On the Near-Optimality of The Reverse Deferred Acceptance Algorithm

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Abstract

This mathematics honors thesis provides an analysis of the optimality of the "Reverse Deferred Acceptance Algorithm" created by Paul Milgrom and Illya Segal for the Federal Communications Commission auction for spectrum licenses. We begin by first providing a general literature review of combinatorial auctions, including algorithmic attempts at solving such optimization problems. A description and an analysis of the reverse deferred acceptance mechanism created by Milgrom and Segal follows. While the mechanism has many interesting economic properties, we focus primarily on the algorithmic properties of the mechanism.

1 Introduction

Auctions have a rich and long history, recorded as early as 500 B.C. Since antiquity, people have used auctions to sell various kinds of items: flowers to books to paintings to carpets, financial assets, and even empires.[1] A rigorous analysis of auctions as games of incomplete information began with

[1] *Auction Theory*, Vijay Krishna

the seminal work of William Vickrey in 1961. Since then there has been much development in the field, ranging from discoveries of new theoretical properties of traditional auctions to the creation of completely new types of auction mechanisms. More importantly, perhaps, is the study of when to use auctions and for what types of allocation problems would auctions be optimal. While auctions have commonly been used to sell single items, recently a more complex auction type has been introduced : combinatorial auctions. Combinatorial auctions are auctions where bidders bid on and compete for bundles of goods rather than just a singular object.

The following section introduces the notion of combinatorial auctions and describes some of the developments in the field. We then proceed to explain the Federal Communication Commission's combinatorial auction for spectrum licenses. Subsequently, I describe the reverse deferred acceptance algorithm in mathematical detail and provide proofs of a few of its applicable economic properties. Section 5 provides the background needed to understand the proofs regarding the optimality of the algorithm. Sections 6 and 7 demonstrate results regarding perfect matchings within graphs, which has interesting applications to the topic of channel assignment. Finally, sections 8 and 9 discuss the near-optimality of the algorithm.

2 Combinatorial Auctions

Combinatorial auctions are auctions that permit bidders to bid on bundles or combinations of products. Such auctions have garnered the interest of mathematicians, computer scientists, and economists. In addition to having many interesting mathematical properties (far more than we can discuss in this thesis), combinatorial auctions have many practical purposes as well: airport takeoff and landing slots, bus routes, freight transportation services, and, of course, radio spectrum.

We begin by outlining the general formulation of such problems. As stated in Vohra and DeVries (2000), the most important problem in combinatorial auctions is determining the set of winning bids.

Let N be the set of bidders and let M be the set of m distinct objects or assets. Since bidders have bids (and preferences) over combinations of goods, let $b^j(S)$ be the bid that agent $j \in N$ has submitted for $S \subset M$.

$$\max \sum_{S \subset M} b^j(S) x_S$$
$$\text{such that } \sum_{S \ni i} x_S \leq 1 \ \forall i \in M$$
$$\text{where } x_S = 0, 1 \ \forall S \subset M$$

One glaring problem with this formulation occurs when the good are substituted. As an example, suppose in the eyes of bidder j, goods A_1 and A_2 are perfect substitutes. Then we have that $b^j(A_1) + b^j(A_2) > b^j(A_1 \cup A_2)$. However, with the above model, it could very well be the case that the optimal solution has bidder j receiving A_1 and A_2, thereby recording a revenue of $b^j(A_1) + b^j(A_2)$. To rewrite the combinatorial auction problem so as it correctly models non-superadditive bids, we have the following linear program. Define a function $y(S, j) = 1$ iff $S \subset M$ is allocated to agent $j \in N$; otherwise, $y(S, j) = 0$. Thus, we have the following optimization problem:

$$\max \sum_{j \in N} \sum_{S \subset M} b^j(S) y(S, j)$$
$$\text{such that } \sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \ \forall i \in M$$
$$\sum_{S \subset M} y(S, j) \leq 1 \ \forall j \in N$$
$$y(S, j) = 0, 1 \ \forall S \subset M, j \in N$$

Unlike the first model, this one does not require bids to be superadditive. The first constraint ensures that two different bundles of goods, S_1 and S_2, can not be assigned if $S_1 \cap S_2 \neq \emptyset$. The second constraint reflects the fact that no agent can receive more than one bundle of goods.

The integer-program combinatorial auction is actually a specific case of a more well known linear program, The Set Packing Problem. The Set packing The Set Packing Problem can be written as follows:

$$\max \sum_{j \in V} c_j x_j$$

$$\text{such that } \sum_{j \in V} a_{ij} x_j \leq 1 \ \forall i \in M$$

$$x_j = 0, 1 \ \forall j \in V$$

It is known that the Set Packing Problem is NP-Hard. In other words, there is no known polynomial time algorithm that solves the Set Packing Problem. However, there has been much literature discussing solvable *instances* of the optimization problem. For example, if the polyhedron formed by the constraint has integral extreme points, than from basic theorems in linear programming, the Set Packing Problem can be solved in polynomial time.[2][3] Other cases where the Set Packing Problem can be solved in polynomial time can be found in DeVries and Vohra (2000). Now, while the exact solution may not be feasible to calculate, approximate solutions may be desired; finding approximate solutions is, obviously, sometimes more viable. Many approximation methods have been explored over the last thirty years.[4] Recently, Borodin and Lucier (2010) reexamined the application of the traditional greedy algorithm to a specific type of combinatorial auction (when bidders could receive up to a total of k objects). The search for approximation algorithms has been exhaustive, and as DeVries and Vohra point out, "anything one can think of for approximating the Set Packing Problem has probably been thought of".

[2] The polyhedron is $\left\{x \colon \sum_{j \in V} a_{ij} x_j \leq 1 \ \forall i \in M, x_j \geq 1 \ \forall j \in V\right\}$

[3] For an in-depth look into Linear Programming and Optimization, one can consult Ye and Luenberger's *Linear and Nonlinear Programming*

[4] Examples of such approximation methods are the following: steepest ascent, genetic algorithms, probabilistic search, and simulated annealing. Examinations of approximation methods for combinatorial auctions can be found in Arkin and Hassen (1998), Dobzinski, Nisan, and Schapira (2005), and Dobzinski and Schapira (2006).

3 FCC Auction for Spectrum

The most well known combinatorial auction to date is the The Federal Communications Commission (FCC) spectrum auction. The goal of this auction is to price and assign spectrum licenses. The FCC first received auction authority in 1993, and ran its first spectrum auction in 1994.[5] These auctions, however, have only been utilized to assign *unused* spectrum. Using an auction for such a purpose is almost universally agreed upon by economists. A detailed account of the history behind the FCC's decision to use an auction mechanism to allocate spectrum can be found in a paper by Thomas Hazlett (1998). The auction that has been used has been the simultaneous ascending auction, first proposed by Paul Milgrom, Robert Wilson, and Preston McAfee. The rules for these auctions have been determined primarily through economic theory. A simultaneous ascending auction is an auction for multiple items in which bidding occurs in rounds. At each round, the bidders submit sealed bids for their desired items. At the end of each round, "results" are posted, which list the highest bid that was offered for each item (along with the corresponding bidder who made the offer). The standing high bid and bidder are replaced in subsequent rounds if and only if a higher bid is submitted. In addition, for each item there is a minimum bid requirement that is a function of the standing high bid. An in-depth analysis and discussion of the properties of the simultaneous ascending auction (i.e. competitive equilibrium and strategy-proofness) can be found in *Putting Auction Theory to Work: The Simultaneous Ascending Auction* by Paul Milgrom (2000).

