# SCATTERING THEORY

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*This thesis is dedicated to my advisor, Andras Vasy.*

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## INTRODUCTION

Scattering theory is a subset of spectral theory that seeks to parametrize the absolutely continuous spectrum of an elliptic operator $H$ on a complete manifold $M$ with uniform structure at infinity. In the simplest case where $M$ is compact, the spectrum of $H$ is discrete, and scattering theory is trivial. On $\mathbb{R}^n$, scattering theory is already non-trivial, and the techniques used in studying scattering on $\mathbb{R}^n$ can be generalized to non-Euclidean manifolds. In particular, perturbation theory is used to observe the behavior of a full Hamiltonian $H$ relative to a free Hamiltonian $H_0$.

The main two problems that scattering theory is concerned with are investigating the asymptotics as $t \to \pm\infty$ of the time-dependent Schrodinger equation

$$i\frac{\partial u}{\partial t} = Hu, \tag{0.1}$$

with initial data $u(0) = f$, and finding conditions for the unitary equivalence of the absolutely continuous operators $H^{(a)}$ and $H_0^{(a)}$. With regard to the first problem, the eigenfunctions of the free Hamiltonian are simply plane waves in the Euclidean setting, and the eigenfunctions of the full Hamiltonian are simply perturbed plane waves. In general, in the setting where $H$ and $H_0$ are self-adjoint operators in arbitrary Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_0$ respectively, with an identification operator $\mathcal{J} : \mathcal{H}_0 \to \mathcal{H}$, the Schrodinger equation has a unique solution

$$u(t) = \exp(-iHt)f \tag{0.2}$$

for the full Hamiltonian, and a similar solution

$$u_0(t) = \exp(-iH_0t)f_0 \tag{0.3}$$

for the free Hamiltonian. The solution $u(t)$ is said to have free asymptotics as $t \to \pm\infty$ if for suitable initial data $f_0^{\pm}$ in the basis of $f$,

$$\lim_{t\to\pm\infty} \|u(t) - u_0^{\pm}(t)\| = 0, \tag{0.4}$$

where

$$u_0^{\pm}(t) = \exp(-iH_0t)f_0^{\pm}. \tag{0.5}$$

This then suggests a correspondence between $f$ and the initial data $f_0^{\pm}$

$$f = \lim_{t\to\infty} \exp(iHt)\exp(-iH_0t)f_0^{\pm}. \tag{0.6}$$

Therefore, a solution $u(t)$ having free asymptotics as $t \to \pm\infty$ is equivalent to the existence of initial data $f_0^{\pm}$ such that equation 0.6 holds, and the operator $W_{\pm}(H, H_0; \mathcal{J}) : f_0^{\pm} \mapsto f$ is known as the wave operator.

We now consider the validity of criterion 0.4 for free asymptotics. In the simple case where $f$ is an eigenfunction of $H$, $Hf = \lambda f$, and $u(t) = \exp(-i\lambda t)f$, which has trivial time dependence. In general, however, eigenvalues are displaced under arbitrarily weak perturbations, so one would not expect the free Hamiltonian to have solutions with the same asymptotic behavior as $t \to \pm\infty$. As we shall soon see, though, a basic result of scattering theory is that $u(t)$ has free asymptotics under general assumptions about $H$ and $H_0$. If the wave operator $W_{\pm}$ exists, then equation 0.4 is equivalent to $f$ being in the range of $W_{\pm}$, and the wave operator is said to be complete if equation 0.4 is valid for any absolutely continuous $f$.

With regard to the second problem, if the wave operator $W_{\pm}$ exists, it is an isometry on $H_0^{(a)}$ and possesses the intertwining property $HW_{\pm} = W_{\pm}H_0$. Therefore, $H^{(a)}$ has a part that is unitarily equivalent to $H_0^{(a)}$, and if the wave operator is complete, then $H^{(a)}$ and $H_0^{(a)}$ are unitarily equivalent.

Another important operator in scattering theory is the scattering operator $S : f_0^- \mapsto f_0^+$ that relates the asymptotics of solutions to the Schrodinger equation as $t \to -\infty$ with the asymptotics as $t \to \infty$, relative to the free Hamiltonian. If the wave operator $W_{\pm}$ exists, the scattering operator commutes with $H_0$, and if the wave operator is complete, then the scattering operator is unitary on $\mathcal{H}_0^{(a)}$. In a representation of $\mathcal{H}_0$ in which $H_0$ is diagonal, $H_0$ acts by multiplication by $\lambda$, and $S$ acts by multiplication by an operator-valued function $S(\lambda)$ known as the scattering matrix.

A simplified approach to scattering theory is the stationary method, which ignores the time-dependence of the Schrodinger equation. In doing so, the unitary groups in the definition of the wave operator are replaced by their corresponding resolvent operators $R_z(H) = (zI - H)^{-1}$ and $R_z(H_0) = (zI - H_0)^{-1}$. Instead of studying the asymptotics as $t \to \pm\infty$, the stationary approach studies the limits of the resolvents as the spectral parameter approaches the real axis. Under this approach, the spectral resolution is obtained as a different of limits of the resolvent family.

## 1. Elementary functional analysis

Before we begin our discussion on scattering theory, we first need some background in functional analysis, which will be provided in this chapter.

### 1.1. **Bounded operator theory.** 

We begin our discussion on functional analysis with the theory of bounded linear operators.

**Definition 1.1.** A linear operator $T : V \to W$ between two Banach spaces is bounded if there exists a constant $C$ such that

$$\|Tx\| \leq C\|x\| \tag{1.1}$$

for all $x \in V$.

**Lemma 1.2.** *A linear operator $T$ has the nice property that it is bounded iff it is continuous.*

*Proof.* If $T$ is bounded, then by equation 1.1 it is Lipschitz continuous. If $T$ is continuous, then boundedness follows by a simple epsilon-delta argument. There exists a $\delta > 0$ such that for all $x \in V$ with $\|x\| < \delta$, $\|Tx\| < 1$. Then for any $x \in V$,

$$\|Tx\| = \Big\| \frac{2\|x\|}{\delta} T\Big(\frac{\delta}{2}\frac{x}{\|x\|}\Big)\Big\| = \frac{2\|x\|}{\delta}\Big\|T\Big(\frac{\delta}{2}\frac{x}{\|x\|}\Big)\Big\| < \frac{2}{\delta}\|x\|. \tag{1.2}$$

□

**Lemma 1.3.** *Furthermore, the space $\mathcal{L}(V,W)$ of all bounded linear operators from $V$ to $W$ is itself a Banach space under the norm*

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}. \tag{1.3}$$

*Proof.* Given a Cauchy sequence $\{T_n\} \subset \mathcal{L}(V,W)$, for any $x \in V$ $\{T_n(x)\} \subset W$ is also a Cauchy sequence. Since $W$ is Banach, this sequence has a limit $\alpha_x$ in $W$. Define $T : V \to W$ by $T(x) = \alpha_x$, and it can easily be verified that $T$ defines a bounded linear map. Therefore, $T_n \to T$ in the operator norm on $\mathcal{L}(V,W)$. □

**Corollary 1.4.** The dual space $V^*$ of a Banach space $V$ is also a Banach space.

**Definition 1.5.** For $T \in \mathcal{L}(V,W)$, the kernel of $T$ ker$T$ is the set of vectors $x \in V$ such that $Tx = 0$, and the range of $T$ ran$T$ is the set of vectors $y \in W$ such that $y = Tx$ for some $x \in V$.

Notice that ker$T$ and ran$T$ are both subspaces of $W$.

**Lemma 1.6.** *kerT is a closed subspace of $W$, but ranT is not necessarily closed.*

*Proof.* ker$T$ is closed by continuity of $T$. To show that ran$T$ is not necessarily closed, we consider a linear operator $T : l^\infty \to l^\infty$. Let $x = \{\xi_n\}_{n=1}^\infty$ denote an element in $l^\infty$, and define a linear operator $T : l^\infty \to l^\infty$ by

$$T(\{\xi_n\}_{n=1}^\infty) = \Big\{\frac{\xi_n}{n}\Big\}_{n=1}^\infty. \tag{1.4}$$

$T$ is clearly bounded with norm 1. Define $x_n = T(\{\sqrt{1}, \sqrt{2}, ..., \sqrt{n}, 0, 0, ...\}) = \Big\{\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, ..., \frac{1}{\sqrt{n}}, 0, 0, ...\Big\}$, and $x_n$ clearly converges to $x = \{n^{-\frac{1}{2}}\}_{n=1}^\infty \in l^\infty$. However, $x \notin$ ran$T$ since its preimage would be $\{n^{\frac{1}{2}}\}_{n=1}^\infty \notin l^\infty$. □

We now turn our attention to adjoint operators. We are mostly interested in bounded linear transformations $T : \mathcal{H} \to \mathcal{H}$ of a Hilbert space $\mathcal{H}$ to itself.

**Definition 1.7.** For a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, is adjoint is the unique bounded linear operator $T^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \tag{1.5}$$

for all $x, y \in \mathcal{H}$.

**Lemma 1.8.** *The adjoint operator has several nice properties:*

(1) $(T^*)^* = T$
(2) $(ST)^* = T^*S^*$
(3) *If $T$ has a bounded inverse, then $T^*$ has a bounded inverse given by $(T^*)^{-1} = (T^{-1})^*$*
(4) $\|T^*T\| = \|T\|^2$

*Proof.* The first three properties follow easily from equation 1.5. To prove (4), we have

$$\|T^*T\| \leq \|T\|\|T^*\| = \|T\|^2 \tag{1.6}$$

and

$$\|T^*T\| \geq \sup_{\|x\|=1} \langle x, T^*Tx \rangle = \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2. \tag{1.7}$$

□

Two types of operators that are of particular interest in mathematical physics are self-adjoint and projection operators.

**Definition 1.9.** A self-adjoint operator $T$ is a bounded linear operator on a Hilbert space such that $T^* = T$.

**Definition 1.10.** A projection operator $P$ is a bounded linear operator on a Hilbert space such that $P^2 = P$. In addition, if $P^* = P$, then $P$ is an orthogonal projection.

With these tools in hand, we now begin our discussion on spectral theory.

1.2. **Spectral theory.**

**Definition 1.11.** For a bounded linear operator $T \in \mathcal{L}(\mathcal{H})$, $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(T)$ of $T$ if $\lambda I - T$ is a bijection with a bounded inverse $R_\lambda(T) = (\lambda I - T)^{-1}$, which we call the resolvent of $T$ at $\lambda$. The spectrum $\sigma(T)$ of $T$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda \notin \rho(T)$.

