(1) (i) Let $G$ be a finite group of order $n$, and let $\rho : G \to \text{Sym}(G)$ be the homomorphism that arises from $G$ acting on itself by left translation. Let $g \in G$ have order $m$. Prove that the sign of $\rho(g)$ is $(-1)^{n+n/m}$.

(ii) Let $G$ be a finite group of order $2k$, with $k$ odd. Prove that $G$ is a semidirect product $N \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $N$ has order $k$. Hint: Using part (i), construct a nontrivial homomorphism $G \to \mathbb{Z}/2\mathbb{Z}$.

(2) Let $G$ be a group, and $k$ a field.

(i) Define the group algebra $k[G]$, and briefly explain why a $k$-linear representation of $G$ is “the same” as a left $k[G]$-module.

(ii) Explain how to make $A \otimes_k B$ naturally into an associative $k$-algebra for any two associative $k$-algebras $A$ and $B$, and construct a natural $k$-algebra isomorphism $k[G] \otimes_k k[H] \simeq k[G \times H]$ for any two groups $G$ and $H$.

(iii) For an associative $k$-algebra $A$, define its opposite algebra $A^{\text{opp}}$ to have underlying $k$-vector space $A$ but the multiplication law $a \times a' := a'a$ (“flipped around”). Explain briefly why this is is an associative $k$-algebra, and for $A = k[G]$ prove that $A^{\text{opp}} \simeq A$. (Hint: use inversion in $G$).

(3) Suppose $P$ is a prime ideal of $\mathbb{C}[x, y]$. Show that $P = (0)$, or $P = (f)$ where $f$ is an irreducible polynomial, or $P = (x - a, y - b)$, where $a$ and $b$ are complex numbers.

$\text{Hint.}$ The following intermediate step may be useful. If $P$ contains two irreducibles $f, g$, not multiples of each other, use the Euclidean algorithm in $\mathbb{C}[(y)][x]$ to find a nonzero $h \in (f, g)$ so that $h \in \mathbb{C}[y]$.

(4) Let $f = X^3 - 2 \in \mathbb{Z}[X]$.

(i) Prove $f$ is irreducible over $\mathbb{Q}$ and that its splitting field $K/\mathbb{Q}$ has Galois group $S_3$.

(ii) For each subgroup $H$ of $S_3$, determine with proof the corresponding field $K^H$.

(iii) Prove that the splitting field of $f$ over $\mathbb{F}_5$ is quadratic and over $\mathbb{F}_7$ is cubic.

(5) Let $G = \text{GL}(2, \mathbb{F}_q)$, where $q = p^n$, $p$ prime. Let $\Pi$ be the set of one-dimensional subspaces in $V = \mathbb{F}_q^2$. Since $G$ acts on $V$ by matrix multiplication, it acts on $\Pi$.

(i) Show that if $\ell \in \Pi$ then the stabilizer of $\ell$ in $G$ contains a unique $p$-Sylow subgroup of $G$. How many $p$-Sylow subgroups does $G$ have, and what is their order?

(ii) Prove that if $\ell_1, \ell_2$ and $\ell_3$ are three distinct one-dimensional subspaces of $V$, then there is an element $g$ of $G$ such that $g\ell_1 = \mathbb{F}_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $g\ell_2 = \mathbb{F}_q \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g\ell_3 = \mathbb{F}_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. 

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(c) Show that if $P_1, P_2$ and $P_3$ are three distinct $p$-Sylow subgroups of $G$, and if $Q_1, Q_2$ and $Q_3$ are another three distinct $p$-Sylow subgroups of $G$, then there exists a $g \in G$ such that
\[ gP_1g^{-1} = Q_1, \quad gP_2g^{-1} = Q_2, \quad gP_3g^{-1} = Q_3. \]
(1) Let $G$ be a subgroup of a finite $p$-group $H$ ($p$ a prime) such that the natural homomorphism $G \to H/[H,H]$ is surjective. Prove that $G = H$ by induction on $|H|$ as follows:

(i) Suppose $N$ is any nontrivial normal subgroup of $H$; show (using the inductive assumption) that $G \cdot N = H$.