As was mentioned before, these auctions were created to assign unused spectrum. Recently, however, there has been a great need for a reassignment of spectrum in order to free up channels for mobile broadband networks. This will require choosing which stations to keep on air and which to leave off. There are 51 available channels and currently, stations are scattered throughout. The goal is to attempt to repackage stations into a smaller block of channels. Such a task obviously requires only allowing certain stations to stay on air and removing the rest. The FCC is concerned with finding out

[5] "Spectrum Auctions", Peter Cramton (2002)

how to minimize the cost of procuring the necessary stations in order to clear sufficiently many channels.[6] This reallocation problem is quite complex as the new channel assignments must be "interference free". In total there are approximately 130,000 interference constraints. Stations are divided up into different Digital Market Areas (DMA regions).[7] Now, no two stations in the same DMA can be assigned the same channel. Moreover, it is possible for stations in neighboring regions to be constrained similarly.

To understand the situation, it is best to model the optimization problem from a graph theoretic perspective. Let $G = (V, E)$ be the following graph:

1. The set of vertices, V, is the set of stations.

2. There is an edge between any two stations if and only if the two stations can not share the same channel. Thus, there is a complete subgraph in every DMA region.

3. Let $w : V \longrightarrow \mathbb{R}$ be a weight function such that $w(v) =$ value of station v.

If we associate a distinct color with every available channel, then the FCC's optimization problem becomes a weighted vertex coloring problem. While the set of graph-coloring problems is known to be NP-complete, algorithms developed specifically for the instances one observes in the FCC's case(s), might be faster.

Because the graph associated with the spectrum auction problem has specific, unique features to it, it seems like there could be an algorithm that approximates the optimal solution well. Recently, a novel auction mechanism has been introduced that could be potentially effective in such an environment. Paul Milgrom and Illya Segal have proposed a variation of a deferred acceptance heuristic to approximate the optimal spectrum assignment.[8]

[6] One might think that the FCC's chief concern would be to maximize the total value of the stations on the air. However, this is not the case. The reason is that the stations values are badly distorted by such factors as must-carry rights, which are valuable to the owners but result from a transfer that is not a social value.

[7] For example, New York City is considered a DMA, as is Chicago, Denver, etc.

[8] A heuristic is a term in computer science that describes a technique used to approximate the optimal solution to a problem.

4 The Reverse Deferred Acceptance Algorithm

The reverse deferred acceptance algorithm is analyzed in great detail in "Deferred-Acceptance Auctions and Radio Spectrum Reallocation Reallocation" by Milgrom and Segal (2014). The authors extensively examine the economic properties of the mechanism. As reflected in the paper, this new class of deferred acceptance auctions is one that works particularly well in the FCC environment. The algorithm is a sealed-bid mechanism that selects the winning bids by iteratively rejecting, after each round, the *active* bid with the highest *score*.[9] In the first round, all bids are active. When a bid gets rejected, it becomes *inactive*. In each round, the algorithm attaches a score to each bid. If, during a round, all remaining active bids have scores of 0, the algorithm terminates and those bids are winning. For application to the FCC's problem, a rejection of a bid implies that that stations gets to remain on the air; the acceptance of a bid implies that the FCC will purchase that station's license. Therefore, the payments are made to the *winning bidders*. The payment awarded to each winning bidder is their threshold price, which is equal to the highest bid that bidder could have submitted without changing the outcome, all else equal.[10]

We now define the algorithm rigorously. The following notation is borrowed from Milgrom and Segal (2014), the developers of the mechanism. This notation will be used in later sections as well.

Let N be the set of bidders. Let $B_i \subset (0, \infty)$ be the set of possible bids for bidder $i \in N$. In the Milgrom and Segal paper, B_i is restricted in the following ways:

1. B_i is finite $\forall i \in N$.

2. $\max B_i > v_i$ where v_i is the payment that makes bidder i indifferent between losing or winning the auction.

[9] Scores are nonnegative.

[10] Again, for application to the FCC spectrum auction, the threshold price is the largest bid a station could submit, holding all other bids fixed, so that it is indifferent between staying on air or selling its license.

Define B to be the set of all bid profiles. In other words, $B = \prod_{i \in N} B_i = B_1 \times B_2 \times \ldots \times B_N$. Define a function $\alpha : B \longrightarrow 2^N$ such that $\alpha(b) \subset N$ is the set of winning bidders/bids given bid profile $b \in B$. This function α is called the allocation rule.

One of the most significant aspects and characteristics of the algorithm is the scoring. After all, a collection of scoring functions is needed to determine the allocation rule α. This collection of scoring functions has cardinality 2^N as each scoring function corresponds to a set of active bidders/bids which in turn is merely a subset of N. For each set of active bidders $A \subset N$, the scoring function for bidder i is $s_i^A : B_i \times B_{N \setminus A} \longrightarrow (0, \infty)$. In addition, it is required that $s_i^A(x, \cdot) \geq s_i^A(x', \cdot)$ whenever $x \geq x'$. The algorithm proceeds as follows:

1. $A_t =$ the set of active bidders at time t.
2. $A_1 = N$ (initially, all bidders are active).
3. Algorithm terminates at time t if and only if $s_i^{A_t}(b_i, b_{N \setminus A_t}) = 0 \ \forall i \in A_t$. Return the allocation rule $\alpha(b) = A_t$.
4. If not all scores are 0, then $A_{t+1} = A_t \setminus \arg\max_{i \in A_t} s_i^{A_t}(b_i, b_{N \setminus A_t})$.

The last part of the algorithm is the payment. The payment rule is a function $p : B \longrightarrow \mathbb{R}^{\mathbb{N}}$. We have $p_i(b) = 0$ whenever $i \in N \setminus \alpha(b)$.

Before proceeding with a discussion regarding the optimality of the algorithm, we provide an example illustrated in Milgrom and Segal (2014) with regards to the application of the algorithm to the FCC problem.[11] In this case, two more terms are introduced: *inessential* and *essential*. The term inessential refers to bids that can be feasibly rejected. Not only must it be the case that the rejected station can be assigned a channel but there must still be sufficient active bids to meet the procurement criteria set out by the FCC. Essential bids are those that are not inessential. These bids are immediately given a score of 0.

[11] This example is labeled *Example 6* in their paper

Let $F \subset 2^N$ be the set of bidders that can be feasibly accepted. In addition, we must assume that $N \in F$, so that the purchase goal is possible. To maintain feasibility, $s_i^A(b_i, b_{N \setminus A}) > 0$ only if $A \setminus \{i\} \in F$. To reiterate, one part of determining a bid's score is deciding whether the rejecting the bid violates an interference constraint or now. As discussed in the previous section, checking feasibility could potentially be difficult as the class of graph-coloring problems is NP-Hard. Thus, one must place a restriction on the computation time for checking feasibility. In this scenario, three outputs are possible when the algorithm runs at time t:

1. It is established that $A_t \setminus \{i\} \in F$.

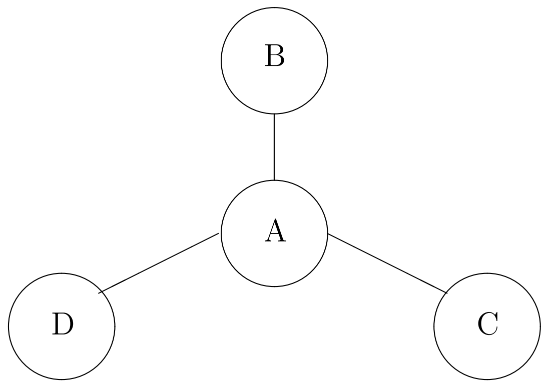
2. It is established that $A_t \setminus \{i\} \notin F$.

3. It times out before establishing either of the cases above.

In case 1, we set $s_i^{A_t}(b_i, b_{N \setminus A}) > 0$. In case 2 and case 3, we set $s_i^A(b_i, b_{N \setminus A}) = 0$. This guarantees that the algorithm terminates and returns a feasible set of acceptable bids.

