Stable Homotopy Groups of Spheres and The Hopf Invariant One Problem

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To my grandfather

Contents

1 Introduction

In algebraic topology, computing homotopy invariants of spaces is of fundamental importance. While cohomology tells much of the story and is relatively easy to compute, there is another, dual, story, told by another invariant. This set of invariants, called homotopy groups, so called because they completely classify a CW complex up to homotopy, are denoted π_i. In addition to this incredibly power, they are also easy to define: the group $\pi_i(X)$ is just the set of basepoint-preserving homotopy classes of maps between the i dimensional sphere S^i and the space X. The homotopy groups measure the "spheriness" of spaces in various dimensions and this measurement gives complete homotopic information. However, this classifying power comes at a cost. The homotopy groups are notoriously difficult to compute, even when X is itself is a sphere.

Luckily, there is a small miracle. When a space X is "connected" through a high enough dimension, there is an isomorphism

$$\pi_n(X) \to \pi_{n+1}(\Sigma X)$$

where Σ is the suspension function which "raises the connectedness" of X by one. For spheres this says

$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for large enough n. Using this, one can control the problem by noting that, as long as the spheres in question are large enough, the homotopy group depends only on the relative dimension k. The notion of X being "connected enough" is encoded by an object called a spectrum. Spectra are a generalization of topological spaces which, in an appropriate sense, behave "the same in each dimension". Surprisingly enough, cohomology functors are, in an appropriate sence, themselves spectra. Spectra have extremely nice formal properties, and they can be used to give information about certain homotopy groups of spaces, known as the "stable" homotopy groups.

To actually compute homotopy groups, we use a rather complicated gadget called the Adams Spectral Sequence. The spectral sequence works by considering an inverse sequence of spectra analogous to a free resolution of a module. The homotopy groups of the spectra in this sequences give an over-approximation of the actual homotopy groups one would like to compute. Then, by analyzing relationships between homotopy classes of maps (called differentials), we can whittle away at the over-approximation bit-by-bit, successively reducing it by taking one sub-quotient at a time. Unfortunately, most of the relationships between homotopy classes of maps are not directly computable. Indeed, the spectra in the resolution cannot, in general, be explicitly constructed. Instead, the geometric relations which we can reason about abstractly often become algebraic relations we can compute concretely. In fact, the initial over-approximation is just a certain Ext group, and the geometric relations correspond to various products and operations on Ext, converging to products and operations in π_*.

One interesting and important side-plot to this story is the so-called Hopf Invariant One problem. It turns out that the algebraic question of when can $\mathbb{R}^n$ be a division algebra is equivalent to a question about existence of certain elements in homotopy groups of spheres. Candidates for these elements are easily recognizable in the Adams Spectral Sequence, so if one can determine which of these candidates survive to become homotopy classes, one can answer this question.

This paper will be organized as follows. In Section 2, we will go over requisite notions from homotopy theory, state classical theorems, define the Hopf Invariant and prove the relation between it and division algebras over $\mathbb{R}$.

In Section 3, we will construct the stable homotopy category and gives its basic properties. Frank Adams said of spectra "To use the machine, it is not necessary to raise the bonnet", and in that spirit we will omit cumbersome and unnecessary proofs. Luckily, the actual construction of the category will not be used again after this section; instead we will rely only on the formal properties from this chapter.

In Section 4, we set up the Adams Spectral Sequence and prove that it converges to the result we want. We will show how one can topologically calculate the differentials and identify elements in the

Spectral Sequence related to the Hopf Invariant One problem. It is helpful, when thinking about the Adams Spectral Sequence, to have a picture in mind, so you know what kind of problems can arise. For this reason, the reader should from this point forward keep Appendix B handy, where diagrams of the Adams Spectral Sequence of spheres can be found. The reader is encouraged to annotate this picture by penciling in differentials as they come up throughout the paper.

In Section 5, we prove additional structures on the Adams Spectral Sequence. These structures can be computed algebraically using Appendix A and Appendix C. We will show, topologically, that the structures on the Adams Spectral Sequence greatly restrict the possible differentials. There is a product structure, converging to the composition product in stable homotopy, with the property that if you know the differentials on a and b, then you know the differentials on ab. This structure alone implies all differentials in the first 13 stable homotopy groups. Next we will show there is are a family of operations called Steenrod Operations on the Adams Spectral Sequence, which we will describe geometrically.

