MORNING ALGEBRA QUAL AUTUMN 2011

(1) Let $p$ be a prime, $G = \text{GL}_3(\mathbb{Z}/p^5\mathbb{Z})$.
   (i) Show that the natural map $G \to \text{GL}_3(\mathbb{Z}/p\mathbb{Z})$ is surjective, and compute the order of the kernel.
   (ii) Compute the size of $G$, and describe an explicit $p$-Sylow subgroup of $G$.

(2) Prove that an algebraically closed field of characteristic zero does not have an automorphism of odd prime order $p$. (Hint: Show that the $p$th roots of unity would belong to the fixed field, and use norms.)

(3) (i) Let $R$ be a commutative Noetherian integral domain. Prove that any nonzero nonunit is a finite product of irreducible elements. (Here an irreducible is a nonzero nonunit $p$ such that in any factorization $p = xy$, $x$ or $y$ is a unit.)
   (ii) Let $R$ be a commutative Noetherian ring. Prove that every ideal $I \neq R$ contains a finite product $P_1 \ldots P_k$ where $P_i$ are prime ideals satisfying $I \subset P_i$.

(4) Let $G$ be a finite group and $H$ a subgroup of index 2.
   (i) Let $V$ be an irreducible complex representation of $G$. Show that the restriction of $V$ to $H$ either remains irreducible or is the direct sum of two non-isomorphic irreducible representations.
   (ii) Assume that whenever two elements of $H$ are conjugate in $G$, they are conjugate in $H$. Prove that the restriction of every irreducible complex representation of $G$ to $H$ remains irreducible.

(5) Let $p$ be a prime.
   (i) Construct a projective resolution of $\mathbb{Z}/p\mathbb{Z}$ as a $\mathbb{Z}/p^2\mathbb{Z}$-module, and use it to compute $\text{Ext}_1^{Z/p^2Z}(\mathbb{Z}/pZ, \mathbb{Z}/pZ)$ for $i > 0$.
   (ii) Prove that $\mathbb{Z}/p^2\mathbb{Z}$ is injective as a module over itself. Compute $\text{Ext}_1^{Z/p^2Z}(\mathbb{Z}/pZ, \mathbb{Z}/pZ)$ for $i > 0$ using a suitable injective resolution.
(1) Let $M$ be a finitely generated nonzero module over a ring $R$ and let $I$ be an ideal of $R$.

(i) Let $\varphi : M \to M$ be an $R$-module endomorphism with $\varphi(M) \subset IM$. Prove there are $a_j \in I^j$ and $n \geq 1$ with the property that

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_n = 0,$$

as endomorphisms of $M$. (Hint: choose generators for $M$ and use “linear algebra.”)

(ii) Suppose that $I$ is contained in every maximal ideal of $R$. Prove Nakayama’s lemma: $IM \neq M$.

(2) Let $G = SL_2(\mathbb{F}_3)$. It is a group of order 24.

(i) Show that $G/\{\pm I\}$ is isomorphic to the alternating group $A_4$. Hint: consider the action of $G$ on one dimensional subspaces of $\mathbb{F}_3^2$ or the conjugation action of $G$ on its set of 3-Sylow subgroups.

(ii) Show that $G$ has three irreducible complex representations of degree 2. Hint: You may use without proof the fact that $G$ has 7 conjugacy classes.

(3) Let $G \subset GL(n, K)$ be a finite $p$-group, where $K$ is a field of characteristic $p$. (K is not assumed to be finite.) Show that $G$ fixes a nonzero vector in its action on $K^n$. (Hint: consider first the action of the center of $G$, and think about minimal polynomials of elements of $G$.)

(4) Let $A$ be a finitely generated integral domain over a field $k$, and $G$ a finite group acting on $A$ as a $k$-algebra.

(i) Prove that $A$ is a finitely generated module over the $k$-subalgebra $A^G$. (Hint: consider $\prod_{g \in G} (X - g(a))$ for $a \in A$.)

(ii) Prove $A^G$ is a finitely generated $k$-algebra.

(5) Suppose the polynomial $f(X) = X^4 + aX^2 + b \in \mathbb{Q}[X]$ is irreducible over the rational numbers. Note the roots of $f(X)$ have the form $\pm \alpha, \pm \beta$. Let $E$ denote a splitting field of $f(X)$, and let $G = \text{Gal}(E/\mathbb{Q})$ be the Galois group of the polynomial $f(X)$.

(i) Show that $|G| = 4$ or $8$ and if $|G| = 4$ then only the identity element of $G$ fixes a root of $f(X)$.

(ii) Show that $G$ is the direct product of two cyclic groups of order 2 if and only if $b$ is a square in $\mathbb{Q}$.

(iii) Show that $G$ is cyclic of order 4 if and only if $\frac{a^2 - 4b}{b}$ is a square in $\mathbb{Q}$.