A natural question arises, though. How does this algorithm differ from a more traditional deferred acceptance or greedy algorithm? An important aspect of this algorithm is the scoring function, which is conditional on the *rejected* bids (i.e. on the bids made by stations that are allowed to stay on the air). In a traditional deferred acceptance algorithm, the score of a particular station's bid is merely its bid amount. This can lead to undesirable results. For instance, suppose the assignment of a certain station causes significant amounts of interference, thereby preventing the assignment of other "inessential" stations. As a specific example, consider an environment where there are four geographic areas, each with one station, and only a single available channel (for the sake of convenience, we assume that the bids submitted by the station are truthful and equal to its value).

Here is a diagram of the constraints:



Let the values of each station be as follows:

$$v_A = 2, v_B = 1, v_C = 1, v_D = 1$$

If we were to run a traditional deferred acceptance algorithm, only the station in the center, station A, would get assigned. The optimal scenario, however, is assigning stations B, C, and D, which gives a total value of 3. Is it possible to achieve such an assignment? What if the scoring function was different? For instance, the scoring function could scale down scores for stations that would cause too much interference if assigned. In the case above, define the score to be the value divided by degree of the vertex. Then the algorithm achieves the optimal assignment.[12]

When applied to the FCC problem, how close to optimal is the reverse deferred acceptance algorithm? Tests of the algorithm run on experimental data provided by the FCC have been promising, showing that the algorithm returns a near optimal assignment. However, we seek a theoretical approach to measuring its optimality.

5 Matroid Theory

As we move on to a discussion of the optimality of the algorithm, some preliminaries will be needed. A brief discussion of matroid theory and its applications to greedy algorithms is necessary.

[12] A more detailed analysis of a similar example is provided by Milgrom and Segal (2014).

Definition 5.1 *A matroid M is a pair of sets (E, I), where E is a finite set called the ground set and I is a collection of subsets of E called the independent sets, which satisfy the following properties:*[13]

1. $\emptyset \in I$
2. *For each* $A \in I$, *if* $A' \subset A$ *then* $A' \in I$
3. $A, B \in I$ *and* $|A| > |B| \implies \exists j \in A$ *such that* $B \cup \{j\} \in I$

If a pair (E, I) only satisfies the first two criterion, then (E, I) is called an independence system.[14]

Definition 5.2 *If $A \subset E$ and $A \notin I$, then A is called a dependent set.*

Definition 5.3 *Let (E,F) be an independence system. For any $S \subset E$, a set $B \subset S$ is called a basis of S or maximal in S if $B \in I$ and $B \cup \{j\} \notin I$ for all $j \in S \setminus B$.*

If (E,I) is an independence system, a set $S \subset E$ is called a circuit or minimally dependent if $S \notin I$ but $S \setminus \{j\} \in I$ for all $j \in S$.

Theorem 5.4 *If (E, I) is a matroid, then all basis of $S \subset E$ have the same size. In other words, if B and B' are a basis of S then $|B| = |B'|$.*

Proof: Suppose not, then for some $S \subset E$ there are two maximal subsets of S, B and B', of different cardinality. Without loss of generality, assume $|B| > |B'|$. Since (E, I) is a matroid $\implies \exists j \in B$ such that $B' \cup \{j\} \in I$. Now, $j \in B \implies j \in S \implies B' \cup \{j\} \subset S \implies B'$ is not maximal, which is a contradiction. Thus, all basis of $S \subset E$ have the same cardinality. ∎

Now, consider the scenario where a weight w_v is assigned to every $v \in E$. Moreover, consider the following constrained optimization problem:

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[13] *Matroid Theory*, J.G. Oxley

[14] Naturally, a matroid is an independence system but not all independence systems are matroids.

$$\max_{S \in I} \sum_{v \in S} w_v$$

An algorithm that could potentially be used to solve this optimization problem is the **greedy algorithm**:

1. Order elements of E by weight (from increasing to decreasing): $w_1 \geq w_2 \geq w_3 \geq \dots w_n$.

2. $A_0 = \emptyset$

3. If $w_k \leq 0$, stop and return A_{k-1}.

4. If $w_k > 0$ and $A_{k-1} \cup k \in I$, define $A_k = A_{k-1} \cup k$.

5. If $w_k > 0$ and $A_{k-1} \cup k \notin I$, define $A_k = A_{k-1}$.

Clearly, this algorithm terminates as E is finite. However, when does this algorithm return the true optimum? It turns out that it finds the optimum if and only if (E, I) is a matroid.

Theorem 5.5 *(E,I) is a matroid if and only if for every possible set of weights, the greedy algorithm solves the optimization problem above.*

Proof: We first prove the forward direction. Define a weight function, $w : E \longrightarrow \mathbb{R}$ such that $w(v) = \textit{weight of } v$. Let $A = \{v_1, v_2, \dots, v_n\}$ be the set returned by the greedy algorithm. By construction, $A \in I$. In addition, without loss of generality, assume $w(v_1) \geq w(v_2) \geq w(v_3) \geq ...w(v_n)$. Let $C = \{u_1, v_2, \dots, u_m\}$ be an independent set of maximum weight. Assume A is not an independent set of maximal weight. Let $F = \{t\colon\ w(v_t) < w(u_t)\}$. Now, define $k = \min F$ if $F \neq \emptyset$. If $F = \emptyset$, then take $k = n + 1$.[15] From the definition of k, none of the elements of $\{u_1, u_2, \dots, u_k\}$ were selected by the greedy algorithm $\Longrightarrow u_j \in \{v_1, v_2, \dots, v_{j-1}\} \vee u_j \cup \{v_1, v_2, \dots, v_{j-1}\} \notin I$ for all $j = 1, 2, \dots, k \Longrightarrow u_j \in \{v_1, v_2, \dots, v_{k-1}\} \vee u_j \cup \{v_1, v_2, \dots, v_{k-1}\} \notin I$ for all $j = 1, 2, \dots, k \Longrightarrow \{v_1, v_2, \dots, v_{k-1}\}$ is a basis for $\{v_1, v_2, \dots, v_{k-1}, u_1, u_2, \dots, u_k\}$.

[15] $F = \emptyset \Longrightarrow |C| > |A|$.

Now, $|\{v_1, v_2, \dots, v_{k-1}\}| = k - 1 < k = |\{u_1, u_2, \dots, u_k\}|$. Therefore, $\{u_1, u_2, \dots, u_k\}$ is an independent subset of the same set but of a larger size than $\{v_1, v_2, \dots, v_{k-1}\}$. This contradicts Theorem 5.4.

Proving the reverse direction is much simpler. Suppose (E, I) is not a matroid $\implies \exists S \subset E$ such that S has two basis, B and B', of differing size. Let $|S| = n_1$ and $|B| = n_2 > n_3 = |B'|$. Consider the following weight function w:

1. $w(v) = -1$ for all $v \in E \setminus S$.
2. $w(v) = 1$ for all $v \in S \setminus B'$.
3. $w(v) = q$ for all $v \in B$, where $q \in (\frac{n_3}{n_2}, 1)$.

The greedy algorithm will return B' as the maximum weight independent set. However, the weight of B' is n_3 while the weight of B is $q \cdot n_2 > \frac{n_3}{n_2} \cdot n_2 = n_3$. Hence, the greedy algorithm is not optimal. Therefore, we must have that (E, I) is a matroid. ■

It is important to introduce a final concept in matroid theory before proceeding, though we will not go into much detail about them: the rank function.