The goal of Section 6 is to derive differential formulas for the Steenrod Operations described in Section 5. Proving these formulas will require reasoning about the cell structure of quotients of the skeleta of spectra corresponding to $\mathbb{R}P^\infty$, which can be very different depending on exactly skeleta involved. This will cause the differential formulas to take different forms depending on the dimensions in question. Lastly, one of the formulas proved in this section will apply to the elements corresponding to potential Hopf Invariant maps. Thus formula will put to rest once and for all the classification of finite dimensional division algebras over $\mathbb{R}$.

In Section 7 we will discuss computational methods which we implemented for computing in the Adams Spectral Sequence. The computations come in two main flavors. The first is deterministically computing necessary homological constructions, including computing Ext over $\mathcal{A}$, products in Ext and Steenrod Operations on Ext. It will turn out that computing Steenrod Operations naively is beyond the reach of modern computing technology. Bruner and Nassau used a trick to reduce the computational load in low-dimensions. We prove that this trick *does not* generalize but instead reduces to general graph-coloring, which is known to be NP-complete. The statement and proof of these complexity results are original, to the best of the author's knowledge. The second flavor of computational methods will be to reason about the differentials by automatically propagating known information through the spectral sequences using techniques inspired by machine verification. It turns out that most of the known differentials can be computed in this way. These methods are also novel, to the best of the author's knowledge.

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1.1 Notation

We adopt the following notations. When we say a topological space, we always mean a CW-complex, to avoid unnecessary pathologies. For functors, we will often use a ? to suppress the argument's name. For all standard functors on topological spaces, for instance the suspension Σ and the cone C, we use the "pointed" or "reduced" version. The symbol $[?, ?]$ should be taken to mean basepoint-preserving homotopy classes of maps, and all homology and cohomology functors are reduced. We use $\mathbb{F}_p$ to be mean the field with p elements, and $\mathbb{Z}/p = \langle\rho|\rho^p = 1\rangle$ to emphasize that it is a multiplicative group on two elements. Throughout this paper, we will only be considering the 2-primary component of the stable homotopy groups. One can actually work p-primary for any prime, but for odd primes the statements of many theorems become vastly more complicated, so we restrict ourselves.

2 Definitions and Prerequisites

Before we can get started on this journey, we will need some preliminary results. We will not present every proof and every detail, but rather list the necessary results with references and make some remarks on their consequences. We will define homotopy groups, state some theorems, define stable groups and introduce an important technical tool: the Steenrod Operations.

2.1 Homotopy Groups

Definition 2.1.1 (Homotopy Groups)**.** *Let S^n be the n-sphere and X be a topological space with basepoint. Define the set*

$$\pi_n(X) = [S^n, X]$$

be the set of homotopy classes of basepoint-preserving maps. Notice that $\pi_1(X)$ is the familiar fundamental group.

We need to define the group operation on these groups, which in fact we can do for maps from the suspension of any space Y. Let Σ be the reduced suspension functor, and

$$f, g : \Sigma Y \to X$$

be pointed maps, so we can form

$$f \vee g : \Sigma X \vee \Sigma Y \to X$$

By collapsing the "equator" of ΣY, which we identity with Y, we can write $\Sigma Y \vee \Sigma Y \cong \Sigma Y/Y$. Letting $p : \Sigma Y \to \Sigma Y \vee \Sigma Y$ be the projection, we define

$$f + g = (f \vee g) \circ p : \Sigma Y \to X$$

It is not hard to see this sum is well-defined on homotopy classes of pointed maps, homotopy-associative, has null-homotopic maps as identity and has inverses up to homotopy, given by "swapping the cones" in the suspension. This means that $[\Sigma Y, X]$ is a group. Better yet, if $\Sigma^2 Y = S^2 \wedge Y$ is an iterated suspension, this is an abelian group, where the homotopy $f + g \sim g + f$ is given by "rotating" the S^2 smash-factor. It is an easy exercise to make these constructions precise, and they can be found in [Hat01]. Since $S^n = \Sigma^n(S^0)$, we have that $\pi_n(X)$ is a group and, if $n \geq 2$, it is an abelian group.