If $T \in \mathcal{L}(\mathbb{C}^n)$ is a linear transformation on a finite dimensional space, then the spectrum of $T$ is simply the set of eigenvalues of $T$, which contains at most $n$ points since $\det(\lambda I - T)$ is a polynomial of degree $n$. For $\lambda \in \mathbb{C}$ not an eigenvalue of $T$, $\lambda I - T$ is invertible since $\det(\lambda I - T) \neq 0$. By the bounded inverse theorem, any bijective bounded linear operator on a Banach space has a bounded inverse so $(\lambda I - T)^{-1}$ is automatically bounded. Over infinite dimensional spaces, spectral theory is more interesting, an we can decompose the spectrum of an operator into its point spectrum and its residual spectrum.

**Definition 1.12.** The set of eigenvalues of a bounded linear operator $T \in \mathcal{L}(\mathcal{H})$ is its point spectrum. If $\lambda \in \mathbb{C}$ is not an eigenvalue and $\operatorname{ran}(\lambda I - T)$ is not dense in $\mathcal{H}$, then $\lambda$ is in the residual spectrum of $T$.

**Lemma 1.13.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator over a Hilbert space. The first resolvent identity states that for any $\lambda, \mu \in \rho(T)$,*

$$R_\lambda(T) - R_\mu(T) = -(\lambda - \mu) R_\lambda(T) R_\mu(T). \tag{1.8}$$

*In particular, $R_\lambda$ and $R_\mu$ commute.*

*Proof.*

$$R_\lambda(T) - R_\mu(T) = R_\lambda(T)((\mu I - T) - (\lambda I - T)) R_\mu(T) \tag{1.9}$$

□

We now examine properties of the resolvent map. The resolvent map is not just differentiable, but it is even analytic, a property that will be valuable in our later analysis.

**Lemma 1.14.** *The resolvent set $\rho(T)$ of an operator $T$ is open, and furthermore the resolvent map is analytic on the resolvent set.*[1]

*Proof.* Let $\lambda, \mu \in \rho(T)$ and examine the formal expression

$$\frac{1}{\lambda - T} = \frac{1}{\mu - T}\frac{1}{1 - \frac{\mu-\lambda}{\mu-T}}. \tag{1.10}$$

This suggests that we consider

$$R_\lambda(T) = R_\mu(T)\sum_{n\in\mathbb{N}}((\mu-\lambda)R_\mu(T))^n. \tag{1.11}$$

This series converges in the operator norm iff

$$\|\mu - \lambda\| < \frac{1}{\|R_\mu(T)\|} \tag{1.12}$$

Such $R_\lambda(T)$ is well defined, and it can easily be verified that this expression is indeed the resolvent of $T$. Therefore, $\rho(T)$ is open, and the resolvent map is analytic on $\rho(T)$ with a power series given by equation 1.11. □

**Lemma 1.15.** *The spectrum $\sigma(T)$ of an operator $T$ is not empty.*[2]

*Proof.* Consider a different formal expression for the resolvent $R_\lambda(T)$.

$$\frac{1}{\lambda - T} = \frac{1}{\lambda}\frac{1}{1 - \frac{T}{\lambda}} \tag{1.13}$$

suggests that we consider

$$R_\lambda(T) = \frac{1}{\lambda}\sum_{n\in\mathbb{N}}\left(\frac{T}{\lambda}\right)^n. \tag{1.14}$$

This series is called the Neumann series for $R_\lambda(T)$, and it converges iff $|\lambda| > \|T\|$. Since $\|R_\lambda(T)\| \to 0$ as $|\lambda| \to \infty$, if $\sigma(T)$ were empty, then $R_\lambda(T)$ would be a bounded entire function. By Liouville's theorem, $R_\lambda(T) = 0$, which is impossible. Therefore, $\sigma(T)$ is not empty. □

To turn our attention towards the spectral theorem, we begin by constructing a spectral measure that underlies its statement. An important result from functional analysis that we will need and that we quote here, without proof, is the Riesz-Markov-Kakutani representation theorem.

**Theorem 1.16.** *Let $X$ be a locally compact Hausdorff space. Any positive linear functional on $C_C(X)$ is of the form $l(f) = \int f d\mu$ for a unique regular Borel measure $\mu$.*

Using this result, we are ready to define a spectral measure. Let $A \in \mathcal{L}(\mathcal{H})$ be a bounded self-adjoint operator, and let $\psi \in \mathcal{H}$. Since $l(f) = \langle\psi, f(A)\psi\rangle$ is a positive linear functional on $C^0(\sigma(A))$, there is a unique measure $\mu_\psi$ on $\sigma(A)$ such that $\langle\psi, f(A)\psi\rangle = \int_{\sigma(A)} f(\lambda)d\mu_\psi$.

**Definition 1.17.** This measure $\mu_\psi$ is the spectral measure associated with the vector $\psi$.

[1] This proof follows the discussion of Theorem VI.5 in Reed and Simon.

[2] This follows from the discussion of a corollary to Theorem VI.5 in Reed and Simon.

To understand explicitly what spectral measures are, we first need a little bit of background on Borel functional calculus.

**Definition 1.18.** Let $p(x) = \sum_{n=0}^{N} a_n x^n$. Then for a bounded self-adjoint operator $A$, $p(A) = \sum_{n=0}^{N} a_n A^n$.

**Lemma 1.19.** *Let $p$ be a polynomial and $A$ be a bounded self-adjoint operator. Then $\sigma(p(A)) = \{p(\lambda) | \lambda \in \sigma(A)\}$.*

*Proof.* Let $\lambda \in \sigma(A)$. Since $\lambda$ is a root of $p(x) - p(\lambda)$, $p(x) - p(\lambda) = (x - \lambda)q(\lambda)$ for some polynomial $q$, so $p(A) - p(\lambda) = (A - \lambda)q(A)$. Since $A - \lambda$ is not invertible, neither is $p(A) - p(\lambda)$, so $p(\lambda) \in \sigma(p(A))$.

Conversely, let $\lambda \in \sigma(p(A))$, and let $p(x) - \lambda = a \prod_{n=1}^{N} x - \lambda_n$. If $\lambda_n \notin \sigma(A)$ for all $n$, then $p(A) - \lambda$ is invertible with inverse $(p(A) - \lambda)^{-1} = a^{-1} \prod_{n=1}^{N} (A - \lambda_n)^{-1}$, which is a contradiction. Therefore, for some $n$, $\lambda_n \in \sigma(A)$, so $\lambda = p(\lambda_n)$. ☐

**Lemma 1.20.** *For a polynomial $p$ and a bounded self-adjoint operator $A$, $\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|$.*

*Proof.* $\|p(A)\|^2 = \|p(A)^* p(A)\| = \|\bar{p}p(A)\| = \sup_{\lambda \in \sigma(\bar{p}p(A))} |\lambda| = \sup_{\lambda \in \sigma(A)} |\bar{p}p(\lambda)| = \left(\sup_{\lambda \in \sigma(A)} |p(\lambda)|\right)^2$. ☐

Using these two lemmas, we can define a Borel functional calculus. Since polynomials are dense in $C_C(\mathbb{C})$, polynomials are also dense in $C^0(\sigma(A))$, allowing us to construct a functional calculus for any Borel function $f$ and any bounded self-adjoint operator $A$ by approximating $f$ by polynomials. In particular, we are interested in the case $f = \chi_\Omega$, the characteristic function of a Borel set $\Omega$.

**Definition 1.21.** A spectral measure $E$ on a measurable space $(X, \Omega)$ is a projection-valued function $E : \Omega \to \{P_\Omega\}$ satisfying

(1) $E(X) = 1$
(2) For any countable collection of pairwise disjoint sets $\{A_n\}_{n=1}^{\infty}$

$$E\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} E(A_n). \tag{1.15}$$

For a projection-valued measure $P_\Omega$, $\langle \phi, P_\Omega \phi \rangle$ is an ordinary measure for any $phi \in \mathcal{H}$. Let $d\langle \phi, P_\lambda \phi \rangle$ denote integration with respect to this measure. Then by the Riesz-Markov Kakutani representation theorem, there is a unique operator $B$ such that $\langle \phi, B\phi \rangle = \int f(\lambda) d\langle \phi, P_\lambda \phi \rangle$. This is the celebrated Spectral Theorem.

## 2. Spectral Theory on Compact Manifolds

We begin by considering the simplest example of scattering, which occurs on compact manifolds. Let us explicitly consider scattering on $\mathbb{S}^1$ for concreteness. Define a Laplacian $\Delta \in \mathcal{L}(H^2(\mathbb{S}^1), L^2(\mathbb{S}^1))$ on $\mathbb{S}^1$ as $\Delta = -\frac{d^2}{dx^2}$, and we begin with looking at properties of the Laplacian.

**Lemma 2.1.** *For $\lambda \in \mathbb{C}$, $\lambda \neq n^2$, $n \in \mathbb{Z}$, $\lambda I - \Delta : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bijection with resolvent $R_\lambda(\Delta) = (\lambda I - \Delta)^{-1} \in \mathcal{L}(L^2(\mathbb{S}^1), H^2(\mathbb{S}^1))$. Furthermore, for $\lambda < 0$, $\|R_\lambda(\Delta)\|_{\mathcal{L}(L^2, L^2)} \leq |\lambda|^{-1}$.*

*Proof.* For any $f \in H^2(\mathbb{S}^1)$, let $g = (\lambda I - \Delta)f$. Taking a discrete Fourier transform gives $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and $g(x) = \sum_{n=-\infty}^{\infty} (\lambda - n^2) c_n e^{inx}$. Clearly, any $g \in L^2$ can be represented in this way for a suitable choice of $f$, so $\lambda I - \Delta$ is a surjection. Since $\lambda - n^2 \neq 0$ for any $n \in \mathbb{Z}$, the coefficients $c_n$ can be recovered from $g$, so $\lambda I - \Delta$ is an injection, and therefore a bijection. Since the Fourier transform preserves norms,

$$\frac{\|f\|_{H^2}^2}{\|g\|_{L^2}^2} = \frac{\sum_{n=-\infty}^{\infty} c_n^2}{\sum_{n=-\infty}^{\infty} (\lambda - n^2)^2 c_n^2} \leq \frac{\sum_{n=-\infty}^{\infty} c_n^2}{\sum_{n=-\infty}^{\infty} \lambda^2 c_n^2} = \lambda^{-2}. \tag{2.1}$$

Therefore, $\|R_\lambda(\Delta)\|_{\mathcal{L}(L^2,L^2)} \leq |\lambda|^{-1}$. □

**Lemma 2.2.** *The inverse of the Laplacian is a compact operator.*

*Proof.* This proof relies on the fact that $\mathbb{S}^1$ is bounded. Let $f_n \in L^2(\mathbb{S}^1)$ be a bounded sequence. Then by application of the Cauchy-Schwarz and Poincare inequalities, $\Delta^{-1}(f_n)$ is bounded in $H^2(\mathbb{S}^1)$. By the Rellich-Kondrachov compactness theorem, $\Delta^{-1}(f_n)$ is relatively compact in $L^2(\mathbb{S}^1)$, so $\Delta^{-1}$ is a compact operator. □

**Lemma 2.3.** *The resolvent operator is a compact operator.*

*Proof.* By the first resolvent identity,

$$R_\lambda = (R_\lambda - R_0) + R_0 = (\lambda - 0) R_\lambda R_0 + R_0 = (\lambda R_\lambda + 1) \circ R_0 \tag{2.2}$$

is the composition of a continuous operator with a compact operator, so it is compact. □

We now introduce a potential $V \in \mathcal{L}(L^2(\mathbb{S}^1))$ and examine the full Hamiltonian $H = \Delta + V \in \mathcal{L}(H^2(\mathbb{S}^1), L^2(\mathbb{S}^1))$.

