ALGEBRA QUALIFYING EXAM, FALL 2009, PART I

1. Let k be a finite field of size q.

   (a) Prove that the number of \(2 \times 2\) matrices over k satisfying \(T^2 = 0\) is \(q^2\).

   (b) Prove that the number of \(3 \times 3\) matrices over k satisfying \(T^3 = 0\) is \(q^6\).

2. (a) Prove that if \(K\) is a field of finite degree over \(\mathbb{Q}\) and \(x_1, \ldots, x_n\) are finitely many elements of \(K\) then the subring \(\mathbb{Z}[x_1, \ldots, x_n]\) they generate over \(\mathbb{Z}\) is not equal to \(K\). (Hint: Show they all lie in \(O_K[1/a]\) for a suitable nonzero \(a\) in \(O_K\), where \(O_K\) denotes the integral closure of \(\mathbb{Z}\) in \(K\).)

   (b) Let \(m\) be a maximal ideal of \(\mathbb{Z}[x_1, \ldots, x_n]\) and \(F = \mathbb{Z}[x_1, \ldots, x_n]/m\). Use (a) and the Nullstellensatz to show that \(F\) cannot have characteristic 0, and then deduce for \(p = \text{char}(F)\) that \(F\) is of finite degree over \(\mathbb{F}_p\) (so \(F\) is actually finite).

3. Let \(E\) be the splitting field of

   \[f(x) = \frac{(x^7 - 1)}{(x - 1)} = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\]

   over \(\mathbb{Q}\). Let \(\zeta\) be a zero of \(f(x)\), i.e. a primitive seventh root of 1.

   (a) Show that \(f(x)\) is irreducible over \(\mathbb{Q}\). (Hint: consider \(f(y + 1)\) and use Eisenstein’s criterion.)

   (b) Show that the Galois group of \(E/\mathbb{Q}\) is cyclic, and find an explicit generator.

   (c) Let \(\beta = \zeta + \zeta^2 + \zeta^4\). Show that the intermediate field \(\mathbb{Q}(\beta)\) is actually \(\mathbb{Q}(\sqrt{-7})\).

   (Hint: first show that \([\mathbb{Q}(\beta) : \mathbb{Q}] = 2\) by finding a linear dependence over \(\mathbb{Q}\) among \(\{1, \beta, \beta^2\}\).)

   (d) Let \(\gamma_q = \zeta + \zeta^q\). Find (with proof) a \(q\) such that \(\mathbb{Q}(\gamma_q)\) is a degree 3 extension of \(\mathbb{Q}\).

   (Hint: use (b).) Is this extension Galois?

4. Let \(G\) be a nontrivial finite group and \(p\) be the smallest prime dividing the order of \(G\). Let \(H\) be a subgroup of index \(p\). Show that \(H\) is normal. (Hint: If \(H\) isn’t normal, consider the action of \(G\) on the conjugates of \(H\).)

5. Let \(G\) be a finite group and \(\pi : G \to \text{GL}(V)\) a finite-dimensional complex representation. Let \(\chi\) be the character of \(\pi\). Show that the characters of the representations on \(V \otimes V\), \(\text{Sym}^2(V)\) and \(\wedge^2(V)\) are \(\chi(g)^2\), \((\chi(g))^2 + \chi(g^2))/2\) and \((\chi(g)^2 - \chi(g^2))/2\). (Hint: Express \(\chi(g)^2\), \((\chi(g))^2 + \chi(g^2))/2\) and \((\chi(g)^2 - \chi(g^2))/2\) in terms of the eigenvalues of \(\pi(g)\).

Date: Thursday, September 17, 2009.
1. Let $V$ be a vector space over a field $F$, and let $B : V \times V \to F$ be a symmetric bilinear form. This means that $B$ is bilinear and $B(x, y) = B(y, x)$. Let $q(v) = B(v, v)$.

(a) Show that if the characteristic of $F$ is not 2 then $B(v, w) = \frac{1}{2}(q(v + w) - q(v) - q(w))$. (This obviously implies that if $q = 0$ then $B = 0$.)

(b) Give an example where the characteristic of $F$ is 2 and $q = 0$ but $B \neq 0$.

(c) Show that if the characteristic of $F$ is not 2 or 3 and if $B(u, v, w)$ is a symmetric trilinear form, and if $r(v) = B(v, v, v)$, then $r = 0$ implies $B = 0$.

2. Let $G$ be a finite group.

(a) Let $\pi : G \to GL(V)$ be an irreducible complex representation, and let $\chi$ be its character. If $g \in G$, show that $|\chi(g)| = \dim(V)$ if and only if there is a scalar $c \in \mathbb{C}$ such that $\pi(g)v = cv$ for all $v \in V$.

(b) Show that $g$ is in the center $Z(G)$ if and only if $|\chi(g)| = \chi(1)$ for every irreducible character $\chi$ of $G$.

3. Let $V$ be a vector space of finite dimension $d \geq 1$ over a field $k$ of arbitrary characteristic. Let $V^*$ denote the dual space.

(a) For any $n \geq 1$, prove that there is a unique bilinear pairing $V^\otimes n \times (V^*)^\otimes n \to k$ satisfying

$$ (v_1 \otimes \cdots \otimes v_n, \ell_1 \otimes \cdots \otimes \ell_n) \mapsto \prod \ell_i(v_i), $$

and by using bases show that it is a perfect pairing (i.e., identifies $(V^*)^\otimes n$ with $(V^\otimes n)^*$).

(b) For any $1 \leq n \leq d$, do similarly with $\wedge^n(V)$ and $\wedge^n(V^*)$ using the requirement

$$ (v_1 \wedge \cdots \wedge v_n, \ell_1 \wedge \cdots \wedge \ell_n) \mapsto \det(\ell_i(v_j)). $$

4. Let $K/k$ be a finite extension of fields with $\alpha \in K$ as a primitive element over $k$. Let $f \in k[x]$ be the minimal polynomial of $\alpha$ over $k$.

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(a) Explain why $K \cong k[x]/(f)$ as $k$-algebras, and use this to relate the local factor rings of $K \otimes_k F$ to the irreducible factors of $f$ in $F[x]$, with $F/k$ a field extension.

(b) Assume $K/k$ is Galois with Galois group $G$. Prove that the natural map $K \otimes_k K \to \prod_{g \in G} K$ defined by $a \otimes b \mapsto (g(a)b)$ is an isomorphism.

5. Let $G$ be a finite abelian group, $\omega : G \times G \to \mathbb{R}/\mathbb{Z}$ a bilinear mapping so that

(i) $\omega(g, g) = 0$ for all $g$ in $G$;
(ii) $\omega(x, g) = 0$ for all $g$ if and only if $x$ is the identity element.

Prove that the order of $G$ is a square. Give an example of $G$ of square order for which no such $\omega$ exists.

Hint: Consider a subgroup $A$ of $G$ which is maximal for the property that $\omega(x, y) = 0$ for $x, y$ in $A$. You may use the following fact without proof: any finite abelian group $X$ admits $|X|$ distinct homomorphisms to $\mathbb{R}/\mathbb{Z}$. 