The calculation of these homotopy groups is notoriously difficult. There is no analogy of Mayer-Vietoris or Seifert-Van Kampen, making it difficult to compute homotopy groups of spaces from the homotopy groups of smaller spaces. Even for the simplest and most ubiquitous spaces, spheres, these computations are a long (but, in my opinion, quite beautiful) journey. We can make a few quick computations, however. First of all, if $k < n$, then a map $S^k \to S^n$ is null-homotopic by cellular approximation, making $\pi_k(S^n) = 0$. We also have that, since S^n is simply-connected for $n > 1$, we have that any map from $S^n \to S^1$ factors through the universal cover, $\mathbb{R}$, which again is contractible. Thus $\pi_n(S^1) = 0$. You might be inclined to hope the rest of the $\pi_k(S^n)$ will fall this easily, but in fact this is the last of the easy computations.

2.2 Some Classical Theorems of Homotopy Theory

We will present three classical and invaluable theorems.

2.2.1 The Hurewicz Map

Just like H_1 is the abelianization of the fundamental group of connected spaces, we can relate the first nontrivial homology group to the first nontrivial homotopy group. We make the following terminology:

Definition 2.2.1. *We say a space X is n-connected if $\pi_i(X) = 0$ for $i \leq n$.*

There is a map

$$h_n : \pi_n(X) \to H_n(X)$$

called the Hurewicz Map, given as follows. Let $[f] \in \pi_n(X)$, and let α generate $H_n(S^n)$. Then $h([f]) = f_*(\alpha) \in H_n(X)$.

Theorem 2.2.2 (Hurewicz Theorem)**.** *Let X be $n-1$ connected for $n > 1$. Then h_n is an isomorphism and h_{n+1} is a surjection.*

This is proved in [Hat01, Thm 4.32].

Corollary 2.2.3. *For all $n > 0$,*

$$\pi_n(S^n) = \mathbb{Z}$$

2.2.2 The Whitehead Theorem

The homotopy groups, and induced maps between them, completely determine the homotopy type of a CW-complexes.

Theorem 2.2.4 (Whitehead Theorem)**.** *Let $f : X \to Y$ be a map of basepointed CW-complexes, such that $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n. Then f is a homotopy equivalence.*

This is proved in [Hat01, Thm 4.5]. The proof comes from the so-called Compression Lemma, which we state for its usefulness. It is proved in [Hat01, Thm 4.6].

Lemma 2.2.5 (Compression Lemma)**.** *Let $f : (X, A) \to (Y, B)$ be a map of CW-pairs. Suppose that for each n such that $X - A$ has an n-cell, $\pi_n(Y/B) = 0$. Then f is homotopic, relative to A, to a map $X \to B$.*

Definition 2.2.6. *In the notation of the Lemma, we say that f is "compressed" to a map into B. If $\pi_n(Y/B)$ is nonzero, we call the nonzero elements obstructions to compression.*

2.2.3 The Freudenthal Suspension Theorem

The homotopy groups of spheres are difficult to compute, but there is a little miracle which makes it possible to compute certain homotopy groups in a stable range. Let Ω be the loop functor, and recall the adjoint relationship between Σ and Ω, which gives a bijection:

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

The miracle is the Freudenthal Suspension Theorem, which is stated as follows:

Theorem 2.2.7 (Freudenthal Suspension Theorem)**.** *Let X be an n-connected space. Then the natural map $X \to \Omega\Sigma X$ induces a map*

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \cong \pi_{k+1}(\Sigma X)$$

which is an isomorphism for $k \leq 2n$ and an epimorphism for $k = n2 + 1$.

Corollary 2.2.8. *If $n > k + 1$, $\pi_{k+n}(S^n)$ is independent of n.*

There are a few proofs of this. A primitive homotopy based proof is given in [Hat01, Cor 4.24]. A proof using the Serre Spectral Sequence is given in [MT68, Ch 12]. There is also a Morse Theory based proof given in [Mil63, Cor 22.3].

Because of this theorem, we can define the so called Stable Homotopy Groups.

Definition 2.2.9. *The k^{th} stable homotopy group of a space X is given*

$$\pi_k^s(X) = \varinjlim_n \pi_{k+n}(\Sigma^n X)$$

We will find that these so called Stable Homotopy Groups are somewhat easier to compute. The computation of these groups will be the goal of the rest of the paper.

2.3 Eilenberg-Maclane Spaces and the Cohomology Operations

Definition 2.3.1 (Eilenberg-Maclane Space)**.** *Let G be an abelian group. We say that a space $K(G,n)$ is an Eilenberg-Maclane Space if*

$$\pi_k(K(G,n)) = \begin{cases} G & n = k \\ 0 & n \neq k \end{cases}$$

The isomorphism between $\pi_n(K(G,n))$ and G is fixed.