**Lemma 2.4.** *Take $V \in \mathcal{L}(L^2(\mathbb{S}^1))$, which in a simple case could be $V \in C^0(\mathbb{S}^1)$ considered as a multiplication operator, and define $H = \Delta + V \in \mathcal{L}(H^2(\mathbb{S}^1), L^2(\mathbb{S}^1))$. Fix $\mu > \|V\|_{\mathcal{L}(L^2)}$, and define $K_\lambda = (V - \lambda - \mu) \circ R_{-\mu}(\Delta)$. Then $K_\lambda \in \mathcal{L}(L^2(\mathbb{S}^1))$ is compact, and $(I - K_\lambda)^{-1}$ exists for $\lambda \notin D$, with $D \subset \mathbb{C}$ discrete.*

*Proof.* Since $R_{-\lambda}(\Delta)$ and $V - \lambda - \mu$ are both bounded linear operators and furthermore $R_{-\lambda}(\Delta)$ is compact, their composition $K_\lambda$ is also compact. To show that $I - K_\lambda$ is invertible, take $\lambda = -\mu$, so that $K_\lambda = V \circ R_{-\mu}(\Delta)$. Since the eigenvalues of the Laplacian are $\lambda_n = n^2$, the resolvent acts on the eigenstates of the Laplacian by division by $n^2 + \mu$. In particular, $\|R_{-\mu}(\Delta)\| \leq \mu^{-1}$. Therefore, $\|K_\lambda\| \leq \|V\| \cdot \|R_{-\mu}(\Delta)\| < 1$, so $K_\lambda$ is invertible at $\lambda = -\mu$. By analytic Fredholm theory, since $K_\lambda$ is invertible at one value of $\lambda$, it is invertible for all $\lambda$. □

**Lemma 2.5.** *For $\lambda \notin D$, $\lambda I - H : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bijection, and $R_\lambda(H) = R_{-\mu}(\Delta)(I - K_\lambda)^{-1} \in \mathcal{L}(L^2(\mathbb{S}^1), H^2(\mathbb{S}^1))$. Therefore, the spectrum of $H$ is discrete.*

*Proof.*

$$I - K_\lambda = (-\Delta - V + \lambda) \circ R_{-\mu}(\Delta) = (\lambda I - H) \circ R_{-\mu}(\Delta) \tag{2.3}$$

is invertible for $\lambda \notin D$, and $R_{-\mu}$ is invertible, so $\lambda I - H$ must be invertible, and it is therefore injective. For any $f \in H^2(\mathbb{S}^1)$, let $g = (\lambda I - H)f$. Taking a discrete Fourier transform gives $g(x) = \sum_{n=-\infty}^{\infty} ((\lambda - n^2)c_n - d_n)e^{inx}$, where $c_n$ and $d_n$ are the Fourier coefficients of $f$ and $V$, respectively. Any $g \in L^2$ can be represented in this way for a suitable choice of $f$, so $\lambda I - H$ is surjective, and therefore a bijection. Rearranging equation 2.3 yields

$$I = (\lambda I - H) \circ R_{-\mu}(\Delta) \circ (I - K_\lambda)^{-1}, \tag{2.4}$$

which is the desired formula for $R_\lambda(H)$. Therefore, $\sigma(H) \subset D$, and in particular, $\sigma(H)$ is discrete. □

This is all that there is to say in the simple case of scattering on compact manifolds, so we now turn our attention to non-compact manifolds, beginning with the simplest example of $\mathbb{R}^n$.

## 3. Scattering Theory on $\mathbb{R}^n$

To generalize the Laplacian to Euclidean space, we define it as $\Delta = \sum_{i=1}^n D_i^2$, where $D_i = \frac{1}{i}\frac{\partial}{\partial x_i}$, and $x_i$ are the canonical coordinates on $\mathbb{R}^n$. Let us consider the Laplacian as acting on the class of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ to avoid issues with convergence.

3.1. **Spectral resolution of the Laplacian.** To begin our discussion of scattering on $\mathbb{R}^n$, we deduce the spectral resolution of the Laplacian. Let us introduce polar coordinates by writing $\zeta \in \mathbb{R}^n$ as $\zeta = \lambda\omega$, where $\lambda = |\zeta|$ and $\omega \in \mathbb{S}^{n-1}$. Until section 3.5, $\lambda$ will always be positive whenever it is used. Using the Fourier transform, an arbitrary Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$ can be decomposed as

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\zeta} \hat{f}(\zeta) d\zeta = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\lambda z\cdot\omega} \lambda^{n-1} \hat{f}(\lambda\omega) d\omega d\lambda. \tag{3.1}$$

By defining an operator $E_0(\lambda)$ as

$$E_0(\lambda) f = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} e^{i\lambda z\cdot\omega} \lambda^{n-1} \hat{f}(\lambda\omega) d\omega, \tag{3.2}$$

this gives rise to a resolution of the identity as

$$I = \int_0^\infty E_0(\lambda) d\lambda. \tag{3.3}$$

Since $\widehat{\Delta f} = |\zeta|^2 \hat{f}$ for $f \in \mathcal{S}(\mathbb{R}^n)$, the spectral resolution of the Laplacian is given by

$$\Delta = \int_0^\infty \lambda^2 E_0(\lambda) d\lambda. \tag{3.4}$$

Now plane waves $\Phi_0(z, \omega, \lambda) = \exp(i\lambda z \cdot \omega)$ are solutions to $(\Delta - \lambda^2)u = 0$, so the range of $E_0(\lambda)$ lies in the kernel of $\Delta - \lambda^2$. $E_0(\lambda)$ can be further decomposed by defining an operator $\Phi_0(\lambda) : C^\infty(\mathbb{S}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$ using these plane wave solutions:

$$(\Phi_0(\lambda) f)(z) = \int_{\mathbb{S}^{n-1}} \Phi_0(z, \omega, \lambda) f(\omega) d\omega. \tag{3.5}$$

The adjoint operator $\Phi_0^*(\lambda) : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{S}^{n-1})$ is simply

$$(\Phi_0^*(\lambda) f)(\omega) = \int_{\mathbb{R}^n} \overline{\Phi_0(z, \omega, \lambda)} f(z) dz = \int_{\mathbb{R}^n} \Phi_0(z, \omega, -\lambda) f(z) dz. \tag{3.6}$$

Then for $\lambda > 0$, equation 3.2 can be rewritten as

$$E_0(\lambda) = (2\pi)^{-n} \lambda^{n-1} \Phi_0(\lambda) \Phi_0^*(\lambda). \tag{3.7}$$

For fixed $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $\Phi_0^*(\lambda)$ is surjective since any smooth function on the sphere of radius $\lambda$ is the restriction of an element of $\mathcal{S}(\mathbb{R}^n)$. Therefore, the range of $E_0(\lambda)$ is the range of $\Phi_0(\lambda)$, which when considered as an operator on the space of distributions on $\mathbb{S}^{n-1}$, is the entire kernel of $\Delta - \lambda^2$ as an operator on $\mathcal{S}'(\mathbb{R}^n)$. It is trivially true that the range of $\Phi_0(\lambda)$ lies in the kernel of $\Delta - \lambda^2$, and to show that the range of $\Phi_0(\lambda)$ is the entire kernel, we use the Fourier transform. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ with $(\Delta - \lambda^2)f = 0$, taking the Fourier transform gives $(|\zeta|^2 - \lambda^2)\hat{f}(\zeta) = 0$. Therefore, in polar coordinates $z = r\theta$, $\hat{f} = \delta(r - |\lambda|)g'(\theta)$ for some distribution $g'$ on the sphere of radius $|\lambda|$. Taking the inverse Fourier transform,

$$f = (2\pi)^{-n}\lambda^{n-1}\Phi_0(\lambda)g' = \Phi_0(\lambda)g \tag{3.8}$$

for $g = (2\pi)^{-n}\lambda^{n-1}g'$. Therefore all solutions of $(\Delta - \lambda^2)f = 0$ of polynomial growth are superpositions of plane wave solutions $\Phi_0(z, \omega, \lambda)$, so the plane waves form a smooth parametrization of the eigenspace.

