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Algebraic completeness of $\frac{Open([0,1])}{Null}$

In a recent lecture[0], D. Scott defines a Heyting algebra from equivalence classes of open sets of the unit interval of reals, namely $\frac{Open([0,1])}{Null}$, to give every proposition a probability. By modifying the proof of topological completeness of the unit interval for S4 given in [1], we show completeness for $\frac{Open([0,1])}{Null}$. The proof in [1] used a mapping from [0,1] to an arbitrary intuitionistic Kripke model K such that it is both open and continuous (from the topology of the unit interval to the topology of R-closed sets on K). We define a similar mapping from $\frac{Open([0,1])}{Null}$ to K which has properties similar to being open and continuous and such that the preimage of each world defines a thick cantor set.

Definition 1: For A, B $\subseteq$ [0,1], define A $\sim$ B if and only if A Δ B = (A\B) $\cup$ (B\A) has measure zero(m(A Δ B) = 0). Also define the equivalence class

$$\widetilde{A} = \{\mathrm{B} \mid \mathrm{B} \in \mathrm{Open}([0,1]),\ \mathrm{B} \sim \mathrm{A}\ \}$$

and collection of classes

$$\frac{Open([0,1])}{Null} = \{\widetilde{A} \mid \mathrm{A} \in \mathrm{Open}([0,1])\}$$

Definition 2: For any $\widetilde{A}$, $\widetilde{B} \in (\frac{Open([0,1])}{Null})$, $\widetilde{A} \leq \widetilde{B}$ if and only if there exists an $S_A \in \widetilde{A}$ such that $S_A \subseteq$ B.

Lemma 3: The ordering $\leq$ is well defined
Proof: Let $\widetilde{A} \leq \widetilde{B}$. Then there is a $S_A \in \widetilde{A}$, $S_A \subseteq$ B. Suppose C is an open set such that m(C Δ B)=0. Take $S'_A = S_A \cap$ C. S'_A is open and $S'_A \subseteq$ C. Also, m(A Δ S'_A)=0 since

$$\begin{aligned} A\Delta S'_A &= (A\backslash S'_A) \cup (S'_A\backslash A) \\ &= (A\backslash(S_A \cap C)) \cup ((S_A \cap C)\backslash A) \\ &= (A\backslash S_A) \cup (A\backslash C) \cup ((S_A \cap C)\backslash A) \end{aligned}$$

where m((A \ S_A)) = m((A \ C)) = m((($S_A \cap$ C) \ A)) = 0.

$\dashv$

Next, we show that $(\frac{Open([0,1])}{Null},\leq)$ is actually a Heyting algebra:

Lemma 4: For any $\widetilde{A}$, $\widetilde{B} \in \frac{Open([0,1])}{Null}$ and lattice operations $\sqcup$, $\sqcap$ and least element $\perp$ defined from $\leq$,
(a)$\widetilde{A} \sqcup \widetilde{B} = \widetilde{A \cup B}$,
(b)$\widetilde{A} \sqcap \widetilde{B} = \widetilde{A \cap B}$,
(c)$\perp = \widetilde{\emptyset}$,

Proof: For (a), we have A $\subseteq$ A $\cup$ B and B $\subseteq$ A $\cup$ B, so $\widetilde{A \cup B}$ is an upper bound. For other upper bound U, there exist $S_A \in \widetilde{A}$, $S_B \in \widetilde{B}$, such that $S_A \subseteq U$ and $S_B \subseteq U$. But m(($S_A \cup S_B$) Δ (A $\cup$ B))= 0 and ($S_A \cup S_B$) is open, so ($S_A \cup S_B$) $\in \widetilde{A \cup B}$.
For (b), repeat arguement (a) with $\cap$ replacing $\cup$, and upper bound replaced by lower bound.
For (c), $\emptyset \in \widetilde{\emptyset}$, and since $\emptyset \subseteq$ A for any set A, $\widetilde{\emptyset}$ is the least element.

$\dashv$

Moreover, for every $\widetilde{A}$, $\widetilde{B} \in \frac{Open([0,1])}{Null}$, there is a greatest $\leq$ element $\widetilde{A} \to \widetilde{B}$ such that $\widetilde{A} \sqcup (\widetilde{A} \to \widetilde{B}) \leq \widetilde{B}$:

Lemma 5: Let

$M_{A,B}$ = { $\widetilde{T} \in \frac{Open([0,1])}{Null}$ | there exists X$\in \widetilde{T}$ such that $X \subseteq$ ([0,1] $\setminus$ A)$\cup$ B }

Then $M_{A,B}$ has a maximal element and this element is $\widetilde{A} \to \widetilde{B}$.

