimes_R \wedge^{n-a}(M'') \simeq X_a/X_{a+1}\) for all \(0 \leq a \leq n\). State and prove a precise functorial property of this isomorphism that justifies calling it “natural”.

(4) Let \(R\) be a commutative ring.

(i) (4 pts) Prove any surjective homomorphism of \(R\)-modules \(f : R^n \rightarrow R^n\) is injective.

(ii) (4 pts) Suppose \(R\) is Noetherian and \(f : R \rightarrow R\) is a ring homomorphism. Prove again that \(f\) is injective if it is surjective. (Hint: consider the kernels of iterates of \(f\).)

(iii) (2 pts) Give a counterexample to (ii) if the Noetherian hypothesis is dropped.

(5) (i) (5 pts) Prove that the prime ideals of \(Q[2^{1/3}] \otimes_Q Q[2^{1/3}]\) are principal by exhibiting an explicit generator for each one (written as a sum of elementary tensors).

(ii) (5 pts) Do the same for \(F_7[\alpha] \otimes_{F_7} F_7[\alpha]\), where \(\alpha^3 = 2\).
6. Let $f : B \to A$ be a surjective homomorphism of $R$-algebras ($R$ a commutative ring) and let $J = \ker(f)$.

   (i) (2 pts) For any $R$-algebra $R'$, let $f' : B' \to A'$ be the $R'$-algebra homomorphism induced by applying scalar extension $R' \otimes_R \cdot$. Show that $f'$ is surjective and that $J \otimes_{R} R' \to B'$ has image $\ker(f')$.

   (ii) (3 pts) Prove that $J \otimes_{R} R' \to k'$ is injective (hence an isomorphism) if $A$ is $R$-flat, and give an example with $R = \mathbb{C}[x]$ and some $A$ that is not $R$-flat for which this injectivity fails.

   (iii) (5 pts) Suppose $R$ and $B$ are local Noetherian, and $A$ is $R$-flat. If the structure map $h : R \to B$ is local (i.e., $h(m_R) \subset m_B$) and $f \mod m_R$ is an isomorphism, prove that $f$ is an isomorphism. Give a counterexample if $A$ is not assumed to be $R$-flat.

7. Let $E$ be a field of characteristic zero.

   (i) (4 pts) Consider a prime $q$ and an element $b \in E^x$ that isn’t a $q$th power. Let $E' = E(a)$ with $a^q = b$ and $E' \neq E$. Show that $X^q - b$ is reducible over $E$ if and only if $[E' : E] < q$, and that in such cases $E'$ contains a primitive $q$th root of unity. Hint: If $1 < d = [E' : E] < q$ then apply the norm $N_{E'/E}$ to the equation $a^d = b$ to infer that $b$ has a $q$th root in $E$, and compare it to $a$.

   (ii) (3 pts) If $K/E$ is a Galois extension of prime degree $p$ and $E'/E$ is an extension such that $K$ does not admit an $E$-embedding into $E'$ then show that $KE'/E'$ is Galois of degree $p$ and the restriction map $\text{Gal}(KE'/E') \to \text{Gal}(K/E)$ is an isomorphism. (This does not use part (i).)

   (iii) (3 pts) Let $E$ be a subfield of $R$ and let $K/E$ be a finite Galois extension of odd degree $> 1$. Prove that $K$ cannot be $E$-embedded into a radical tower that is a subfield of $R$. (Hint: Use group theory to reduce to the case when $K/E$ has prime degree $p > 2$, and use (i) and (ii) to obtain a contradiction by using that $R$ does not contain nontrivial roots of unity of odd prime order.)

8. Suppose $A$ is a Noetherian ring.

   (i) (5 pts) Prove that the ring of formal power series $A[x]$ is also Noetherian.

   (ii) (5 pts) Let $I = (a_1, \ldots, a_n)$ be an ideal of $A$. Let $\hat{A}$ be the completion of $A$ for the $I$-adic topology. Prove the existence of a surjective ring homomorphism $A[[x_1, \ldots, x_n]] \to \hat{A}$, and deduce that $\hat{A}$ is Noetherian.

9. Let $G$ be a group, $V$ a nonzero finite-dimensional vector space over an algebraically closed field $k$ (allowing $\text{char}(k) > 0$), and $\rho : G \to \text{GL}(V)$ a representation. Let $E = \text{End}_G(V)$ be the algebra of endomorphisms of $V$ commuting with $\rho(g)$ for all $g \in G$. Say $V$ is decomposable if $V$ is isomorphic to a direct sum $V_1 \oplus V_2$ of two nonzero G-representations.
(i) (4 pts) Prove that \( V \) is not decomposable if and only if every element of \( E \) is (uniquely) expressible in the form \( S + N \), where \( S \) is a scalar transformation (i.e. a multiple of the identity) and \( N \) is nilpotent. (Hint: Adapt the proof of Schur’s lemma, using generalized eigenspaces.)

(ii) (3 pts) Suppose that \( V \) is not decomposable. Show that every element of \( E \) is either invertible or nilpotent, and prove that the nilpotent elements of \( E \) form a two-sided ideal \( I \) inside \( E \).

(iii) (3 pts) Prove that \( I^n = 0 \) for some \( n \geq 1 \) (i.e., \( x_1 \ldots x_n = 0 \) with any \( x_i \in I \)).

(10) Let \( G = \text{SL}_3(\mathbb{F}_p) \), where \( p \) is an odd prime. Let \( \ell \) be a prime divisor of \( p^2 + p + 1 \).

(i) (5 pts) Suppose \( \ell > 3 \). Prove that the \( \ell \)-Sylow subgroups of \( G \) are cyclic.

(ii) (5 pts) Suppose that \( \ell = 3 \). Prove that the \( \ell \)-Sylow subgroups of \( G \) are not cyclic.