3.2. **The Poisson operator.** To understand the asymptotic behavior of these solutions as $|z| \to \infty$, we use polar coordinates to rewrite equation 3.5 as

$$\Phi_0(\lambda)f(z) = \int_{\mathbb{S}^{n-1}} e^{ir\lambda\theta\cdot\omega} f(\omega)d\omega. \tag{3.9}$$

Since the phase of equation 3.9 is stationary at $\omega = \pm\theta$ and the Hessian is non-degenerate at these points, the stationary phase lemma gives a complete asymptotic expansion as

$$\Phi_0(\lambda)f(z) \sim \left(\frac{2\pi}{\lambda r}\right)^{\frac{1}{2}(n-1)} \sum_{\pm} e^{\pm i\lambda r \mp \frac{1}{4}(n-1)i} \sum_{j\geq 0} r^{-j} h_j^{\pm}(\theta), \tag{3.10}$$

where $h_0^{\pm}(\theta) = f(\pm\theta)$, and for $j > 0$, $h_j^{\pm}$ are polynomials of degree $j$ in the Laplacian applied to $f(\pm\theta)$. In fact, for any $h \in C^\infty(\mathbb{S}^{n-1})$, there is a unique solution to $(\Delta - \lambda^2)u = 0$ with asymptotic expansion

$$u(z) = r^{-\frac{1}{2}(n-1)}(e^{i\lambda r}h(\theta) + e^{-i\lambda r}h'(\theta)) + O(r^{-\frac{1}{2}(n+1)}), \tag{3.11}$$

for some $h' \in C^\infty(\mathbb{S}^{n-1})$. Therefore, distributions on the sphere at infinity parametrize the generalized eigenspace with eigenvalue $\lambda^2$. Using the complete asymptotic expansion 3.10, the map $P_0(\lambda) : C^\infty(\mathbb{S}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$ that gives rise to the solution 3.11 is

$$u(z) = P_0(\lambda)h = \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}(n-1)} e^{\frac{1}{4}\pi(n-1)i}\Phi_0(\lambda)h. \tag{3.12}$$

Such an operator is called the Poisson operator for the boundary problem $(\Delta - \lambda^2)u = 0$ for given $h$.[3]

3.3. **Resolvent of the Laplacian.** We now generalize our results for the resolvent of the Laplacian on $\mathbb{S}^1$ to $\mathbb{R}^n$. Since we must now use the continuous Fourier transform instead of the discrete Fourier transform, $\sigma I - \Delta$ is invertible for $\sigma \in \mathbb{C}\backslash[0, \infty)$. Since the spectrum of the Laplacian lies in the positive real numbers, we will use $\sigma = \lambda^2$ as a modified spectral parameter. For such $\sigma$, the resolvent is given by the Fourier transform as

$$R_\sigma(\Delta)f(z) = (\sigma - \Delta)^{-1}f(z) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iz\cdot\zeta}(\sigma - |\zeta|^2)^{-1}\hat{f}(\zeta)d\zeta. \tag{3.13}$$

[3]This discussion follows from section 1.3 in Melrose.

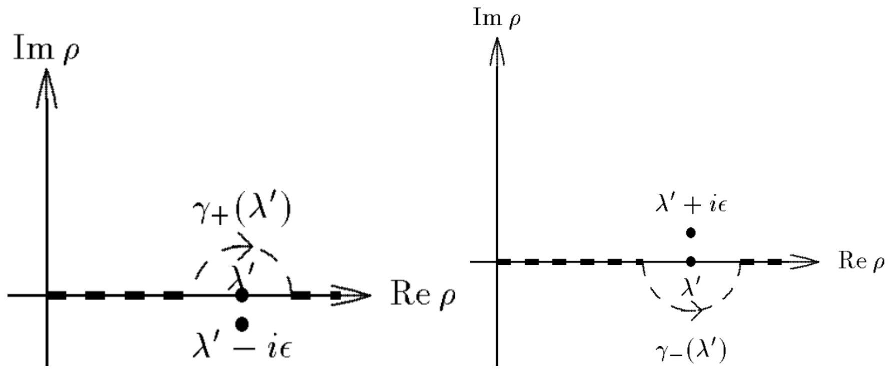


FIGURE 1. The new contours of integration $\gamma_\pm(\lambda')$.

Since each $\sigma \in \mathbb{C}\backslash[0,\infty)$ gives rise to two values of $\lambda$, by convention we will choose $\lambda$ such that $\text{Im}\lambda < 0$. We will then slightly adjust the definition of the resolvent in 3.13 to be in terms of $\lambda$:

$$R_\lambda(\Delta) = (\lambda^2 - \Delta)^{-1}. \tag{3.14}$$

Equation 3.13 then becomes

$$R_\lambda(\Delta)f(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\zeta}(\lambda^2 - |\zeta|^2)^{-1}\hat{f}(\zeta)d\zeta. \tag{3.15}$$

3.4. **Stone's Theorem.** Using the resolvent family of the Laplacian, we can use Stone's Theorem to produce another expression for the spectral resolution. Stone's Theorem generally states that the spectral resolution can be obtained from the difference of the limits from above and below of the resolvent family on the spectrum. To begin with, we rewrite the resolvent as an integral operator

$$R_\lambda(\Delta, z, z') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} \frac{d\zeta}{\lambda^2 - |\zeta|^2}, \tag{3.16}$$

so equation 3.15 becomes

$$R_\lambda(\Delta)f(z) = \int_{\mathbb{R}^n} R_\lambda(\Delta, z, z')f(z')dz'. \tag{3.17}$$

By introducing polar coordinates $\zeta = \rho\omega$, this becomes

$$R_\lambda(\Delta, z, z') = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_0^\infty e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1}d\rho}{\lambda^2 - \rho^2} d\omega. \tag{3.18}$$

To compute the difference of the limits of the resolvent as the spectral parameter $\sigma = \lambda^2$ approaches $[0,\infty)$ from above and below, we consider the limits as $\text{Im}\lambda \nearrow 0$. To do so, we decompose $\lambda = \pm\lambda' - i\epsilon$ into its real and imaginary components, with $\lambda' > 0$. Since the integrand of equation 3.18 is holomorphic in $\rho$ away from $\rho = \pm\lambda$, by Cauchy's theorem, the contour of integration in equation 3.18 can be moved from $\mathbb{R}^+$ to $\gamma_\pm(\lambda')$ as in figure 1.[4] Then equation 3.18 becomes

$$R_\lambda(\Delta, z, z') = (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\gamma_\pm(\lambda')} e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1}d\rho}{\lambda^2 - \rho^2} d\omega \tag{3.19}$$

[4]This figure is figure 1 in Melrose.

for $\lambda = \pm\lambda' - i\epsilon$. The difference in the limits of the resolvent is then given by

(3.20)
$$\begin{aligned} R_\lambda(\Delta, z, z') - R_{-\lambda}(\Delta, z, z') &= (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\gamma_+(\lambda)-\gamma_-(\lambda)} e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1} d\rho}{\lambda^2 - \rho^2} d\omega \\ &= (2\pi)^{-n} \int_{\mathbb{S}^{n-1}} \int_{\gamma} e^{i\rho(z-z')\cdot\omega} \frac{\rho^{n-1} d\rho}{\lambda^2 - \rho^2} d\omega \\ &= \frac{i}{2}(2\pi)^{-(n-1)} \lambda^{n-2} \int_{\mathbb{S}^{n-1}} e^{i\lambda(z-z')\cdot\omega} d\omega, \end{aligned}$$

where $\gamma$ is a clockwise circle of small radius, and the last equality follows from Cauchy's theorem. Finally, equation 3.2 can then be written in terms of equation 3.20 as

$$E_0(\lambda) = \frac{\lambda}{i\pi}(R_\lambda(\Delta) - R_{-\lambda}(\Delta)), \tag{3.21}$$

which is the desired result of Stone's Theorem.[5]

3.5. **Analytic continuation of the resolvent.** Notice that equation 3.19 makes sense for $\epsilon$ small relative to the radius of the contour, regardless of its sign. This indicates that the resolvent can be analytically continued through the real axis away from 0. To demonstrate this rigorously, we define a new operator $M(\lambda)$ that resembles equation 3.20 as

$$M(\lambda) = \frac{i}{2}(2\pi)^{-(n-1)} \int_{\mathbb{S}^{n-1}} e^{i\lambda(z-z')\cdot\omega} d\omega. \tag{3.22}$$

Hitherto, we have only considered the case where $\lambda > 0$ in all of our formulations. However, note that $M(\lambda)$ extends to an entire function. By equation 3.20, we can define the resolvent near the positive real axis with $Im\lambda > 0$ by

$$R_\lambda(\Delta) = R_{-\lambda}(\Delta) + \lambda^{n-2} M(\lambda). \tag{3.23}$$

This is holomorphic not only near $(0, \infty)$, but also for all of $\mathbb{C}\backslash(-\infty, 0]$. By applying the antipodal map, it is clear that $M(\lambda) = M(-\lambda)$, so by repeated application of equation 3.23, we obtain

$$\lim_{\epsilon \searrow 0} R_{-\lambda'+i\epsilon}(\Delta) - \lim_{\epsilon \nearrow 0} R_{-\lambda'+i\epsilon}(\Delta) = \begin{cases} 0 & n \text{ odd} \\ 2\lambda'^{n-2} M(\lambda') & n \text{ even} \end{cases} \tag{3.24}$$

For $n = 1$, the resolvent kernel is meromorphic with a simple pole at 0. For $n > 1$ odd, the resolvent kernel is locally integrable in $z$ and $z'$ and is an entire function of $\lambda$. For $n$ even, since the kernel is only entire on the logarithmic covering of the complex plane $\Lambda$, the physical domain for the resolvent is defined to be $\mathcal{P} = \mathbb{R} \times i(-\pi, 0)$. When the preimage of the point $\lambda = e^\tau$ is shifted by the transformation $\tau \to \tau + i\pi$, the resolvent transforms by

$$R_{\tau+i\pi}(\Delta) = R_\tau(\Delta) + e^{(n-2)\tau} M(e^\tau).\text{[6]} \tag{3.25}$$

[5]This discussion follows section 1.5 in Melrose.
[6]This discussion follows section 1.6 of Melrose.

## 4. Potential Scattering on $\mathbb{R}^n$

Now that scattering theory has been established on Euclidean space, we consider the simplest perturbations of the Laplacian, which are given by potentials. For simplicity, consider a real potential $V \in C_c^\infty(\mathbb{R}^n)$ as an operator that acts by multiplication, and the corresponding perturbed Laplacian $\Delta + V$. To begin with, we generalize our discussion of the resolvent.