Definition 5.6 *Suppose we have matroid M = (E, I). A real valued function r defined on subsets of the ground set E is called a* **rank function** *for M if and only if* $r(A) = \max\{|T| : T \subseteq A, T \in I\}$. *This function has the following properties:*

1. *r is integer valued*
2. $r(\emptyset) = 0$
3. $r(\{i\}) \leq 1$ *for* $i \in E$
4. *If* $|A_1| > |A_2|$, *then* $r(A_1) \geq r(A_2)$ *for any* $A_1, A_2 \subseteq E$
5. $r(A_1 \cup A_2) + r(A_1 \cap A_2) \leq r(A_1) + r(A_2)$ *for any* $A_1, A_2 \subseteq E$[16]

[16]Such a function r is called supermodular

We will make use of the following theorem about rank functions in the next section.

Theorem 5.7 *Let $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ be two matroids (defined on the same ground set E) with associated rank functions r_{M_1} and r_{M_2}. Then*

$$\max\{|A|\colon A \in I_1 \cap I_2\} = \min_{B \subseteq E} r_{M_1}(B) + r_{M_2}(E \setminus B)$$

6 Hall's Marriage Theorem

A theorem that will be important in the discussion of the near-optimality of the deferred acceptance algorithm, is Hall's Marriage Theorem. There are many proofs of the theorem but I present a proof of the theorem using the matroid theory results from the previous section. Moreover, in the subsequent section, theorems and proofs related to graph-theoretic generalization of Hall's theorem are provided.

Now, given a graph $G = (V, E)$, a *matching* is a set of pairwise non-adjacent edges. A *perfect matching* is a matching that matches every vertex. That is, for every vertex $v \in V$, there is some edge in the matching that is incident to v. A very related concept, specific to bipartite graphs, is a saturated matching. Given bipartite graph $G = (X \cup Y, E)$ with $|X| \leq |Y|$, an $X-$*saturated matching* is a matching that matches every vertex in X. Hall's Marriage theorem provides necessary and sufficient condition for the existence of such a matching in a bipartite graph.

Theorem 6.1 *Hall's Marriage Theorem: Let $G = (X \cup Y, E)$ be a bipartite graph, with X and Y as the two disjoint sets of vertices. Furthermore, assume that for every $x \in X$, there exists some $y \in Y$ such that $(x, y) \in E$. Then an $X-$saturated matching exists if and only if for every $A \subset X$, $|A| \leq |N_G(A)|$, where $N_G(A)$ is the set of vertices adjacent to some vertex in A.*[17]

Proof: To prove the necessary condition, let $G = (X \cup Y, E)$ contains a perfect matching M. Suppose there exists some set $A \subset X$ such that

[17] In the situation where $|X| = |Y|$, an $X-$*saturated* matching is a perfect matching.

$|A| > |N_G(A)|$. Let $M_A \subset M$ be the set of all edges in the matching that are incident to some vertex in A. Then by the pigeon hole principle, there exists $e_1, e_2 \in M_A$ such that e_1 and e_2 are incident to the same vertex in $N_G(A)$. This is clearly a contradiction by the definition of a perfect matching. Hence, we have proved the "only if" direction.

To prove sufficiency, we may assume that $N_G(\{x\}) \geq 2$ for every $x \in X$. The reason for this is that if $N_G(\{x\}) = 1$ for some vertex $x \in X$, then a perfect matching exists in G if and only if a perfect matching exists in the graph G' obtained by *removing* x and its incident edges.

Consider the set I_X whose elements are subsets $T \subseteq E$ such that no vertex in X is incident to more than one edge in T. Likewise, define the set I_Y in the same way except with regards to the vertex set Y. Then we have that $M_X = (E, I_X)$ and $M_Y = (E, I_Y)$ are both matroids.

Now, note that if $T \in I_X \cap I_Y$ then T is a matching. Thus, it suffices to prove that there exists some $T \in E$ such that $T \in I_X \cap I_Y$ and $|T| = |X|$. For a given set $T \subset E$ let T_X be the set of vertices in X incident to some edge in T. Define T_Y similarly. Likewise, let T_X^C and T_Y^C be $X \setminus T_X$ and $X \setminus T_Y$, respectively.

Let us define the functions $r_X : \mathcal{P}(E) \longrightarrow \mathbb{R}$, $r_Y : \mathcal{P}(Y) \longrightarrow \mathbb{R}$, where $r_X(T) = |T_X|$ and $r_Y = |T_Y|$ for every $T \subseteq E$. It is easy to see that these two functions are rank functions for matroids M_X and M_Y, respectively.

For any $T \subseteq E$ we have the following:

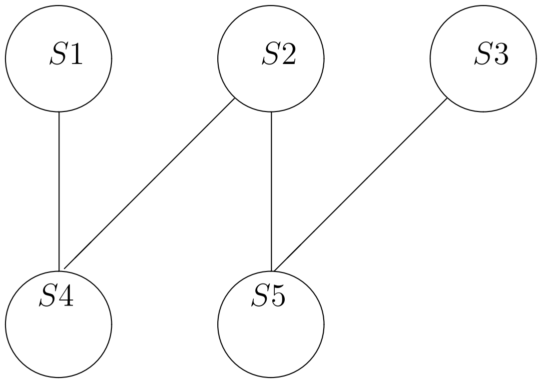
$$r_Y(E \setminus T) = |\{y \in Y \colon (x, y) \in E \setminus T\}| \geq \left|\left\{y \in Y \colon y \in N_G(\{x\}), x \in T_X^C\right\}\right|$$

$$= \left|\bigcup_{x \in T_X^C} N_G(\{x\})\right| \geq |X| - |T_X|$$

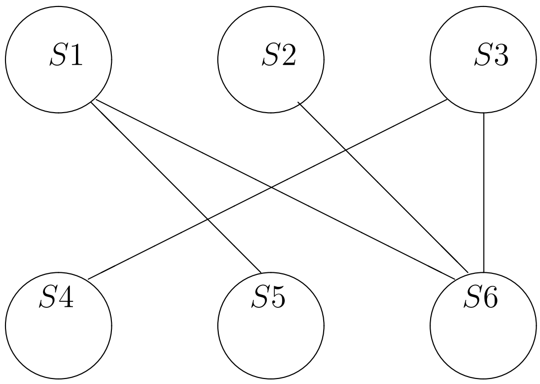
$$\Longrightarrow r_X(T) + r_Y(E \setminus T) \geq |X|$$

Since this is true for every $T \subseteq E$, by Theorem 5.6, we have that $\max\{|A| \colon A \in I_X \cap I_Y\} \geq |X|$. Equality must hold (as it is trivially impossible for the expression to be greater than $|X|$ for any set $A \in I_X \cap I_Y$), and thus there is an X-saturated matching.

Example 1 *Consider the following situation: there are* 3 *channels and* 5 *stations, three in one geographic area and two in another. The diagram below depicts the interference constraints (obviously, stations in the same geographic area can not be assigned the same channel).*[18]



The key insight in this problem is to add a "fake" station, $S6$, to the geographic area with only two stations. Moreover, assume $S6$ does not interfere with $S1$, $S2$, or $S3$. Now, take the complement of the graph. The complement graph reveals which stations can be assigned to the same channel. It looks like this:



This graph satisfies the conditions for Hall's Theorem, and so there is a perfect matching of size 3*: $\{(S1, S5), (S2, S6), (S3, S4)\}$. Assign the first channel to $S1$ and $S5$, the second to $S3$ and $S4$, and the remaining channel to $S2$ ($S6$ was "fake"). Thus, we have a feasible channel assignment.*

[18]A similar example is discussed by Alexey Kushnir, however, there are differences in my presentation of it.

Remark: Professor Alexey Kushnir found other conditions for which a channel assignment would be feasible. However, they seem quite restrictive. For example, one such condition is that no cycles are present between the geographic areas (the different cliques).