Proof: $\leq$ is a partial ordering on $M_{A,B}$ and arbitrary chain C $\subseteq M_{A,B}$ has upper bound $\widetilde{U}$ where U = $\bigcup_{\widetilde{D}\in C}$D. So by Zorn's lemma, a $\leq$-maximal element $\widetilde{M}$ exists. $\widetilde{M}$ satisfies $\widetilde{A} \sqcap \widetilde{M} \leq \widetilde{B}$: For X $\in \widetilde{M}$ such that X $\subseteq$ ([0,1] $\setminus$ A)$\cup$ B,

$$\begin{aligned} A \cap X &\subseteq A \cap (([0,1]\setminus A) \cup B) \\ &= (A \cap ([0,1]\setminus A)) \cup (A \cap B) \\ &\subseteq B \end{aligned}$$

Thus, $\widetilde{A} \sqcap \widetilde{M} \leq \widetilde{B}$. $\widetilde{M}$ is also the greatest such element: $\widetilde{A} \sqcap \widetilde{Y} \leq \widetilde{B}$ if and only if there is a Y $\in \widetilde{Y}$ such that A $\cap$ Y $\subseteq$ B. Then (A $\cap$ Y) $\cup$ (Y$\setminus$A) $\subseteq$ B $\cup$ (Y$\setminus$A). But then this means Y $\subseteq$ B $\cup$ (Y$\setminus$A). Thus, Y $\subseteq$ B $\cup$ ([0,1]$\setminus$A). But $\widetilde{M}$ is the maximal $\leq$ element containing such Y. Thus, $\widetilde{M}$ is $\widetilde{A} \to \widetilde{B}$.

$\dashv$

By Lemma 4 and 5, ($\frac{Open([0,1])}{Null}$,$\leq$) is a Heyting algebra. Recall the following defintion of algebraic semantics[2]:

Definition 6: Let $(X, \leq)$ be a Heyting algebra. Then a valuation V is a function defined from propositional variables to X and has the following extension to all propositional formulas:

$$\begin{gathered} V(\alpha \wedge \beta) = V(\alpha) \sqcap V(\beta) \\ V(\alpha \vee \beta) = V(\alpha) \sqcup V(\beta) \\ V(\alpha \to \beta) = V(\alpha) \to V(\beta) \\ V(\bot) = \bot \end{gathered}$$

We can then define truth in model $(X, \leq)$ under valuation V as $(X,\leq,V)\models$ A if and only if V(A) = $\top$. We will also say that A is *valid* in $(X, \leq)$, denoted $(X, \leq)\models$ A, if A is true under any valuation.

Now we can interpret the measure m(V(α)) as the probability of the sentence α. In particular, $\perp$ is assigned probability zero and any true statement under the valuation is assigned probability 1.

Next we prove a completeness result: a formula is derivable in the intuitionistic propositional calculus NJp if and only if it is valid (algebraically) in $(\frac{Open([0,1])}{Null},\leq)$. Recall that a formula is derivable in NJp if and only if it is valid in every finite pointed intuitionistic Kripke model. Moreover, we know the weaker result, a formula is derivable if and only if it is valid in all Heyting algebras[2]. Thus, to show algebraic completeness for $(\frac{Open([0,1])}{Null},\leq)$, it suffices to show that if A is valid in this algebra, then it is valid in every finite pointed intuitionistic Kripke model. But because we are dealing with algebraic semantics, we transform every Kripke frame K in a standard way[2] into an algebra B = (B_K,$\leq$), such that a formula is valid in K if and only if it is valid in B.

Definition 7: Let K= <W,R,V> be a Kripke model and V a valuation. We say that A $\subseteq$ W is R-closed if and only if for any w $\in$ A and any world y $\in$ W, wRy implies y $\in$ A. We then define

$$B_K := \{ \text{A} \subseteq \text{W} \mid \text{A is R-closed} \}$$
$$\text{A} \leq \text{B} := \text{A} \subseteq \text{B}$$
$$\text{V'(p)} := \{\text{w} \mid \text{V(p,w)=1}\}$$

In the algebra $(B_K,\leq)$, operations $\cup$, $\cap$, are lattice join and meet, $\emptyset$ is the least element, and M$\rightarrow$N is the largest R-closed subset of (W$\setminus$M) $\cup$ N.