### 4.1. **Resolvent of the perturbed Laplacian.**

**Lemma 4.1.** *For $\lambda \in \mathbb{C}\backslash D$, with $D \subset \mathbb{C}$ discrete, there is a unique operator $R_\lambda(\Delta + V) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ such that*

$$(\lambda^2 - \Delta - V) \circ R_\lambda(\Delta + V) = Id. \tag{4.1}$$

*Proof.* Formally, the existence of the perturbed resolvent is equivalent to the validity of the following expression for the unperturbed resolvent:

$$\begin{aligned} R_\lambda(\Delta) &= R_\lambda(\Delta) \circ (\lambda^2 - \Delta - V) \circ R_\lambda(\Delta + V) \\ &= R_\lambda(\Delta + V) - R_\lambda(\Delta) \circ V \circ R_\lambda(\Delta + V) \\ &= (Id - R_\lambda(\Delta) \circ V) \circ R_\lambda(\Delta + V) \end{aligned} \tag{4.2}$$

Therefore, if $Id + R_\lambda(\Delta) \circ V$ is invertible, then the perturbed resolvent exists and is given by

$$R_\lambda(\Delta + V) = (Id - R_\lambda(\Delta) \circ V)^{-1} R_\lambda(\Delta) \tag{4.3}$$

Consider $R_\lambda(\Delta)$ as an operator on $L^2_c(B(R))$. Then

$$R_\lambda(\Delta) : L^2_c(B(R)) \to e^{T|z|} L^2(\mathbb{R}^n) \tag{4.4}$$

is a compact operator depending holomorphically on $\lambda$ for $|Im\lambda| < T$. Since $V$ is compactly supported, for large $R$, $V$ is an operator $V : e^{T|z|}L^2(\mathbb{R}^n) \to L^2_c(B(R))$. Since $\|R_\lambda(\Delta)\| \to 0$ as $\lambda \to -i\infty$ in the physical domain $\mathcal{P}$, the compact operator $R_\lambda(\Delta) \circ V : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ acting on the full space is norm-approximable by finite rank operators. Then the local invertibility of $Id - R_\lambda(\Delta) \circ V$ is equivalent to the invertibility of a matrix, and by the analytic Fredholm theorem, since $Id + R_\lambda(\Delta) \circ V$ is invertible at one point, its inverse is meromorphic, and all of its residues are operators of finite rank. Therefore, $(Id - R_\lambda(\Delta) \circ V)^{-1} : e^{T|z|}L^2(\mathbb{R}^n) \to e^{T|z|}L^2(\mathbb{R}^n)$ exists for $\lambda \notin D$, with $D$ discrete. Finally, by uniqueness of this operator in the physical region $\mathcal{P}$, enlarging $T$ gives an extension of the resolvent family to the entire complex plane minus a discrete set.[7] □

A slightly different form for equation 4.3 can be obtained by using a different version of the identity 4.2. Writing the unperturbed resolvent as

$$\begin{aligned} R_\lambda(\Delta) &= R_\lambda(\Delta + V) \circ (\lambda^2 - \Delta - V) \circ R_\lambda(\Delta) \\ &= R_\lambda(\Delta + V) - R_\lambda(\Delta + V) \circ V \circ R_\lambda(\Delta) \\ &= R_\lambda(\Delta + V) \circ (Id - V \circ R_\lambda(\Delta)) \end{aligned} \tag{4.5}$$

gives an expression for the perturbed resolvent as

$$R_\lambda(\Delta + V) = R_\lambda(\Delta)(Id - V \circ R_\lambda(\Delta))^{-1}. \tag{4.6}$$

[7] This discussion follows section 2.1 in Melrose.

Equations 4.3 and 4.6 then give decompositions of the perturbed resolvent in terms of the unperturbed resolvent and a meromorphic operator.

The poles of the perturbed resolvent are similar to the eigenvalues of the Laplacian, and they are associated with generalized eigenfunctions. For $n$ odd, equation 4.6 implies that if the resolvent has a pole at $\lambda$, then there is an eigenfunction of $(\Delta + V - \lambda^2)u = 0$ of the form

$$u = \exp\Big(\frac{-i\lambda}{x}\Big)x^{\frac{1}{2}(n-1)}w \tag{4.7}$$

with $w \in C^\infty(\mathbb{S}^n_+)$. For $n$ even, this is true only in the physical region $\mathcal{P}$, but the generalized eigenfunctions can still be characterized by the following lemma.

**Lemma 4.2.** *For $n$ even, the resolvent has a pole at $\lambda$ if and only if there is a non-trivial eigenfunction $u \in \mathcal{C}(\mathbb{R}^n)$ of $(\Delta + V - \lambda^2)u = 0$ with $u = -R_\lambda(\Delta)Vu$.*

*Proof.* If $R_\lambda(\Delta+V)$ has a pole, then by equation 4.6, so must $(Id - V \circ R_\lambda(\Delta))^{-1}f$ for some $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Since the residue $u' \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ of $(Id - V \circ R_\lambda(\Delta))^{-1}f$ must satisfy $u' + VR_\lambda(\Delta)u' = 0$, $u = R_\lambda(\Delta)u'$ satisfies $u = -R_\lambda(\Delta)VR_\lambda(\Delta)u' = -R_\lambda(\Delta)Vu$. In the other direction, if there is such $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, then $g = Vu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfies $g = -VR_\lambda(\Delta)g$. Therefore, $Id + V \circ R_\lambda(\Delta)$ has a non-trivial kernel and is thus non-invertible, so $(Id - V \circ R_\lambda(\Delta))^{-1}$ must have a pole, and thus $R_\lambda(\Delta + V)$ must as well.[8] □

4.2. **Absence of embedded eigenvalues.** Building on the previous lemma, this section will be dedicated to proving that there are no embedded eigenvalues of the perturbed Laplacian. That is, for any $n$ and $V \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, there is no non-trivial eigenfunction of $(\Delta + V - \lambda^2)u = 0$ with $\lambda \in \mathbb{R}$ non-zero. The first major step in proving this is by showing if $u$ satisfies equation 4.7, then $u \in \mathcal{C}^\infty(\mathbb{S}^n_+)$.

Consider a more general expansion of the solution to equation 4.7, namely

$$u = \sum_\pm u_\pm = \sum_\pm \exp\Big(\pm\frac{i\lambda}{x}\Big)x^{\frac{1}{2}(n-1)}w_\pm, \tag{4.8}$$

where $w_\pm \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ are the leading coefficients, and $(\Delta + V - \lambda^2)u = f \in \mathcal{C}^\infty(\mathbb{S}^n_+)$. Then the coefficients $w_\pm$ of this formal solution are linked by a boundary pairing relationship.

**Lemma 4.3.** *For solutions $u_1$, $u_2$ to equation 4.8, and $\lambda \in \mathbb{R}$ non-zero,*

$$-2i\lambda\int_{\mathbb{S}^{n-1}}(v_1^+\overline{v_2^+} - v_1^-\overline{v_2^-})dz = \int_{\mathbb{R}^n}(f_1\overline{u_2} - u_1\overline{f_2})dz, \tag{4.9}$$

*where $v_i^\pm$ is the restriction $w_i^\pm \mid \partial\mathbb{S}^n_+$.*

*Proof.* This can be shown by finding a cut-off function $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(x) = 1$ for $|x| < 1$ and support in $|x| \leq 2$. Then the integral

$$I_\epsilon = \int_{\mathbb{R}^n}(f_1\overline{u_2} - u_1\overline{f_2})\phi(\epsilon|z|)dz \tag{4.10}$$

converges to the right-hand side of equation 4.9 as $\epsilon \searrow 0$. Application of Green's formula for the Laplacian then shows convergence to the left-hand side of equation 4.9 as well.[9] □

[8] This discussion also follows section 2.1 of Melrose.

[9] This discussion follows section 2.3 in Melrose.

By applying equation 4.9 to $u$ as in equation 4.7, we see that the leading coefficient $v = v^+$ vanishes identically. Now let us consider the full asymptotic expansion of $u$.

**Lemma 4.4.** *For any $h \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\lambda \in \mathbb{R}$ non-zero, there exists $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ in the formal null space of $\Delta - \lambda^2$ with an asymptotic expansion*

$$u \sim exp(i\lambda r) r^{-\frac{1}{2}(n-1)} \sum_{j \geq 0} r^{-j} h_j(\theta), \tag{4.11}$$

*where $z = r\theta$ is expressed in polar coordinates, and $h_0 = h$.*

*Proof.* The stationary phase lemma gives a complete asymptotic expansion with leading coefficient $h_0$ as the leading coefficient in $u_+$, as in equation 3.10. By an application of Borel's lemma on $\partial\mathbb{S}^n_+$, the coefficients $r^{-j}h_j^+$ can be summed to give a solution of the desired form 4.11. Furthermore, this summation is unique modulo $\mathcal{S}(\mathbb{R}^n)$, since if a solution $u$ has leading coefficient $h_0$ identically zero, then $u \in L^2$. Therefore, the Fourier transform of $(\Delta - \lambda^2)u = f$ dictates that $(|\zeta|^2 - \lambda^2)^{-1}\hat{f} \in L^2$, so $\hat{f} = 0$ for $|\zeta|^2 = \lambda^2$. This shows that $u \in \mathcal{S}(\mathbb{R}^n)$, and we are done.[10] □

The final step in showing that there are no embedded eigenvalues is unique continuation of $u$.

**Lemma 4.5.** *For $\lambda \in \mathbb{R}$ non-zero, any eigenfunction $u \in \mathcal{C}^\infty(\mathbb{S}^n_+)$ of $(\Delta + V - \lambda^2)u = 0$ vanishes identically.*

*Proof.* Consider an eigenfunction $u \in \mathcal{S}(\mathbb{R}^n)$, which must satisfy $(\Delta - \lambda^2)u = -Vu = f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. By expanding $u$ in spherical harmonics, each coefficient is a rapidly decreasing function of $r$ which satisfies a form of Bessel's equation with no non-trivial rapidly decreasing solution. This then implies that $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, and unique continuation of solutions to second order elliptic equations then implies that $u$ must vanish identically. □

### 4.3. Perturbed plane waves.

The plane wave eigenfunctions of the unperturbed Laplacian can be perturbed to generate eigenfunctions of the perturbed Laplacian. Since $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $(\Delta + V - \lambda^2)e^{i\lambda z\cdot\omega} = Ve^{i\lambda z\cdot\omega} = f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ as well. Then define a plane wave solution

$$\Phi_V(z, \omega, \lambda) = \exp(i\lambda z \cdot \omega) + R_\lambda(\Delta + V)f. \tag{4.12}$$

This is in the kernel of $\Delta + V - \lambda^2$, and is in fact of the form

$$\Phi_V(z, \omega, \lambda) = \exp(i\lambda z \cdot \omega) + \exp(-i\lambda r) r^{-\frac{1}{2}(n-1)} \phi_V(z, \omega, \lambda), \tag{4.13}$$

where $\phi_V \in \mathcal{C}^\infty(\mathbb{S}^n_+)$. This is, in fact, the unique solution of this form, since the difference of any two such solutions would be of the form 4.7. Furthermore, since the coefficient $\phi_V(z, \omega, \lambda) \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1} \times (\mathbb{R}\backslash\{0\}))$ depends smoothly on the parameters, $\Phi_V$ can be treated as an operator into the kernel of $\Delta + V - \lambda^2$

$$\Phi_V(\lambda) f = \int_{\mathbb{S}^{n-1}} \Phi_V(z, \omega, \lambda) f(\omega) d\omega. \tag{4.14}$$

[10] This discussion follows section 2.4 in Melrose.