7 Matching Generalizations

Under what conditions will a graph $G = (V, E)$ have a perfect matching? As a corollary to Hall's Marriage Theorem, we have the following theorem, which is a special case of the Tutte-Berge formula:

Theorem 7.1 *Given a graph $G = (V, E)$, define $C_O(G)$ to be the number of* odd components *of G, that is, the number of connected components of G with an odd number of vertices. Moreover, for every set $U \subset V$, let G_{V-U} denote the subgraph induced by removing U from the vertex set. Then G has a perfect matching if and only if for every $U \subset V$, $C_O(G_{V-U}) \leq |U|$.*[19][20]

Proof: We first prove the "only if" direction as it is much simpler. Suppose $G = (V, E)$ has a perfect matching. Given $U \subset V$, consider the subgraph induced by G_{V-U}. Every odd component of G_{V-U} must have at least one vertex matched to a vertex in U. Thus, $C_O(G_{V-U}) \leq |U|$.

Proving the sufficiency of the condition is more difficult. To begin, we prove some basic preliminary results.

Lemma 7.2 $|V| \equiv 0 \mod 2$.

Proof: By taking $U = \emptyset$, $C_O(G) \leq 0 \implies$ all connected components have an even number of vertices. ∎

Lemma 7.3 *If $C_O(G_{V-U}) \geq |U| - 1$, then $C_O(G_{V-U}) = |U|$.*

[19] Let this condition be denoted as $(*)$

[20] There is an implicit assumption that $E \neq \emptyset$

Proof: Since $|V|$ is even, we must have that $|U| + \sum_{i=1}^{C_O(G_{V-U})} C_i \equiv 0 \mod 2$, where C_i is the number of vertices in connected component i. Assume $C_O(G_{V-U}) = U - 1 \implies |U| + \sum_{i=1}^{|U|-1} C_i \equiv 0 \mod 2$. If $|U| \equiv 1 \mod 2$, then $\sum_{i=1}^{|U|-1} C_i \equiv 1 \mod 2$. However, each C_i is odd and so we have that the sum of an even number of odd numbers is odd, which is a contradiction. Now, if $|U| \equiv 0 \mod 2$ then we have that the sum of an odd number of odd numbers is even, which is also a contradiction. Therefore, $C_O(G_{V-U}) = |U|$. ■

Now we proceed via induction. Any graph $G = (V, E)$ with $|V| = 2$ that satisfies $(*)$. G trivially has a perfect matching. Our base case is satisfied. Assume that whenever $|V| \leq M - 1$ and $G = (V, E)$ satisfies $(*)$, then G contains a perfect matching. We take a look at $G = (V, E)$ satisfying $(*)$ with $|V| = M$. Once more, we should emphasize that $|V| \equiv 0 \mod 2$ by Lemma 7.2. Therefore, we may assume that M is even as well.

Let $S = \{T \colon C_O(G_{V-T}) = |T|\} \implies S = \{T \colon C_O(G_{V-T}) \geq |T| - 1\}$ by Lemma 7.3. Clearly, S is non-empty as $\emptyset \in S$. Moreover, S is a finite set which means it contains a maximal element U. Consider G_{V-U}. If G_{V-U} contained a component with an even number of vertices then $C_O(G_{V-(U \cup \{x\})}) = |U| + 1 = |U \cup \{x\}| \implies$ contradiction by the supposed maximality of U in S. As a result, G_{V-U} has only odd components, which we denote as $G^{(1)}, G^{(2)}, \ldots, G^{(|U|)}$.

For each $G^{(j)}$ select a vertex x_j from the vertex set of $G^{(j)}$, denoted $V_{G^{(j)}}$. Define a graph $G' = (V', E')$ as follows:

1. $V' = U \cup \{x_1, x_2, \ldots, x_{|U|}\}$
2. $E' = \{e = (v_1, v_2) \in E \colon v_1$ and v_2 are not *both* in $U\}$.

Thus, G' is a bipartite graph. Suppose G' does not admit a perfect matching $\implies \exists W \subset \{x_1, x_2, \ldots, x_{|U|}\}$ such that $|N_{G'}(W)| < |W|$. This means that $C_O(G_{V-N_{G'}(W)}) \geq |W| > |N_{G'}(W)|$, which is a contradiction. Therefore, G' admits a perfect matching.

Since G' admits a perfect matching then there is a matching in G that covers all the vertices in U as well as vertex x_j for each $1 \leq j \leq |U|$. The question is now whether each $G^{(j)}_{V_{G^{(j)}}-\{x_j\}}$ admits a perfect matching? If so, then G admits a perfect matching.

Assume one can not find a perfect matching in $F_j = G^{(j)}_{V^{(j)}-\{x_j\}}$. F is trivially a subgraph of G and so has less than M vertices. By the inductive hypothesis, F must violate condition $(*)$ $\implies$ there exists a subset U' of the vertex set of F such that $C_O(F_{V_F - U'}) > |U'|$. However, then $N_O(G_{V-(U \cup U' \cup \{x_j\})}) \geq |U| + |U'| \implies U \cup U' \cup \{x_j\} \in S \implies$ contradiction by the maximality of U in S. Thus, F_j admits a perfect matching for each j and hence so does G. ■

Implications: The previous theorem provides us with a condition such that if satisfied, would make it possible to assign every station to a channel with less than or equal to $\frac{T}{2}$ channels, where T is the total number of stations.

Realistically, one DMA will only interfere with approximately three or four other digital market areas (i.e. the three or four surrounding geographic regions). Hence this bound of $\frac{T}{2}$ could be reduced substantially in practice, as we essentially only need to look at the total number of stations in the four surrounding DMAs.

While in practice we will know the interference constraints beforehand (as that will be provided by the FCC), it would be interesting to see what can be found *ex ante*. To do this, we incorporate the idea of random graphs. Suppose now that stations in neighboring geographic areas interfere with a probability p. Then the interference graph can be interpreted as a random graph. We will use the traditional Erdos-Renyi model for random graphs, where a graph on n vertices with m edges is drawn uniformly from the set of all such graphs. The models are equivalent as a graph with n vertices and m edges can be viewed as a graph where there is a probability of $\frac{m}{\binom{n}{2}}$ that there is an edge between any two vertices.

We now present an extension of Hall's Marriage theorem to the case of random graphs. The first is due to Bollobas and Thomason, and is restricted to the case of random bipartite graphs.

Theorem 7.4 *Let G be a bipartite random graph with vertex sets X and Y, with $|X| = |Y| = n$ and at least $n \log n$ edges. Then G contains a perfect matching with probability greater than $1 - O(\frac{1}{\sqrt{n \log n}})$.*[21]

Proof: Let p denote the probability that there is an edge between any two vertices in X and Y, respectively. Then $p = \frac{\log n}{n}$. Assume G does not have a perfect matching. By Hall's Marriage Theorem, there exists a subset $X' \subset X$ with $|X'| > N_G(X)$. By symmetry, there is $Y' \subset Y$ with $|Y'| > N_G(Y)$. Let A be the minimum of the set of all X' and Y'. It is easy to see that the subgraph $G_{A \cup N_G(A)}$ is connected (otherwise, A could be replaced with a proper subset of itself, thereby contradicting the minimality of A). In addition, by the minimality of A, we must have that $N_G(A) = |A| - 1$. Using the fact that a connected set contains a spanning tree, we determine that the subgraph $G_{A \cup N_G(A)}$ has at least $2(|A| - 1)$ edges.

Without loss of generality, suppose $A \subset X$. Then $Y \setminus N_G(A)$ violates Hall's Marriage Theorem. Thus, $|A| \leq |B| = |V| - |N_G(A)| = n - |A| + 1 \longrightarrow |A| \leq \frac{n+1}{2}$.