There is a construction of Eilenberg-Maclane spaces which has no i cells for $i < n$, but unfortunately most are infinite-dimensional and complicated. However, these spaces have enormous theoretic importance, for reasons we will see in soon. However, given any two, a map between them can be found inducing isomorphisms in homotopy groups, meaning $K(G,n)$ is unique up to homotopy.

Now, consider

$$H^n(K(G,n);G)$$

By the universal coefficient theorem and the Hurewicz Theorem, this is the same as

$$\mathrm{Hom}(H_n(K(G,n)),G) \cong \mathrm{Hom}(\pi_n(K(G,n)),G) \cong \mathrm{Hom}(G,G)$$

This means there is a unique cohomology class in $\iota_n \in H^n(K(G,n);G)$ corresponding to the identity in $\mathrm{Hom}(G,G)$, which we will call the "fundamental class". Note that this is only well defined because the isomorphism $\pi_n(K(G,n)) = G$ is fixed. Define

$$\Phi : [?, K(G,n)] \to H^n(?)$$

be the natural transformation defined by

$$\Phi([f]) = f^*(\iota_n)$$

Theorem 2.3.2. Φ *is an isomorphism*

Proof. First we notice that this works for S^n, since

$$H^n(S^n;G) \cong \mathrm{Hom}(\mathbb{Z},G) \cong G$$

Let X_i be the i-skeleton of X. Then by CW-approximation, any $f : X_n \to K(G,n)$ is homotopic to a map of pairs

$$f : (X_n, X_{n-1}) \to (K(G,n), *)$$

since $K(G,n)$ can be given a cell-structure with no $n-1$ cells. This means f factors through a wedge of spheres

$$\bigvee S^n$$

and so in homotopy, f is essentially wedge sum of elements of $\pi_n(K(G,n)) = G$. Thus Φ surjects onto $C^*(X;G)$ at the cochain level, so also at the cohomology level. Since, at the cochain level, maps yielding the same cochain are homotopic, and since Φ is surely well-defined on cohomology, two maps yielding the same cohomology class are homotopic, so Φ is also injective.

Letting $i : X_{n+1} \to X$ and $j : X_n \to X_{n+1}$ be the inclusions, we have a diagram

$$\begin{tikzcd}
[X, K(G,n)] \arrow[r, "i^*"] \arrow[d, "\Phi"] & {[X_{n+1}, K(G,n)]} \arrow[r, "j^*"] \arrow[d, "\Phi"] & {[X_n, K(G,n)]} \arrow[d, "\Phi"] \\
H^n(X;G) \arrow[r, "i^*"] & H^n(X_{n+1};G) \arrow[r, "j^*"] & H^n(X_n;G)
\end{tikzcd}$$

Now, if a map $f : X_n \to K(G,n)$ can be extended to X_{n+1}, it can be extended all the way up to X, since $\pi_i(K(G,n)) = 0$ for $i > n$, so the top i^* is an isomorphism. The top j^* is an injection, since obstructions to extending a homotopy lie in $\pi_{n+1}(K(G,n))$, so homotopic maps from X_n are homotopic in X_{n+1}. Likewise, the lower i^* is an isomorphism, again by cellular homology and j^* is an injection. Thus, because the diagram commutes (Φ is natural), Φ is an isomorphism for X. □

This is a strange and remarkable theorem. Since we cannot usually visualize $K(G, n)$, this does not help with computation, but it makes the formal properties of H^* obvious.

We now introduce cohomology operations.

Definition 2.3.3. *Let $O(n, \pi, m, G)$ be the set of natural transformations*

$$H^n(?; \pi) \to H^m(?; G)$$

We call these cohomology operations

Corollary 2.3.4.

$$O(n, \pi, m, G) = H^m(K(\pi, n); G)$$

Proof. Seeing $H^m(K(\pi, n); G)$ as maps $[K(\pi, n), K(G, m)]$, post-composition obviously gives natural transformations. Given such a natural transformation, we can apply it to $i \in H^n(K(\pi, n); \pi) = [K(\pi, n), K(\pi, n)]$, yielding an element in $H^m(K(\pi, n); G)$. Composition of these maps yields identity, since i is the identity in $[K(\pi, n), K(\pi, n)]$. □

This fact will become very important later.