Just like in the unperturbed case, this operator has a complete asymptotic expansion with leading coefficients

$$\left(\frac{2\pi}{\lambda r}\right)^{\frac{1}{2}(n-1)} \sum_{\pm} e^{\pm\lambda r \mp \frac{1}{4}\pi(n-1)i} g_{\pm}(\theta) + u', \tag{4.15}$$

with $u' \in L^2(\mathbb{R}^n)$.

4.4. **$L^2$ eigenfunctions.** $L^2$ eigenfunctions of $\Delta + V$ are known as bound states, and they may arise from negative eigenvalues or zero eigenvalues. In the case of negative eigenvalues, since $R_\lambda(\Delta + V)$ is meromorphic and acts on $\mathcal{S}(\mathbb{R}^n)$ on a deleted neighborhood of a pole, its residue operators also act on $\mathcal{S}(\mathbb{R}^n)$, with range equal to the corresponding eigenspace. In the physical domain $\mathcal{P}$, by self-adjointness, all poles are simple and lie on the imaginary axis. Conversely, for any negative eigenvalue $\sigma$ of $\Delta + V$ and corresponding $L^2$ eigenfunction, the resolvent has a pole at the point $\lambda \in \mathcal{P}$ with $\sigma = \lambda^2$.

The case of zero eigenvalues is more complicated, and it depends on the dimension $n$ of the space. For $n$ odd, it follows from self-adjointness that the resolvent can have at most a double pole at 0. The leading coefficient in the expansion of the resolvent maps onto the kernel of $\Delta + V$ acting on $L^2(\mathbb{R}^n)$, and the residue space consists of elements of the kernel of $\Delta + V$ that are not in $L^2(\mathbb{R}^n)$. For $n = 3$, the residue space is at most one-dimensional, and for $n \geq 5$ this space is empty.

## 5. General scattering theory

5.1. **The wave operator.** After much ado, we finally arrive at general scattering theory, which seeks to draw information about a Hamiltonian operator through perturbation theory. More specifically, we seek to investigate the asymptotics at large time $t$ of solutions to the time-dependent Schrodinger equation

$$i\frac{\partial u}{\partial t} = Hu \tag{5.1}$$

with given initial data $u(0) = f$. Let $H_0$ denote the unperturbed Hamiltonian, and let $H$ denote the full Hamiltonian. Suppose these are self-adjoint operators on Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$ respectively, and there is a bounded identification operator $\mathcal{J} : \mathcal{H}_0 \to \mathcal{H}$. Furthermore, let $P : \mathcal{H} \to \mathcal{H}_H^{(a)}$ be the projection onto the absolutely continuous subspace $\mathcal{H}_H^{(a)}$, and let $P_0$ be the corresponding operator for $H_0$. Then Schrodinger's equation has a simple unique solution $u_0(t) = \exp(-iH_0t)f_0$ for the unperturbed Hamiltonian and a similar solution $u(t) = \exp(-iHt)f$ for the full Hamiltonian. We say that $u(t)$ has free asymptotics as $t \to \pm\infty$ if for appropriate initial data $f_0^{\pm}$

$$\lim_{t\to\pm\infty} \|u(t) - u_0^{\pm}(t)\| = 0, \tag{5.2}$$

where

$$u_0^{\pm}(t) = \exp(-iH_0t)f_0^{\pm}. \tag{5.3}$$

Equations 5.2 and 5.3 give rise to a relation between the initial data $f_0^{\pm}$ and $f$:

$$f = \lim_{t\to\pm\infty} \exp(iHt)\exp(-iH_0t)f_0^{\pm}. \tag{5.4}$$

This then motivates our definition of the wave operator (WO), which is the basic object of scattering theory.

**Definition 5.1.** The WO for self-adjoint operators $H_0$ and $H$ with identification operator $\mathcal{J}$ is the operator

$$W_{\pm}(H, H_0; \mathcal{J}) = \text{s-}\lim_{t\to\pm\infty} U(-t)\mathcal{J}U_0(t)P_0, \tag{5.5}$$

where $U(t) = \exp(-iHt)$, if the strong limit exists. For finite time $t$, we define a related operator

$$W(t) = W_{\mathcal{J}}(t) = U(-t)\mathcal{J}U_0(t). \tag{5.6}$$

Of course, the existence of the wave operator imposes strong restrictions on the pair $H_0$ and $H$, but a discussion of this will be postponed to a later section. For now, we will assume the existence of the wave operator and study its properties. First of all, it is clear that the singular subspaces of $H_0$ are contained in the kernel of the wave operator. That is,

$$\mathcal{H}_0^{(s)} = \mathcal{H}_0 \ominus \mathcal{H}_0^{(a)} \subset N(W_{\pm}). \tag{5.7}$$

Furthermore,

**Lemma 5.2.** *The wave operators $W_{\pm}$ are bounded, with*

$$\|W_{\pm}f\| \leq \|\mathcal{J}\|\|P_0 f\|. \tag{5.8}$$

*A necessary and sufficient condition for $W_{\pm}$ to be an isometry on $\mathcal{H}_0^{(a)}$ is that for any $f \in \mathcal{H}_0^{(a)}$,*

$$\lim_{t\to\pm\infty} \|\mathcal{J}U_0(t)f\| = \|f\|. \tag{5.9}$$

*Proof.* Since the norm is a continuous functional with respect to strong convergence, the unitarity of $U(t)$ gives

$$\|W_{\pm}f\| = \lim_{t\to\pm\infty} \|W_{\mathcal{J}}(t)P_0 f\| = \lim_{t\to\pm\infty} \|\mathcal{J}U_0(t)P_0 f\|. \tag{5.10}$$

Therefore, for $f \in \mathcal{H}_0^{(a)}$, equation 5.9 is equivalent to $\|W_{\pm}f\| = \|f\|$, and equation 5.8 follows from the unitarity of $U_0(t)$. □

A more convenient sufficient criterion for the validity of equation 5.9 is

**Lemma 5.3.** *The wave operator $W_{\pm}$ is an isometry on $\mathcal{H}_0^{(a)}$ if*

$$s\text{-}\lim_{t\to\pm\infty} (\mathcal{J}^*\mathcal{J} - I)U_0(t)P_0 = 0. \tag{5.11}$$

*For this, it is sufficient that $\mathcal{J}^*\mathcal{J} - I$ is compact or that*

$$(\mathcal{J}^*\mathcal{J} - I)E_0(\Lambda) \in \mathcal{G}_{\infty} \tag{5.12}$$

*for any bounded interval $\Lambda$.*

*Proof.* We can rewrite

$$\|\mathcal{J}U_0(t)f\|^2 - ((\mathcal{J}^*\mathcal{J}U_0(t)f, U_0(t)f) + \|f\|^2. \tag{5.13}$$

Therefore, equation 5.9 follows from equation 5.11 and the unitarity of $U_0(t)$. It suffices that equation 5.11 holds on a dense set of elements of the form $f = E_0(\Lambda)f$, or that

$$\text{s-}\lim_{t\to\pm\infty} (\mathcal{J}^*\mathcal{J} - I)E_0(\Lambda)U_0(t)P_0 = 0 \tag{5.14}$$

for arbitrary bounded $\Lambda$.[11] □

[11] This discussion follows section 2.1 in Yafaev.

If the operator $\mathcal{J}$ is sufficiently nice, there are other sufficient criteria that demonstrate that $W_\pm$ is an isometry. If $\mathcal{J}$ itself is an isometry, then of course equation 5.9 holds, and $W_\pm$ is an isometry. In particular, for the easy case of $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{J} = I$, $W_\pm$ is automatically an isometry when it exists.

Another important property of the wave operator is the intertwining property.

**Lemma 5.4.** *For any bounded Borel function* $\phi$,

$$\phi(H)W_\pm(H, H_0; \mathcal{J}) = W_\pm(H, H_0; \mathcal{J})\phi(H_0), \tag{5.15}$$

*and in particular, for any Borel set* $\Lambda \subset \mathbb{R}$,

$$E(\Lambda)W_\pm(H, H_0; \mathcal{J}) = W_\pm(H, H_0; \mathcal{J})E_0(\Lambda). \tag{5.16}$$

*Proof.* The sesquilinear form of the unitary group $U_H(t)$ is given by the Fourier transform of the spectral measure as

$$(U(t)f, g) = \int_{-\infty}^{\infty} \exp(-i\lambda t) d(E(\lambda)f, g). \tag{5.17}$$

By the definition of the wave operator,

$$U(s)W_\pm = \text{s-}\lim_{t\to\pm\infty} U(s-t)\mathcal{J}U_0(t)P_0 = \text{s-}\lim_{t\to\pm\infty} U(-t)\mathcal{J}U_0(t+s)P_0 = W_\pm U_0(s), \tag{5.18}$$

so equation 5.15 trivially holds for the case $\phi(\lambda) = e^{-i\lambda s}$. Then for any $f_0 \in \mathcal{H}_0$, $f \in \mathcal{H}$, and $s \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{-is\lambda} d(E(\lambda)W_\pm f_0, f) = \int_{-\infty}^{\infty} e^{-is\lambda} d(E_0(\lambda)f_0, W_\pm^* f), \tag{5.19}$$

and equation 5.16 follows by uniqueness of the Fourier transform. Integrating equation 5.16 then gives equation 5.15.[12] □

**Corollary 5.5.** Let $\mathcal{H}_0^{(\pm)} = \mathcal{H}_0 \ominus N(W_\pm)$ and $\mathcal{H}^{(\pm)} = \overline{R(W_\pm)}$, and let $H_0^{(\pm)}$ and $H^{(\pm)}$ denote the restriction of $H_0$ and $H$ to these subspaces. Then the operator $H^{(\pm)}$ is absolutely continuous, so $\overline{R(W_\pm)} \subset \mathcal{H}^{(a)}$ and $PW_\pm = W_\pm$. Furthermore, if $N(W_\pm) = \mathcal{H}_0^{(s)}$, then $H^{(\pm)}$ is unitarily equivalent to $H_0^{(a)}$.

Our next result is an important property of the wave operator known as the chain rule.