Let E_k denote the event that there is a subset A of X or Y such that $k = |A| \leq \frac{n-1}{2}$, $|A| = |N_G(A)| + 1|$, and the subgraph $G_{A \cup N_G(A)}$ is connected. The probability that there exists subsets $A_1 \subset X$, $A_2 \subset Y$ with $k = |A_1| = |A_2| + 1$ such that $G_{A_1 \bigcap A_2}$ has at least $2(k-1)$ edges and $N_G(A_1) \subseteq A_2$, is at most

$$\binom{k(k-1)}{2(k-1)} p^{2(k-1)} (1-p)^{k(n-k+1)}$$

There are $2\binom{n}{k}$ choices for A_1 and $\binom{n}{k-1}$ choices for A_2, so we have

$$Pr\left(\bigcup_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} E_k\right) \leq 2 \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{k} \binom{n}{k-1} \binom{k(k-1)}{2(k-1)} p^{2(k-1)} (1-p)^{k(n-k+1)}$$

[21] *Assignment Problems*, Burkard, Dell'Amico, Martello

Applying Stirling's Approximation and substituting $p = \frac{\log n}{n}$, we get

$$Pr(\bigcup_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} E_k) \leq 2 \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} (\frac{en}{k})^k (\frac{en}{k-1})^{k-1} (\frac{ek}{2})^{2(k-1)} (\frac{\log n}{n})^{2(k-1)} (1 - \frac{\log n}{n})^{k(n-k+1)}$$

Now, $(1 - \frac{\log n}{n})^{k(n-k+1)} \leq (\frac{1}{n})^{k - \frac{k^2}{n} + \frac{k}{n}} \leq n^{\frac{k^2}{n} - k}$. Thus,

$$Pr(\bigcup_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} E_k) \leq \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} \frac{e^{4k-3} k^{k-2} (\log n)^{2(k-1)} n^{1-k+\frac{k^2}{n}}}{(k-1)^{k-1} 2^{2k-3}} \leq \sum_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} (e \log n)^{3k} n^{1-k+\frac{k^2}{n}}$$

With a little more algebraic manipulation, it is easy to see that there are constants C_1, C_2 such that $(e \log n)^{3k} n^{1-k+\frac{k^2}{n}} \leq \frac{C_1}{\sqrt{n \log n}}$ for $k = 2$ and $(e \log n)^{3k} n^{1-k+\frac{k^2}{n}} \leq \frac{C_2}{n\sqrt{n \log n}}$ for $k > 2$. Hence, $Pr(\bigcup_{k=2}^{\lfloor \frac{n-1}{2} \rfloor} E_k) \leq O(\frac{1}{\sqrt{n \log n}})$, which proves the theorem. ■

A more general theorem regarding the existence of a perfect matching in a random graph was proven by Erdos and Renyi. The two mathematicians utilized Theorem 7.1 (the generalization of Hall's Marriage Theorem presented earlier in the section) in order to prove their assertion.

Theorem 7.5 *Let $G = (U, E)$ be a random graph with $|U| = n$, n even. Suppose that G has $\frac{n \log n + n f(n)}{2}$ edges, where $f(n)$ is a function satisfying* $\lim_{n \to +\infty} f(n) = +\infty$. *If we define S to be the event that G contains a perfect matching, then* $\lim_{n \to +\infty} Pr(S) = 1$.[22]

Discussion: It is important to first note that when applying the two theorems above to our problem, the graph that needs to be analyzed is the *complement* of the interference graph. In both the bipartite and general case, it becomes likely that a perfect matching exists as soon as one has the sufficiently many random edges for the minimum degree to be at least one with high probability.[23] An implicit assumption in the proofs is that

[22] This result was proved by Erdos and Renyi in their paper *On the Existence of a Factor of Degree One of a Connected Random Graph.*

[23] *Perfect matchings in random graphs with prescribed minimal degree*, Alan Frieze and Boris Pittel

the probability of an edge between two vertices tends to 0 as n gets large. Now, in our situation, if p is the probability that a station interferes with any adjacent station, then the probability that a vertex has minimum degree one in the complement graph is precisely $1 - p^k$ where k is the number of vertices where an edge could potentially exist.[24] Thus, if p is small and k is large, which is true in practice, the probability a vertex has at least one edge incident to it in the complement graph tends to 1.

8 Optimality: Hall's Theorem Approach

The notion of perfect matchings are useful in discussing whether a feasible channel assignment is possible provided one has sufficient channel space, which might or might not be the case. Hence, the theorems in the previous section do have their limitations. However, they do provide insight and can function as tools when discussing the optimality of the class of reverse deferred acceptance algorithms. Recall that interference constraints are represented by a graph, in which two stations share an edge if they cannot be assigned the same channel. There is a finite number of channels available to assign (but at least three), and we say that a particular set of stations can be feasibly assigned if there exists some assignment that satisfies all of the constraints. Each station has a "value" and the goal is to maximize the value of the stations that are assigned.

A "deferred acceptance algorithm" in this context is a greedy algorithm for deciding which stations to assign, but not for deciding the channel assignment. In the auction processed through this algorithm, bids are "accepted" from stations that are not assigned to any channel (i.e. the license for that station is purchased and hence spectrum is freed up).

We can represent the constraint graph with each station in one of a finite number of geographic areas ("digital market areas" and "DMAs"). The geographic areas themselves are described by a graph, with a link between two areas indicating physical adjacency. Thus, within each DMA, there is a

[24]Recall that a station interferes with all the other stations in the same geographic region, and so the 'randomness' occurs only between stations in adjacent geographic areas.

complete graph in which all stations within that area share an edge. Let T denote the set of all stations. Since T is obviously finite, we can index its element using the natural numbers. Now, we can represent the interference conditions with a simple adjacency matrix, A, where $A_{ij} = 1$ if and only if stations $i, j \in T$ have an edge between them. Let us first consider the case when we know the interference conditions. The other case will be a random graph approach. In that scenario, the relevant constraint graph will be constructed as follows. Stations in the same geographic area are linked with probability $q = 1$. Two stations in adjacent geographic areas are linked with probability p. Otherwise, stations are unlinked. There are at least 3 stations in each region. This approach will be discussed in the next and final section of the paper.

As described in section 3, the FCC is attempting to repackage stations into a specific channel block that will be determined in the auctions. Clearly, there are only a finite number of channels then that can be assigned. For theoretical purposes, call this number n. In any channel assignment for some set of stations, we can not have the interference constraints violated. Thus, a feasible channel assignment is a function $f : S \subseteq T \longrightarrow \{1, 2, 3, \ldots, n\}$ satisfies the condition that $f(i) \neq f(j)$ for $i, j \in S$ whenever $A_{ij} = 1$.

Thus, the optimization problem revolves around maximizing the total value of the stations assigned subject to the constraint that the set of stations is an element of the set $\{S \subset T \colon \exists f : S \longrightarrow \{1, 2, 3, \ldots, n\} \text{ where } f \text{ is feasible}\}$

Now, as stated before, every station is in a specific DMA, that is, the set of stations T can be partitioned into m cliques. We use the term "clique" because within each digital market area, the interference constraints can be modeled by a complete graph. Let us define a function g that maps each stations to its DMA. In other words, $g : T \longrightarrow \{1, 2, 3, \ldots, m\}$.