2.4 The Hopf Invariant

The Hopf Invariant is a somewhat mysterious invariant of maps between spheres of certain sizes. The existence of maps whose Hopf Invariant is equal to one is closely related to beautiful and elementary algebraic facts, in particular, the existence of finite dimensional division algebras over $\mathbb{R}$. We can define the invariant as follows. Let n be an integer

$$f : S^{4n-1} \to S^{2n}$$

Letting f be the attaching map of a $4n$ cell, so we can form the complex

$$S^{2n} \cup_f D^{4n}$$

It is easy to see, by cellular cohomology, that the reduced cohomology group is

$$H^*(S^{2n} \cup_f D^{4n}) = \Sigma^{2n}\mathbb{Z} \oplus \Sigma^{4n}\mathbb{Z}$$

If we let α be the cohomology class in dimension $2n$ and β be the dimension of the cohomology class in $4n$, we have that there is an integer h with

$$\alpha \smile \alpha = h\beta$$

Definition 2.4.1. *Way say that h is the Hopf Invariant of the (unstable) map $f \in \pi_{4n-1}(S^{2n})$. This is only well defined up to sign, or choice of generator.*

It is worth trying to describe a map of Hopf Invariant One. Let

$$\eta : S^3 \to S^2$$

be given as follows by considering $S^2 = \mathbb{C}P^2 = S^3/S^1$, also written as a fibration

$$S^1 \to S^3 \to S^2$$

This is known as the Hopf Fibration. Notice that

$$S^2 \cup_\eta D^4 = \mathbb{C}P^3$$

and since

$$H^*(\mathbb{C}P^3) = \mathbb{Z}[\alpha]/(\alpha^3)$$

we have that the Hopf Invariant of η is one.

This is low-dimensional enough to visualize. Because the preimage of any point is a circle, the preimage of a circle is a torus, but somehow twisted, making the preimage of a line is a mobius strip. This is described in detail in [Hat01, Example 4.45], and there are numerous visualizations which can be found.

We will state an easy lemma.

Lemma 2.4.2. *The Hopf Invariant is a homomorphism of groups*

$$H : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$$

As promised, there is a fantastic equivalence

Theorem 2.4.3. *When n is even, the following are equivalent:*

1. *There is an element of Hopf Invariant 1 in $\pi_{2n-1}(S^n)$.*
2. *There is an n-dimensional division algebra over $\mathbb{R}$ (not necessarily commutative or associative).*
3. *There is a map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ and a point $e \in S^{n-1}$ with $\mu(e, x) = \mu(x, e) = x$ for all $x \in S^{n-1}$. We say that S^{n-1} is an H-space in this case.*

Note that when n is odd, the hairy ball theorem makes (2) and (3) false.

Proof. To see that (2) implies (3), set

$$\mu(x, y) = xy/||xy||$$

where $||.||$ is the Euclidean norm in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a division algebra, we know $xy \neq 0$ when $x \neq 0$ and $y \neq 0$, so this is well defined and continuous if x and y are on S^{n-1}. Also, we can assume that $1 = (1, 0, ..., 0)$ is the identity of the division algebra structure on $\mathbb{R}^n$, so μ has an identity element $e = 1$.

To see that (3) implies (1), suppose we have a μ and write

$$S^n = D^n_+ \cup D^n_-$$

where we identify the boundaries and

$$S^{2n-1} = \partial(D^n \times D^n) = (\partial D^n) \times D^n \cup D^n \times (\partial D^n)$$

And define

$$f(x, y) = \begin{cases} ||x||\mu(\frac{x}{||x||}, y) \in D^n_+ & (x, y) \in (\partial D^n) \times D^n \\ ||y||\mu(x, \frac{y}{||y||}) \in D^n_- & (x, y) \in D^n \times (\partial D^n) \end{cases}$$

Once can check that this is well defined and continuous. We claim that the Hopf invariant of f is ± 1, or equivalently, that the cup product

$$H^n(S^n \cup_f D^{2n}) \otimes H^n(S^n \cup_f D^{2n}) \to H^{2n}(S^n \cup_f D^{2n})$$

is surjective.