$(B_K, \leq)$ has the following relation with Kripke structure K:

Theorem 8: For every propositional formula A, and Kripke model K= <W,R,V>,

$$(B_K, \leq, V')\models \text{A if and only if (K,V)} \Vdash \text{A}$$

Consequently, $(B_K, \leq)\models$ A if and only if K $\Vdash$ A. The proof of the preceeding theorem and the verification that B_K is indeed an algebra can be found in [2].

So if a formula A is not derivable in some Kripke model, then it is not valid in its induced algebra and valuation. Then, using the induced algebra, we build a valuation that refutes A in $(\frac{Open([0,1])}{Null},\leq)$, which implies completeness. To describe relations between algebras (similar to how open and continuous mappings relate topological spaces), we use Heyting algebra morphisms:

Definition 9: Let $(X_1, \leq_1), (X_2, \leq_2)$ be two Heyting algebras. A morphism f between two Heyting algebras is a function from X_1 to X_2 such that for any x, y $\in X_1$,

$$\begin{aligned} f(\bot) &= \bot \\ f(\top) &= \top \\ f(x \sqcup y) &= f(x) \sqcup f(y) \\ f(x \sqcap y) &= f(x) \sqcap f(y) \\ f(x \to y) &= f(x) \to f(y) \end{aligned}$$

Theorem 10: Let f be a morphism from $(X_1, \leq_1)$ to $(X_2, \leq_2)$ and V_1 be a valuation for algebraic semantics on X_1. Also let $V_2(\alpha) = f(V_1(\alpha))$ for all formulas α.

$$\text{If } (X_1, \leq_1, V_1) \models \alpha, \text{ then } (X_2, \leq_2, V_2) \models \alpha.$$

Proof: Induction on the structure of α.

$\dashv$

The principal step of our completeness proof is constructing a morphism from **equivalence classes** of open sets to R-closed sets. Just like in the proof of topological completeness[1], we build a coloring scheme on the unit interval (i.e a mapping from the unit interval to Kripke worlds) and then extend it into a morphism on open sets. We will still need the coloring be a continuous and open mapping between the standard topology defined on the unit interval, and the Kripke induced topology where the open sets are still the R-closed sets. This is because we are using the contrapositive of Theorem 10 (X_2 is B_K and X_1 is $\frac{Open([0,1])}{Null}$), so given $V_2(p)$, we have to construct $V_1(p)$ so that it maps to $V_2(p)$ under morphism S. To ensure that we can find the open sets that map to these specific R-closed sets, we will need to use continuity and openess.

Moreover, we design the coloring so that the preimage of each world has measure greater than zero. This will ensure that when we move from open sets to equivalence classes of open sets, the coloring scheme remains well defined (i.e, if the coloring scheme maps two open sets to different R-closed sets, then the two open sets must differ by a set of measure greater than zero).

To define the main coloring, we are going to define a sequence of partitions of the unit interval and define a coloring for each partition. On the first partition, we are going to break [0,1] into N intervals and color the even intervals with the root world and the odd intervals with a different related world. On the second partition, we are going to take each interval labeled w and consider it the new root world. We will then break each interval of the first partition into N^2 intervals and label, once again, the even intervals with the root node and the odd intervals with a different related world (and when we run out of different worlds, we label with the root node). Likewise, the third partition will break each interval of the second partition into N^3 pieces and so forth. The main coloring will be the pointwise limit of the sequence of these colorings.

Definition 11: Define for interval [a,b] and positive integer L, the partition of [a,b] into L pieces as

$$\Psi^L_{[a,b]} = \{x_0^{[a,b]}, x_1^{[a,b]}, ..., x_L^{[a,b]}\}$$

where $x_i^{[a,b]}$ = a + i$\frac{b-a}{L}$, i = 0, 1, ... L.

Then for a Kripke model with domain W such that |W|=N, recursively define the sequence of partitions as

$$\Psi_1 = \Psi^{2N-1}_{[0,1]}$$
$$\Psi_{n+1} = \bigcup_{x_i^{[a,b]} \in \Psi_n} \Psi^{(2N-1)^{n+1}}_{[x_i^{[a,b]}, x_{i+1}^{[a,b]}]}$$

Henceforth, we call Ψ_n the n-th partition and its elements the *edgepoints* of the n-th partition.