Theorem 8.1 *Let T denote the set of bidders (i.e the set of stations). Suppose there is no adjacency interference between stations; in other words, stations only interfere with stations within the same DMA. If we let $F \subset 2^T$ be the set of all feasible bids that can be accepted, then the cost-minimizing optimization problem:*

$$\alpha(b) \in \arg\min_{A \in F} \sum_{i \in A} b_i$$

can be solved using the reverse deferred acceptance algorithm.[25]

Proof: Let us recall that if a station has its bid rejected, then that station must be assigned a channel. Thus, $A \in F$ if and only if there exists a feasible channel assignment, f, for $T \setminus A$. Also, let g be the mapping from stations to DMA regions.

Define the scoring function as follows:

$$s_i^A(b_i, b_{T\setminus A}) = \begin{cases} b_i & \text{when } A \setminus \{i\} \in F \\ 0 & \text{otherwise} \end{cases}$$

Now, consider the set family $R = \{H \subseteq T \colon T \setminus H \in F\}$ of feasible rejected bids.

1. It is clear that $\emptyset \in R$.

2. In addition, for $H \in R$, there exists a feasible channel assignment f for the stations in H. Thus, there clearly exists a channel assignment for any $H' \subset H \implies T \setminus H' \in F \implies H' \in R$.

3. Suppose, $H_1, H_2 \in R$ with $|H_1| > |H_2|$. Assume there is a station $s \in H_1$ such that $g(s) \neq g(s')$ for all $s' \in H_2$. Let f_{H_2} be the feasible channel assignment for H_2 and fix some $j \in H_2$. Then consider the following channel assignment

   $$f_{H_2 \cup \{s\}}(i) = \begin{cases} f_{H_2}(i) & \text{for } i \in H_2 \\ f_{H_2}(j) & \text{for } i = s \end{cases}$$

   Since s is in a different DMA region from any of the stations in H_2, it does not interfere with any of them. Hence the channel assignment f_{H_2} is feasible $\implies H_2 \cup \{s\} \in R$.

[25] The notation used is the same as in section 3. For instance, α, is the *allocation rule.*

The second case is that there is no $s \in H_1$ such that $g(s) \neq g(s')$ for all $s' \in H_2$. This means that for every $s \in H_1$ there is some $s' \in H_2$ such that $g(s) = g(s')$. Thus, $g(H_1) \subseteq g(H_2)$. Define $f_{H_2}^{min}$ to be the minimum feasible channel assignment on H_2. Define $f_{H_1}^{min}$ in the same way. It is clear that $|f_{H_1}^{min}(H_1)| > |f_{H_2}^{min}(H_2)|$. We can then use $|f_{H_2}^{min}(H_2)|$ channels of $f_{H_1}^{min}(H_1)$ to create a new feasible channel assignment for H_2 (call these set of channels C). Then for $s \in H_1 \setminus H_2$, we can assign it a channel $c \in f_{H_1}^{min}(H_1) \setminus C$. Thus, we have a feasible channel assignment for $H_2 \cup \{s\} \Longrightarrow H_2 \cup \{s\} \in R$.

We can see then that (T, R) is a matroid, with N being the ground set and R the family on independent sets. In addition, using the reverse deferred acceptance algorithm with the scoring function above is equivalent to employing a greedy algorithm on the family of set of rejected bids. After all, finding the allocation rule that minimizes the cost of the accepted bids is equivalent to finding the maximum value over the set of all bids that can be feasibly rejected. Hence by Theorem 5.5, the reverse deferred acceptance algorithm will output the optimal solution. ■

Now, in reality, there is inter-area spectrum interference. Thus, we must consider the scenario when there are other interference constraints besides the interference within each DMA. We have to tackle the problem slightly differently. The following is the approach taken by Milgrom and Segal in their paper. First, we define the following additional sets, $F_{An}, F_{mn}^d,$ and F_{Amn}^d as follows:

$$F_{An} = \{S \subseteq T \colon \exists f \colon S \longrightarrow \{1, 2, \ldots, n\} \text{ such that } \forall s, s' \in S, f(s) \neq f(s') \text{ whenever } A_{ss'} = 1\}$$

$$F_{mn}^d = \left\{S \subseteq T \colon S \cap g^{-1}(j) \leq n - d \text{ for all } j \in \{1, 2, \ldots, m\}\right\}$$

$$F_{Amn}^d = F_{An} \cap F_{mn}^d$$

Conceptually, F_{An} is the family of sets of stations that can be feasibly assigned to channels 1 through n. F_{mn}^d is the family of sets of stations that satisfy the property that no more than $n-d$ stations are in each clique. F_{Amn}^d

is self-explanatory. However, its significance appears when one considers the case where $F^d_{Amn} = F^d_{mn}$, that is, when the interference constraints essentially become redundant in checking feasibility.

Before presenting Milgrom and Segal's near-optimality theorem, we define a few more sets:

$$S'_j = S \cap g^{-1}(j)$$

$$S_j = S \cap g^{-1}(\{i \colon i \leq j\})$$

$$Z^*(S'_j) = \bigcap_{s \in S'_j} \{t \in S_{j-1} : A(s,t) = 1\}$$

The first two are self-explanatory, while the latter is the set of all stations in S_{j-1} that interfere with every station in S'_j.

Theorem 8.2 *Suppose there exists some $d < n$ such that for each $S \in F^d_{mn}$, $|S'_j| + |Z^*(S'_j)| \leq n$ for all j. Then there exists a reverse deferred acceptance algorithm that, for any set of bids, first rejects the most valuable stations in each clique $g^{-1}(j)$, iterates in that way as long as that is feasible, and then continues. For every such algorithm, the value of stations assigned is at least a fraction $\frac{n-d}{n}$ of the optimal value.*

Proof: A detailed proof of this result can be found in the Appendix of the Milgrom and Segal paper. The main step in the proof is to show that the condition outlined above guarantees that $F^d_{Amn} = F_{An} \cap F^d_{mn} = F^d_{mn}$.

Consider $S \in F^d_{mn}$. We must show that $S \in F_{An}$. In other words, we must find a feasible channel assignment for S. The proof of the existence of such a channel assignment is done using induction. Begin constructing a channel assignment c by assigning a different channel to each station in $S_1 = S'_1$, which is possible because $|S_1| = |S'_1| \leq n$. Suppose we can feasibly assign channels to all stations $s \in S_j$. Consider $S_{j+1} = S_j \cup S'_{j+1}$. Then for any $X \subset S'_{j+1}$, we have,

$$|c(Z^*(X))| + |X| \leq |Z^*(X)| + |X| \leq n$$

$$\implies |\,\{1, 2, 3, \ldots, n\} \setminus c(Z^*(X))| \geq |X|$$

.

Then by Hall's Marriage Theorem, there exists an injective mapping $c\colon S'_{j+1} \longrightarrow \{1,2,3,\ldots,n\}$ such that $c(s) \in \{1,2,3,\ldots,n\} \setminus c(Z^*(\{s\}))$ for all $s \in S'_{j+1}$. Thus, we have created a feasible channel assignment for S_{j+1}.

Finally, to establish the proposition, we construct a reverse deferred acceptance algorithm. Define the scoring function as follows. At any round t, if the set of stations already assigned is X, then any station that is essential gets a score of zero. Among inessential stations at round t, the score for any station s is given by:

$$n - |N \cap g^{-1}(g(s))| + \frac{b(s)}{1+b(s)}$$

For clarity, note that since g is not injective, $g^{-1}(g(s))$ is the *preimage* of $g(s)$, which is indeed a set. Under this scoring function, the algorithm will first assign the most valuable station in each area, then the second most valuable station in each area, and so on until at least the $n-d$ most valuable stations in each area are assigned. The result follows immediately. ■

Remark: Clearly, we would want to find the value of d that gives us the largest fraction of the optimal value. Hence we define the following variable:

$$d_{mn} = \min\left\{d \geq 0\colon \forall S \in F^d_{mn}, |S'_j| + |Z^*(S'_j)| \leq n, \text{ for all } j\right\}$$

.