(ii) Let $Z$ be the center of $H$. Using (i) we have $G \cdot Z = H$; explain why $G \cap Z$ cannot be trivial. Now set $N = G \cap Z$ in (i).

(2) Let $G$ be a finite group and $K \subset L$ an extension of fields of any characteristic. For a $K$-vector space $W$, let $W_L$ denote $L \otimes_K W$. Let $V, V'$ be $n$-dimensional $K$-linear representations of $G$ (with $n \geq 1$).

(i) Prove that there is a nonzero $K[G]$-linear map $V' \to V$ if there is a nonzero $L[G]$-linear map $V'_L \to V_L$.

(ii) Prove (for any $N \geq 1$) that a nonzero polynomial over $K$ in $N$ variables cannot vanish on $K^N$ if $K$ is infinite, and deduce that if $K$ is infinite then

$$V'_L \cong V_L \text{ as } L[G]\text{-modules} \implies V' \cong V \text{ as } K[G]\text{-modules}.$$ 

(3) Let $K$ be a field and $L, L'$ two finite extensions of $K$.

(i) Prove that if $L/K$ is separable then $L \otimes_K L'$ is isomorphic to a product $\prod_{i=1}^r L'_i$ of finitely many fields. (Hint: use the primitive element theorem to write $L = K[t]/(f)$ for a suitable monic $f$.)

(ii) Give an example to show this need not be true without the separability assumption.

(4) Let $A$ be a commutative ring. We say an $A$-module $M$ is finitely presented if it is isomorphic to the cokernel of an $A$-module homomorphism $A^{\oplus m} \to A^{\oplus n}$ for some $m, n \geq 1$.

(i) Let $B$ be a flat $A$-algebra, $M$ a finitely presented $A$-module, and $N$ any $A$-module. Prove the natural map $\text{Hom}_A(M, N) \otimes_A B \to \text{Hom}_B(M \otimes_A B, N \otimes_A B)$ is an isomorphism.

(ii) Suppose

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of $A$-modules. Suppose that for each maximal ideal $m \subset A$, the localized sequence

$$0 \to M'_m \to M_m \to M''_m \to 0$$

is split. If $M''$ is finitely presented, show that (*) is split. 

*Hint:* Prove (*) splits if and only if the map $\text{Hom}_A(M'', M) \to \text{Hom}_A(M'', M'')$ induced by the right map $M \to M''$ of (*) is surjective.
(5) Let $F$ be a field, and let $V$ be the 4-dimensional vector space $F^4$ with the skew-symmetric bilinear form

$$\langle x, y \rangle = x_1 y_3 + x_2 y_4 - x_3 y_1 - x_4 y_2.$$ 

A subspace $U$ of $V$ is isotropic if $\langle u_1, u_2 \rangle = 0$ for $u_1, u_2 \in U$. For example, the two-dimensional subspace $U_0$ of $x = (x_1, x_2, x_3, x_4)$ with $x_3 = x_4 = 0$ is isotropic. Let $G$ be the group of $g \in \text{GL}_4(F)$ such that $\langle gx, gy \rangle = \langle x, y \rangle$ for all $x, y \in V$.

(a) Prove that if $U$ is a two-dimensional isotropic subspace then there exists $g \in G$ such that $gU = U_0$. *Hint:* let $u_1, u_2$ be any basis of $U$. Show that there exist $v_1$ and $v_2$ such that $\langle v_1, v_2 \rangle = 0$ and $\langle u_i, v_j \rangle = \delta_{ij}$.

(b) Suppose that $F = \mathbb{F}_q$. Show that the number of two dimensional isotropic subspaces of $V$ is $(q^2 + 1)(q + 1)$. *Hint:* First count the number of pairs $(u_1, u_2)$ so that $u_1, u_2$ are linearly independent and $\langle u_1, u_2 \rangle = 0$. 