Definition 12: Let (K,W,R) be a finite Kripke model with a root. Fix a labelling 0, 1, 2, ... N-1 such that wRy implies w $\leq$ y (in particular 0 is the root). For every w $\in$ K, define

$$K_w := \{w' | wRw'\} = \{\ k_{w_0}, k_{w_1}, ..., k_{w_{p_w}}\ \}$$

where k_{w_0} = w and $k_{w_0} < k_{w_1} < k_{w_2} ... < k_{w_{p_w}}$. For every n > p_w, extend this to a sequence of size n by padding with the root w, i.e,

$$k_{w_{(p_w+j)}} := \text{w for j} \geq 1.$$

Using Definition 11 and 12, we can now define the sequence of colorings S_1, S_2,...of partitions:

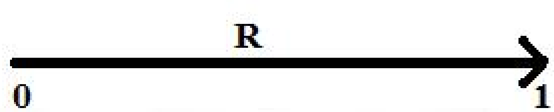
Definition 13: For S_k: [0,1]$\rightarrow$W,

$$S_1(x) = \begin{cases} 0 \text{ if x} \in \Psi_1 \text{ or there exists an i such that i is even and for } x_i^{[0,1]} \in \Psi_1,\ x_i^{[0,1]} < \text{x} < x_{i+1}^{[0,1]} \\ \frac{i+1}{2} \text{ where i is such that } x_i^{[0,1]} \in \Psi_1 \text{ and } x_i^{[0,1]} < \text{x} < x_{i+1}^{[0,1]} \text{ otherwise} \end{cases}$$

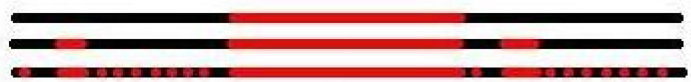
and for n>1 ,

$$S_{n+1}(x) = \begin{cases} S_n(x) \text{ if x} \in \Psi_n \text{ or there exists a, b, and i where i is even and there is an } x_i^{[a,b]} \in \Psi_n \\ \qquad \text{such that } x_i^{[a,b]} < \text{x} < x_{i+1}^{[a,b]} \\ \text{k}_{w_j} \text{ where w} = (S_n(\text{x})) \text{ and j} = \frac{i+1}{2} \text{ where a, b, and i are such that there is} \\ \qquad x_i^{[a,b]} \in \Psi_n \text{ and } x_i^{[a,b]} < \text{x} < x_{i+1}^{[a,b]} \text{ otherwise} \end{cases}$$

So, for example, the two element Kripke model,



has the following first three colorings, where black denotes color 0 and red denotes color 1:



We now check that the pointwise limit of the sequence of colorings exists:

Lemma: 14 For every x $\in$ [0,1], there is an n such that

$$S_k(\text{x}) = S_n(\text{x}) \text{ for all k} \geq \text{n.} \qquad (*)$$

Proof: Fix x. By construction, $S_n(x) R S_{n+1}(x)$ and since |W| is finite and R is a partial ordering, the value of $S_n(x)$ must stabilize to some world.

$\dashv$

We define our coloring scheme S as the "stabilized points" of our coloring sequence:

Definition 15: Define n(x) as the least n such that (*) holds. The coloring scheme S: [0,1] $\rightarrow$ W is then

$$S(x) = \; S_{n(x)}(x)$$

Note, when we return to algebras, we will overload the notation and define S: $\frac{Open([0,1])}{Null} \rightarrow B_K$ where

$$S(A) = \; \{S(x)|x \in A\}$$

We proceed to show that S(x) is a continuous and open mapping to the topology induced by <K,R,W>. But first, we need one short lemma:

Lemma 16: If x is an edgepoint of the m-th partition, then S_m(x) = S(x).
Proof: An edgepoint of the m-th partition is an edgepoint of all future partitions. Thus S_n(x) = S_m(x) for all n > m, and so S_m(x) = S(x).