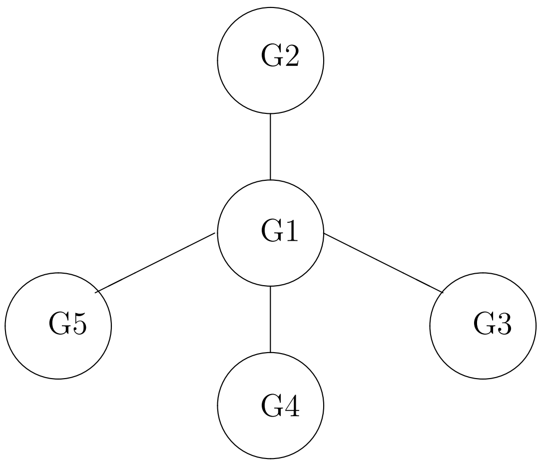
9 Optimality Continued: The Random Graph Approach

Again, while in practice we will know the interference constraints beforehand (as that will be provided by the FCC), it would be interesting to see what can be found *ex ante*. We turn once more to random graphs. Recall that the relevant constraint graph will be constructed as follows. Stations in the same geographic area are linked with probability q = 1. Two stations in adjacent geographic areas are linked with probability p. Otherwise, stations are unlinked. There are at least three stations in each region. In this approach, we can still utilize adjacency matrices except we start with a set Z of *all possible* adjacency matrices satisfying the condition:

For i and j in neighboring geographic areas, we have that

$$\frac{|\{A\colon A \in S \text{ and } A_{ij} = 1\}|}{|Z|} = p$$

We will now make some important but realistic assumptions about the interference between regions. In general, stations in a single geographic area G_1, only interfere with stations in adjacent areas. Therefore, for a region G_1 in the middle of the country, we can imagine it being surrounded by four other areas, G_2, G_3, G_4, and G_5. For the densely populated regions along the coast, we can imagine the regions being linked in a chain. We proceed with the former case first.



Label the cliques G_1, G_2, G_3, G_4, G_5 as $1, 2, 3, 4, 5$, respectively. Using similar notation to that in the previous section, define the following:

$$S'_j = S \cap g^{-1}(j) \text{ for } j = 1, 2, 3, 4, 5$$

$$S_j = S \cap g^{-1}(\{i\colon i \leq j\}) \text{ for } j = 1, 2, 3, 4, 5$$

$$F^k = \left\{S \subset T\colon S \cap g^{-1}(1), S \cap g^{-1}(3), S \cap g^{-1}(3), S \cap g^{-1}(4), S \cap g^{-1}(5) \leq n - k\right\}$$

$$Z^*(S_j^{'})|A = \bigcap_{s \in S_j'} \{t \in S_{j-1} : A(s,t) = 1\}$$

It is important to point out that in this situation, Z^* is a *random variable* because the adjacency matrix is randomly determined. The goal then is to calculated the *expected value* of d_{mn}:

$$\mathbb{E}(d_{mn}) = \min \left\{ k \geq 0 \colon \forall S \in F^k, |S_j'| + \mathbb{E}(|Z^*(S_j')|) \leq n, \text{ for all } j \right\}$$

Now, let us start by fixing $S \in F^k$. Let $|S_1'| = q_1, |S_2'| = q_2, |S_3'| = q_3, |S_4'| = q_4$, and$|S_5'| = q_5$. We can now calculate $\mathbb{E}(Z^*(S_j'))$ for each j explicitly:

$$\mathbb{E}(|Z^*(S_1')|) = 0$$

$$\mathbb{E}(|Z^*(S_2')|) = q_2 p^{q_1}$$

$$\mathbb{E}(|Z^*(S_3')|) = q_3 p^{q_1+q_2}$$

$$\mathbb{E}(|Z^*(S_4')|) = q_4 p^{q_1+q_3}$$

$$\mathbb{E}(|Z^*(S_5')|) = q_5 p^{q_1+q_2+q_4}$$

By Theorem 8.2 in the previous section, the necessary and sufficient condition required is that $|S_j'| + \mathbb{E}(|Z^*(S_j')|) \leq n, \forall j$, for each $S \in F^k$. Now, let us look at the worst-case scenario, which is when $|S_j'| + \mathbb{E}(|Z^*(S_j')|)$ is maximized. Using the expressions above, maximizing the function clearly requires that $q_1 = 1$.

For the constraint $|S_j'| + \mathbb{E}(|Z^*(S_j')|) \leq n$ to be violated, it suffices to look at the case when $j = 2$. The reason for this is that the exponent of p in $\mathbb{E}(Z^*(S_j'))$ are greater than 1 for $j > 2$.

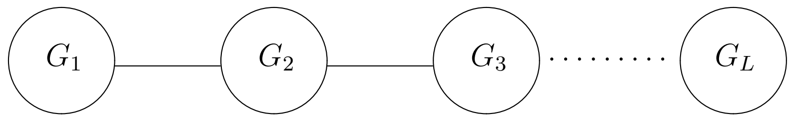
$$|S_2'| + \mathbb{E}(|Z^*(S_2')|) \leq (n-k)(1+p)$$

$$\implies (n-k)(1+p) \leq n \text{ if and only if } p \leq \frac{k}{n-k}$$

$$\implies \mathbb{E}(d_{mn}) \leq \min \left\{ k \geq 0 \colon p \leq \frac{k}{n-k} \right\}$$

By Theorem 8.2, there is some reverse deferred acceptance algorithm that assigns stations with a total value of $1 - \frac{\min\left\{k \geq 0 \colon p \leq \frac{k}{n-k}\right\}}{n}$ of the optimal value.

Now we examine the "coastal case", where regions form an interference chain of sorts. Assume that there are exactly L DMAs along the coasts. Then we can represent the interference between regions with the following diagram:



We use the same notation as in the previous case as well as set $|S'_j| = q_j$ for each j. Again, we proceed to calculate $\mathbb{E}(|Z^*(S'_j)|) = 0$ explicitly:

$$\mathbb{E}(|Z^*(S'_1)|) = 0$$

$$\mathbb{E}(|Z^*(S'_j)|) = q_j p^{q_j - 1} \text{ for } 1 < j \leq L$$

Thus, using the same worst-case scenario analysis as in the previous case we get the following:

$$\mathbb{E}(d_{mn}) \leq \min\left\{k \geq 0 \colon p \leq \frac{k}{n-k}\right\}$$

Hence, regardless of which one of the two interference patterns the DMAs exhibit, there is some reverse deferred acceptance algorithm that assigns stations with a total value of $1 - \frac{\min\left\{k \geq 0 \colon p \leq \frac{k}{n-k}\right\}}{n}$ of the optimal value.

10 Conclusion

One of the archetypal principles used in the design of mechanisms for allocating scarce resources in the shadow of private information is "deferred acceptance". Milgrom and Segal (2014) have sewn this principle in their "reverse deferred acceptance auction" for reallocating spectrum. The resource allocation problem considered by Milgrom and Segal is an instance of a packing problem which are well known to be NP-hard.

While the reverse deferred acceptance auction has good incentive properties, its low complexity means that it cannot be guaranteed to return a provably optimal packing. Nevertheless, when run on experimental data it seems to perform quite well. The previous theorems and analyses help to explain why the algorithm has been successful. However, there is still work to be done. One limitation of the Milgrom and Segal algorithm is that each bidder is assumed to be interested in a single license when in fact a bidder might want to sell or purchase multiple spectrum licenses. This is likely in the case when the FCC needs to sell the spectrum to mobile networks. Thus, subsequent examination of such classes of deferred acceptance mechanism and how they perform in such scenarios will be important for future research.

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