**Lemma 5.6.** *If the wave operators* $W_\pm(H_1, H_0; \mathcal{J}_0)$ *and* $W_\pm(H, H_1; \mathcal{J}_1)$ *exist, then the wave operator* $W_\pm(H, H_0; \mathcal{J}_1\mathcal{J}_0)$ *also exists, with*

$$W_\pm(H, H_0; \mathcal{J}_1\mathcal{J}_0) = W_\pm(H, H_1; \mathcal{J}_1)W_\pm(H_1, H_0; \mathcal{J}_0). \tag{5.20}$$

*Proof.* We can rewrite the wave operator as

$$U(-t)\mathcal{J}_1\mathcal{J}_0U_0(t)P_0 = U(-t)\mathcal{J}_1U_1(t)[P_1 + (I - P_1)]U_1(-t)\mathcal{J}_0U_0(t)P_0. \tag{5.21}$$

By corollary 5.5, for any $f_0 \in \mathcal{H}_0^{(a)}$,

$$\|(I - P_1)U_1(-t)\mathcal{J}_0u_0(t)f_0\| \to 0 \tag{5.22}$$

as $t \to \pm\infty$. The remaining term on the right hand side is the product of two factors that converge to $W_\pm(H, H_1; \mathcal{J})$ and $W_\pm(H_1, H_0; \mathcal{J}_0)$.[13] □

[12] This discussion also follows section 2.1 in Yafaev.
[13] This discussion also follows section 2.1 of Yafaev.

### 5.2. **The scattering operator.**

**Definition 5.7.** The scattering operator relates the past asymptotics of a wave with its future asymptotics, and the scattering operator is given in terms of the wave operator as $\mathcal{S} = W_+^* W_-$.

The scattering operator has several nice properties which follow from properties of the wave operator. By lemma 5.3, the scattering operator vanishes on $\mathcal{H}_0^{(s)}$ and has range in $\mathcal{H}_0^{(a)}$. By lemma 5.2, the scattering operator is bounded, with $\|\mathcal{S}\| \leq \|\mathcal{J}\|^2$. By the intertwining property, lemma 5.4, $\mathcal{S}$ and $H_0$ commute.

The subspace $\mathcal{H}_0^{(a)}$ can be decomposed into a direct integral of auxiliary Hilbert spaces $\mathfrak{h}_0(\lambda)$ that diagonalizes $H_0^{(a)}$ as

$$\mathcal{H}_0^{(a)} \leftrightarrow \mathfrak{h}_0^{(a)} = \int_{\hat{\sigma}_0} \bigoplus \mathfrak{h}_0(\lambda) d\lambda, \tag{5.23}$$

where $\hat{\sigma}_0 = \hat{\sigma}(H_0)$ is a core of the spectrum of $H_0$. Since $\mathcal{S}$ and $H_0^{(a)}$ commute, under the correspondence 5.23, the scattering operator becomes multiplication by an operator-value function $\mathcal{S}(\lambda; H, H_0; \mathcal{J}) : \mathfrak{h}_0(\lambda) \to \mathfrak{h}_0(\lambda)$, which is known as the scattering matrix. Many of the properties of the scattering operator can be reformulated in analogous terms of the scattering matrix.

**Example 5.8.** To give a demonstration of the results of the previous sections, we now explicitly formulate scattering theory for a simple example. Let $\mathcal{H} = L^2(\mathbb{R})$ and $H_0 = -i\frac{d}{dx}$, where the domain of $H_0$ is the set of absolutely continuous functions $f$ such that $f, f' \in L^2(\mathbb{R})$. Therefore, $H_0$ is unitarily equivalent to the operator $A$ of multiplication by the independent variable in the dual space $L^2(\Xi)$. That is, $H_0 = \Phi^* A \Phi$, where $\Phi$ is the Fourier transform. The spectrum of $H_0$ is absolutely continuous, covers the entire spectral axis, is simple, and its unitary group acts by

$$(U_0(t)f)(x) = f(x - t). \tag{5.24}$$

Define the perturbed Hamiltonian $H$ as

$$Hf = -if' + qf, \tag{5.25}$$

where $q$ is any locally summable real function, and the domain of $H$ is the set of absolutely continuous functions $f$ such that $f, -if' + qf \in L^2(\mathbb{R})$. In this example, let the identification operator simply be the identity $\mathcal{J} = I$. Furthermore, define an operator $Y$ by

$$(Yf)(x) = \exp\Big(-i \int_0^x q(y)dy\Big) f(x). \tag{5.26}$$

Then $Y$ is unitary in $\mathcal{H}$, maps the domain of $H_0$ onto the domain of $H$, and satisfies

$$HYf = YH_0 f. \tag{5.27}$$

In particular, $H$ is self-adjoint. The wave operator 5.6 can then be expressed as

$$(W(t)f)(x) = (YU_0(-t)Y^*U_0(t)f)(x) = \exp\Big(i \int_x^{x+t} q(y)dy\Big) f(x) \tag{5.28}$$

If we assume that the integral of $q$ over the semiaxis $\mathbb{R}_\pm$ converges, then the wave operator $W_\pm(H, H_0)$ exists and is given by multiplication by

$$\exp\Big(i \int_x^{\pm\infty} q(y)dy\Big). \tag{5.29}$$

Furthermore, this wave operator is unitary and complete. It then follows that the scattering operator $\mathcal{S}$ is simply given by multiplication by

$$\exp\Big( - i \int_{-\infty}^{\infty} q(x) dx \Big). \tag{5.30}$$

Similarly, the scattering matrix $\mathcal{S}(\lambda)$ is independent of $\lambda$ and is also given by multiplication by 5.30.[14]

5.3. **The stationary approach.** Just like in the previous two chapters, the stationary approach can be used to define the wave operator in terms of the resolvents of $H_0$ and $H$.

**Lemma 5.9.** *If the wave operator $W_\pm(H, H_0; \mathcal{J})$ exists, then for any $f_0 \in \mathcal{H}_0$ and any $f \in \mathcal{H}$,*

$$(W_\pm(H, H_0; \mathcal{J}) f_0^{(a)}, f^{(a)}) = \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} (\mathcal{J} R_{\lambda + i\epsilon}(H_0) f_0^{(a)}, R_{\lambda + i\epsilon}(H) f^{(a)}) d\lambda. \tag{5.31}$$

*Proof.* Let $\theta(t)$ denote the Heaviside step function, and apply Parseval's equality

$$\int_{-\infty}^{\infty} (\hat{f}_1(\lambda), \hat{f}_2(\lambda)) d\lambda = \int_{-\infty}^{\infty} (f_1(t), f_2(t)) dt \tag{5.32}$$

with $f_1 = \theta(t) e^{-\epsilon t} \mathcal{J} U_0(\pm t) f_0^{(a)}$ and $f_2 = \theta(t) e^{-\epsilon t} U(\pm t) f^{(a)}$. The sesquilinear form of the unitary group

$$(U(t) f, g) = \int_{-\infty}^{\infty} \exp(-i\lambda t) d(E(\lambda) f, g) \tag{5.33}$$

is the Fourier transform of the spectral measure, and when this is combined with equation 3.15 for the resolvent, it gives rise to the following equation for the resolvent

$$R_{\lambda + i\epsilon}(H) = \pm i \int_0^{\infty} \exp(-\epsilon t \pm i\lambda t) U(\pm t) dt. \tag{5.34}$$

Using this relation, Parseval's equality then gives

$$2\epsilon \int_0^{\infty} e^{-2\epsilon t} (\mathcal{J} U_0(\pm t) f_0^{(a)}, U(\pm t) f^{(a)}) dt = \frac{\epsilon}{\pi} \int_{-\infty}^{\infty} (\mathcal{J} R_{\lambda \pm i\epsilon}(H_0) f_0^{(a)}, R_{\lambda \pm i\epsilon}(H) f^{(a)}) d\lambda, \tag{5.35}$$

as desired.[15] □

**Definition 5.10.** This then lends itself to the definition of the stationary wave operator $\mathcal{W}_\pm(H, H_0; \mathcal{J})$ in its sesquilinear form of

$$(\mathcal{W}_\pm f_0, f) = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} (\mathcal{J} R_{\lambda \pm i\epsilon}(H_0) f_0, R_{\lambda + i\epsilon}(H) f) d\lambda, \tag{5.36}$$

which is defined a.e. $\lambda \in \mathbb{R}$.

**Lemma 5.11.** *A more useful representation of the stationary wave operator that will be commonly used is*

$$(\mathcal{W}_\pm f_0, \mathcal{W}_\pm g_0) = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \frac{\epsilon}{\pi} (\mathcal{J} R_{\lambda \pm i\epsilon}(H_0) f_0, \mathcal{J} R_{\lambda \pm i\epsilon}(H_0) g_0) d\lambda, \tag{5.37}$$

[14]This discussion follows section 2.4 of Yafaev.

[15]This discussion follows section 2.7 in Yafaev.

*with* $f_0, g_0 \in \mathcal{H}_0$.