$\dashv$

Theorem 17: S is continuous: for every x $\in$ [0,1], there exists a $\delta > 0$ such that for every x' $\in$ [0,1], if |x-x'|<δ, S(x') $\in$ $K_{S(x)}$
Proof: If x is not an edgepoint, then let I be an interval defined by the n(x)-th partition such that x $\in$ interior(I). Since $S_n(x')$ = S(x) for all x' $\in$ interior(I), S(x)RS(x') for every x' $\in$ I, i.e, x' $\in$ $K_{S(x)}$. Thus choose δ such that $I_\delta(x) \subseteq$ I, where $I_\delta(x)$ = {y| |y-x| < δ }. Now consider the case x is an edgepoint. If n(x) = 1, the edgepoint is colored 0 and the result trivially holds since the model is pointed. If n(x) $\neq$ 1, then x cannot be an edgepoint of partition $S_{n(x)}$ (since an edgepoint takes on the coloring of the previous partition, contradicting the minimality of n(x)). Thus, x is contained in some interval I such that for any x' $\in$ I, $S_{n(x)}(x') = S(x)$, so the result follows from the same arguement as in the case x is not an edgepoint.

⊣

Theorem 18: S is open: for every x ∈ [0,1], every w' ∈ $K_{S(x)}$ and ϵ >0, there exists a x' ∈ [0,1] such that |x-x'| < ϵ and S(x')=w'.
Proof: If x is not an edgepoint, then on any partition after the n(x)-th partition, x is contained in an interval colored S(x). Choose n such that n > n(x) and the edgepoints of Ψ_n defines an interval I such that x ∈ I and I ⊆ $I_\epsilon(x)$. Then on the next partition, Ψ_{n+1}, one of the intervals I' ⊆ I is colored w'. On partition Ψ_{n+2}, edgepoints in I are colored w', so let x' be one of these edgepoints. Then |x-x'| < ϵ and S(x')=w' by Lemma 16. If x is an edgepoint of some partition n, choose m such that m > n and the difference between adjacent edgepoints of Ψ_m is less than $\epsilon/2$. Since either the left or right interval of x is colored S(x) by construction, on the next partition, m+1, one of the intervals I' ⊆ $I_\epsilon(x)$ is colored w'. On partition m+2, edgepoints in I' are colored w', so let x' be one of these edgepoints. Once again, by Lemma 16, we have the same conclusion.

⊣

Note that these two properties imply topological completeness of the unit interval[1]. But before we can use these two properties to construct $V_2(p)$ and apply Theorem 10, we have to prove S(x) is a well defined morphism from ($\frac{Open([0,1])}{Null}$,≤) to (B_K, ≤). We will need the following lemma to prove S(x) is well defined:

Lemma 19: S(x) will be well defined if the following property holds: for every w ∈ W, every x ∈ [0,1], every δ >0, if x ∈ S^{-1}(w) and $I_\delta(x) \cap S^{-1}$(w) ≠ ∅, then m($I_\delta(x) \cap S^{-1}$(w))>0.
Proof: Suppose S(x) was still ill-defined. Let A, A' ∈ $\widetilde{A}$ and S(A)= K, and S(A') = K' with some w ∈ K but w ∉ K'. Then the set (S^{-1}(w) ∩ A) ⊆ A'ΔA. Since A is open and A ∩ S^{-1}(w) ≠ ∅ because S(A)=K, we have m(S^{-1}(w)∩ A)>0, implying m(A' Δ A) >0, a contradiction.

⊣

To use this lemma, we will first prove for any S(x), that the preimage of 0 defines a set of positive measure. We will then use this result, as well as the symmetry of the coloring, to show that on any open interval such that the intersection with the preimage of w is nonempty, this intersection has positive measure. Particularly, because the preimage of 0 defines a set of positive measure and the preimage of w contains a scaled copy of the 0 case for some Kripke model with root 0 replaced by root w, then the preimage of w contains a set of positive measure.

Lemma 20: m(S^{-1}(0)) > 0
Proof: Since the measure is trivially 1 when N = 1, we can assume N ≠ 1. To calculate the measure, we can interpret any nonzero coloring as permanently

"removing" intervals since the relation R is antisymmetric. On the first partition, we remove N-1 intervals (corresonding to nonroot worlds) of size $\frac{1}{2N-1}$. On the n+1 partition, we remove, for each interval colored zero in the nth partition, N-1 intervals. Since we can show, through induction, that the size of intervals in the nth partition is $\prod_{i=1}^{n} \frac{1}{2N-1^{i}}$ and the number of zeros in partition n for n > 1, is $\prod_{i=1}^{n-1} (2N-1)^{i} - (N-1)$, we can calculate a lower bound for the measure as follows:

$$
\begin{aligned}
m(S^{-1}(0)) &= 1 - \text{m(Intervals removed in the first partition)} \\
&- \sum_{n=2}^{\infty} \text{(Number of intervals removed in partition n)*(N-1)*(Size of interval in partition n)} \\
&= 1 - \frac{N-1}{2N-1} - (N-1) * \sum_{n=2}^{\infty} \prod_{i=1}^{n-1} ((2N-1)^{i} - (N-1)) * \prod_{i=1}^{n} \frac{1}{2N-1^{i}} \\
&= 1 - \frac{N-1}{2N-1} - (N-1) * \sum_{n=2}^{\infty} \frac{1}{(2N-1)^{n}} * \prod_{i=1}^{n-1} 1 - \frac{N-1}{(2N-1)^{i}} \\
&\geq 1 - \frac{N-1}{2N-1} - (N-1) * \sum_{n=2}^{\infty} \frac{1}{(2N-1)^{n}} * \prod_{i=1}^{n} \frac{1}{2N-1^{i}} \\
&= 1 - \frac{N-1}{2N-1} - (N-1) * \sum_{n=2}^{\infty} \frac{1}{(2N-1)^{n}} \\
&= 1 - \frac{N-1}{2N-1} - \frac{N-1}{(2N-1)^{2}} * \frac{1}{1 - \frac{1}{2N-1}} \\
&= 1 - \frac{N-1}{2N-1} - \frac{N-1}{(2N-1)(2N-2)} \\
&= .5
\end{aligned}
$$

⊣

We can then extend this result to prove:

Lemma 21: For every w ∈ W, every x ∈ [0,1], every δ >0, if x ∈S^{-1}(w) and $I_{\delta}(x) \cap S^{-1}$(w), then m($I_{\delta}(x) \cap S^{-1}$(w))>0.
Proof : Let x, w, and δ be such that the above conditions hold. By construction of S(x), there is some partition that produces an interval I of size $\prod_{i=1}^{n} \frac{1}{(2N-1)^{i}}$ such that $S_n(I) = w, x \in I, I \subseteq I_{\delta}(x)$. Restricting our attention to the first$(2|K_w| - 1) * \prod_{i=1}^{n+1} \frac{1}{(2N-1)^{i}}$ of this interval, say I', notice that w is

now the "0" node of a scaled unit interval. Thus, by the preceeding lemma, $S^{-1}(w) \cap$ I' has positive measure, and since I' $\subseteq I \subseteq I_\delta(x)$, we have m$(S^{-1}$(w) $\cap$ $I_\delta(x))>0$.

$\dashv$

Thus, Lemma 19 is satisfied and so S(x) is well defined. S(x) is a morphism follows simply because the join and meet in both algebras are union and intersection, and because S(x) = {S(y) : y $\in$ x }. Thus,

Theorem 22: S(x) is a well defined morphism from $(\frac{Open([0,1])}{Null}, \leq)$ to $(B_K, \leq)$

To finish the proof, form B_k and S(x) from a Kripke model $<K,R,V,> \not\models A$. We can then find a mapping $V_1(p)$ s.t S($V_1(p)$)=V_2(p) = R where R is formed from union or intersection of K_{w_1}, K_{w_2}, ...: use continuous property of S(x) to find open intervals such that, for the i-th interval, there is a point on that interval that maps to w_i and the entire interval maps into each K_{w_i} (which exists because on the second partition, for each world, there is a set of edgepoints that are specifically colored to that world and by Lemma 16, when endpoint x is specifically colored w, S(x)=w. Then by open property, the open interval maps to the entire K_{w_i}. Thus, under this $V_1(p)$ and $V_2(p)$ and applying Theorem 10, we have:

Theorem 23: If $<K,R,V,> \not\models A$, then $(\frac{Open([0,1])}{Null}, \leq) \not\models$ A.

And because a formula is valid in NJp if and only if it is valid in all Heyting algebras, we have completeness:

Corollary 24: A formula is derivable in NJp iff it is valid in $(\frac{Open([0,1])}{Null}, \leq)$

Future Work

Now that we have proven algebraic completeness of $(\frac{Open([0,1])}{Null}, \leq)$ with respect to propositional formulas, next, we would like to add a $\Box$ modality to our language (interpreted as "interior"). We are currently extending the proof to show completeness of this algebra with S4: not only do we have to adapt our semantics and definition of morphism, but the Kripke models are only reflexive and transitive. Instead of single stabilized worlds, we will need to consider stabilized *clusters* of worlds. The final coloring is then a representative of each cluster. The rest of the proof, using edgepoints to prove openess and continuity and thick cantor sets for well definedness, remain essentially the same. After this, we will work on the first order case.

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