*Proof.* Let

$$P(\lambda,\epsilon) = (2\pi i)^{-1}[R_{\lambda+i\epsilon}(H) - R_{\lambda-i\epsilon}(H)] = \frac{\epsilon}{\pi} R_{\lambda+i\epsilon}(H) R_{\lambda-i\epsilon}(H) \tag{5.38}$$

be the Poisson kernel from section 3.4. From functional analysis,

$$\lim_{\epsilon\to 0}(P(\lambda,\epsilon)f,g) = \frac{d(E(\lambda)f,g)}{d\lambda} \tag{5.39}$$

for a.e. $\lambda \in \mathbb{R}$. Then for any Borel set $\Lambda$, we have

$$(E(\Lambda)\mathcal{W}_\pm f_0, f) = \int_\Lambda \lim_{\epsilon\to 0}(P(\lambda,\epsilon)\mathcal{W}_\pm f_0, f) d\lambda. \tag{5.40}$$

From the original definition of the stationary wave operator 5.36, it can be shown that the intertwining property

$$E(\Lambda)\mathcal{W}_\pm = \mathcal{W}_\pm E_0(\Lambda) \tag{5.41}$$

holds, and that furthermore, there is the representation

$$(\mathcal{W}_\pm E_0(\Lambda_0) f_0, E(\Lambda) f) = \int_{\Lambda_0\cap\Lambda} \lim_{\epsilon\to 0} \frac{\epsilon}{\pi}(\mathcal{J} R_{\lambda\pm i\epsilon}(H_0) f_0, R_{\lambda+i\epsilon}(H) f) d\lambda. \tag{5.42}$$

Equation 5.40 then becomes

$$\lim_{\epsilon\to 0}(P(\lambda,\epsilon)\mathcal{W}_\pm f_0, f) = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi}(\mathcal{J} R_{\lambda\pm i\epsilon}(H_0) f_0, R_{\lambda\pm i\epsilon}(H) f). \tag{5.43}$$

Rearranging, this implies that as $\epsilon \to 0$,

$$P(\lambda,\epsilon)\mathcal{W}_\pm f_0 \sim \frac{\epsilon}{\pi} R_{\lambda\mp i\epsilon}(H)\mathcal{J} R_{\lambda\pm i\epsilon}(H_0) f_0 \tag{5.44}$$

Applying the resolvent identities, we have

$$\begin{aligned} \frac{\epsilon}{\pi}(\mathcal{J} R_{\lambda\pm i\epsilon}(H_0) f_0, R_{\lambda\pm i\epsilon}(H)\mathcal{W}_\pm g_0) &= ([\mathcal{J} + V R_{\lambda\pm i\epsilon}(H_0)] f_0, P(\lambda,\epsilon)\mathcal{W}_\pm g_0) \\ &\sim \frac{\epsilon}{\pi}([\mathcal{J} \\ &\quad + V R_{\lambda\pm i\epsilon}(H_0)] f_0, R_{\lambda\mp i\epsilon}(H)\mathcal{J} R_{\lambda\pm i\epsilon}(H_0) g_0) \\ &= \frac{\epsilon}{\pi}(\mathcal{J} R_{\lambda\pm i\epsilon}(H_0) f_0, \mathcal{J} R_{\lambda\pm i\epsilon}(H_0) g_0). \end{aligned} \tag{5.45}$$

Finally, taking $f = \mathcal{W}_\pm g_0$ yields the desired representation.[16] □

We can now construct the sesquilinear form of the stationary scattering operator $\mathcal{W}_+^*\mathcal{W}_-$, which can be generalized to $\mathcal{W}_+^* E(\Lambda)\mathcal{W}_-$, for $\Lambda$ an arbitrary Borel set. Beginning with equation 5.44, we have

$$\begin{aligned} &\frac{\epsilon}{\pi}(\mathcal{J} R_{\lambda-i\epsilon}(H_0) f_0, R_{\lambda-i\epsilon}(H)\mathcal{W}_+ g_0) \\ &\quad = ([\mathcal{J} + V R_{\lambda-i\epsilon}(H_0) f_0, P(\lambda,\epsilon)\mathcal{W}_+ g_0) \\ &\quad \sim \frac{\epsilon}{\pi}([\mathcal{J} + V R_{\lambda-i\epsilon}(H_0)] f_0, R_{\lambda-i\epsilon}(H)\mathcal{J} R_{\lambda+i\epsilon}(H_0) g_0). \end{aligned} \tag{5.46}$$

[16]This discussion also follows section 2.7 in Yafaev.

Representation 5.42 then gives

$$(E(\Lambda)\mathcal{W}_- f_0, \mathcal{W}_+ g_0) = \int_\Lambda \lim_{\epsilon\to 0} \frac{\epsilon}{\pi}([\mathcal{J} + VR_{\lambda-i\epsilon}(H_0)]f_0, R_{\lambda-i\epsilon}(H)\mathcal{J}R_{\lambda+i\epsilon}(H_0)g_0)d\lambda. \tag{5.47}$$

This integral can be transformed by the identity

$$\begin{aligned} &\frac{\epsilon}{\pi}R_{\lambda-i\epsilon}(H_0)\mathcal{J}^*R_{\lambda+i\epsilon}(H)(\mathcal{J}+VR_{\lambda-i\epsilon}(H_0)) \\ &\quad = (\mathcal{J}^* + R_{\lambda-i\epsilon}(H_0)V^*)P(\lambda,\epsilon)(\mathcal{J}+VR_{\lambda-i\epsilon}(H_0)) \\ &\quad = \frac{\epsilon}{\pi}(\mathcal{J}^* + R_{\lambda-i\epsilon}(H_0)V^*)R_{\lambda+i\epsilon}(H)\mathcal{J}R_{\lambda-i\epsilon}(H_0). \end{aligned} \tag{5.48}$$

By defining new operators $T_\pm$ as

$$T_+(\lambda+i\epsilon) = \mathcal{J}^*V - V^*R_{\lambda+i\epsilon}(H)V \tag{5.49}$$

$$T_-(\lambda+i\epsilon) = V^*\mathcal{J} - V^*R_{\lambda+i\epsilon}(H)V, \tag{5.50}$$

the definition of the resolvent operator 5.38 gives the transformation

$$\begin{aligned} &\frac{\epsilon}{\pi}(R_{\lambda+i\epsilon}(H)\mathcal{J}R_{\lambda-i\epsilon}(H_0)f_0, [\mathcal{J}+VR_{\lambda+i\epsilon}(H_0)]g_0) \\ &\quad = \frac{\epsilon}{\pi}(\mathcal{J}R_{\lambda-i\epsilon}(H_0)f_0, R_{\lambda-i\epsilon}(H)[\mathcal{J}+VR_{\lambda-i\epsilon}(H_0)]g_0) \\ &\qquad - 2i\epsilon(R_{\lambda+i\epsilon}(H)\mathcal{J}R_{\lambda-i\epsilon}(H_0)f_0, VP(\lambda,\epsilon)g_0) \\ &\quad = \frac{\epsilon}{\pi}(\mathcal{J}R_{\lambda-i\epsilon}(H_0)f_0, \mathcal{J}R_{\lambda-i\epsilon}(H_0)g_0) - 2\pi i(T_-(\lambda+i\epsilon)P(\lambda,\epsilon)f_0, P(\lambda,\epsilon)g_0). \end{aligned} \tag{5.51}$$

By means of equations 5.48 and 5.51, the representation 5.37 can be rewritten as

$$(E(X)\mathcal{W}_- f_0, \mathcal{W}_+ g_0) = (E_0(X)\mathcal{W}_\pm f_0, g_0) - 2\pi i\int_X \lim_{\epsilon\to 0}(T_\pm(\lambda+i\epsilon)P(\lambda,\epsilon)f_0, P(\lambda,\epsilon)g_0)d\lambda. \tag{5.52}$$

Under the decomposition of $\mathcal{H}_0^{(a)}$ in equation 5.23, we have the association $f_0 \leftrightarrow \tilde{f}_0(\lambda)$, and a family of bounded operators $u_\pm(\lambda) : \mathfrak{h}_0(\lambda) \to \mathfrak{h}_0(\lambda)$. This then gives rise to the identity

$$(E(\Lambda)\mathcal{W}_- f_0, \mathcal{W}_+ g_0) = (E_0(\Lambda)\mathcal{W}_\pm^{(0)} f_0, g_0) + \int_\Lambda ((S(\lambda) - u_\pm(\lambda))\tilde{f}_0(\lambda), \tilde{g}_0(\lambda))d\lambda. \tag{5.53}$$

Since $\Lambda$ is arbitrary, comparison with equation 5.52 yields

$$([S(\lambda) - u_\pm(\lambda)]\tilde{f}_0(\lambda), \tilde{g}_0(\lambda)) = -2\pi i \lim_{\epsilon\to 0}(T_\pm(\lambda+i\epsilon)P(\lambda,\epsilon)f_0, P(\lambda,\epsilon)g_0) \tag{5.54}$$

for a.e. $\lambda \in \hat{\sigma}_0$. We now assume that $\lim_{\epsilon\to 0} T_\pm(\lambda+i\epsilon)$ exists, and that $P_0T_\pm(\lambda + i0)P_0$ has a well-defined kernel $t_\pm(\mu,\nu;\lambda+i0)$, where $\lambda$, $\mu$, and $\nu$ independently run through a set of full measure in $\hat{\sigma}_0$. Then the value of the kernel of $P_0T_\pm(\lambda+i0)P_0$ is well defined on the diagonal $\lambda = \mu = \nu$, with

$$\lim_{\epsilon\to 0}(T_\pm(\lambda+i\epsilon)P(\lambda,\epsilon)f_0, P(\lambda,\epsilon)g_0) = (t_\pm(\lambda,\lambda;\lambda+i0)\tilde{f}_0(\lambda), \tilde{g}_0(\lambda)). \tag{5.55}$$

Equation 5.54 yields

$$S(\lambda) = u_\pm(\lambda) - 2\pi i t_\pm(\lambda,\lambda;\lambda+i0) \tag{5.56}$$

for a.e. $\lambda \in \hat{\sigma}_0$. If $\mathcal{W}_\pm$ is an isometry on $\mathcal{H}_0^{(a)}$, then $u_\pm(\lambda) = I(\lambda)$, so

$$S(\lambda) = I(\lambda) - 2\pi i t(\lambda,\lambda;\lambda+i0). \tag{5.57}$$

In the case where $\mathcal{J} = I$, we can define $T = T_+ = T_-$ and $t = t_+ = t_-$, so the scattering matrix further reduces to

$$S(\lambda) = I(\lambda) - 2\pi i t(\lambda, \lambda; \lambda + i0).^{17} \tag{5.58}$$

## 6. Closing remarks

We have formulated a solution to the Schrodinger equation in terms of the spectral projection $E_0(\Lambda)$ as

$$u(t) = \exp(-iHt)f = \int_{-\infty}^{\infty} \exp(-i\lambda t) dE_0(\Lambda) f. \tag{6.1}$$

In this formulation, the temporal variable $t$ receives special treatment, but in the fully relativistic Dirac equation

$$i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0, \tag{6.2}$$

where $\gamma^\mu$ are the gamma matrices, the spatial and temporal variables are treated equally. This is because we do not consider the limit just as $t \to \pm\infty$, but rather the limit as both $z, t \to \pm\infty$. In the case of scattering over $\mathbb{R}^n$, we studied the asymptotics as $\|z\| \to \infty$, and in the case of scattering over non-Euclidean manifolds, we studied the asymptotics as $t \to \pm\infty$. In both cases, we used the stationary phase lemma to obtain an expansion for the solution, but in the larger picture, we want to take both limits simultaneously, with $\frac{z}{t}$ fixed as we take the limit. We saw that this limit is equivalent to taking the limit of the resolvents $R_z(H)$ as the spectral parameter $z$ approaches the real axis, and the spectral projection is given as a difference in the limits of the resolvents.

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[17] This discussion follows section 2.8 in Yafaev.