Let $\Phi : D^{2n} \to S^n \cup_f D^{2n}$ be the characteristic map of the $2n$ cell. I claim we have the following diagram

$$\begin{tikzcd}
H^n(S^n \cup_f D^{2n}) \otimes H^n(S^n \cup_f D^{2n}) \arrow[rr] & & H^{2n}(S^n \cup_f D^{2n}) \\
H^n(S^n \cup_f D^{2n}, D^n_+) \otimes H^n(S^n \cup_f D^{2n}, D^n_-) \arrow[u, "\approx"] \arrow[rr] \arrow[d, "\Phi^* \otimes \Phi^*"'] & & H^{2n}(S^n \cup_f D^{2n}, S^n) \arrow[u, "\approx"'] \arrow[d, "\Phi^*"', "\approx"] \\
H^n(D^n \times D^n, (\partial D^n) \times D^n) \otimes H^n(D^n \times D^n, D^n \times (\partial D^n)) \arrow[rr] \arrow[d, "\approx"'] & & H^{2n}(D^n \times D^n, \partial(D^n \times D^n)) \\
H^n(D^n \times \{e\}, (\partial D^n) \times \{e\}) \otimes H^n(\{e\} \times D^n, \{e\} \times (\partial D^n)) \arrow[rr, "\approx"] & & H^{2n}(D^n \wedge D^n, \partial(D^n \times D^n)) \arrow[u, "\approx"']
\end{tikzcd}$$

The top row is the cup product, which we want to show is a surjection. The second row is the relative cup product. Since the left subcomplexes are both contractible, the left map is an isomorphism. The right map is an isomorphism because S^n has no effect on H^{2n} The next row is a obtained by applying the characteristic map Φ. The right vertical map is an isomorphism because of the short exact sequence from the cofibration (the connecting maps must be zero!)

$$0 \to H^*(S^{2n}) \to H^*(S^n \cup_f D^{2n}) \to H^*(S^{2n}) \to 0$$

which implies that

$$\Phi^* : H^*(S^n \cup_f D^{2n}, S^2) \to H^*(S^{2n})$$

is an isomorphism. Finally, the bottom row is Künneth, the bottom left vertical map is an isomorphism because it is a deformation retract and the bottom right vertical map is induced by a homeomorphism.

Finally, (1) will imply (2) once we have classified all maps of Hopf Invariant One and associated to them a division algebra.

□

2.5 The Steenrod Operations

It turns out that we can explicitly describe the cohomology operations in $\mathbb{F}_2$ cohomology in terms of operations called Steenrod Squares. Loosely speaking, the Steenrod Squares measure the failure of the cup product square to be commutative. The $\mathbb{F}_2$-algebra they generate is known as the Steenrod Algebra, denoted $\mathcal{A}$. It will show up throughout the calculations of the stable homotopy groups.

We will give a (non-minimal) axiomatic description of the Steenrod Squares. The let H^n denote mod-2 cohomology.

Theorem 2.5.1 (Steenrod Squares)**.** *For each $i \geq 0$ there is a natural map*

$$Sq^i : H^n(?) \to H^{n+i}(?)$$

Such that

1. *Sq^0 is the identity map*
2. *Sq^1 is the Bockstien Map*
3. *Sq^n is the cup-product square*
4. *Sq^i is 0 for all $i > n$.*

5. *The Cartan formula holds:*
$$Sq^k(a \smile b) = \sum_{i+j=k} Sq^i(a) \smile Sq^j(b)$$
6. *Adem relations hold: if $i < 2j$*
$$Sq^iSq^j = \sum_{k=0}^{i/2} \binom{j-k-1}{i-2k} Sq^{i+j-k}Sq^k$$
7. *The Steenrod Squares commute with the natural isomorphism $H^n(X) \to H^{n+1}(\Sigma X)$, that is, they are "stable"*

The construction of these operations specifically is given in [MT68, Ch 2]. The construction is similar to A.2, but in a different setting.

We want to consider some of the structure of the Steenrod Algebra $\mathcal{A}$. First notice that, by the Adem relations, it is spanned, as an $\mathbb{F}_2$-vector space, by $Sq^{i_1}Sq^{i_n}...Sq^{i_k}$, where $i_j \geq i_{j+1}$. It is shown in [MT68, Ch 6] that these elements are actually a basis, known as the Serre-Cartan Basis. The Steenrod Algebra is also, importantly, a Hopf Algebra, with comultiplication given by
$$Sq^i \mapsto \sum_{i=j+k} Sq^j \otimes Sq^k$$
One can check that this, with the obvious unit and counit (sending Sq^0 to 1, other squares to 0 and 1 to Sq^0) is a Hopf Algebra.

An easy application of the definition is to show the nonexistence of certain elements of Hopf Invariant One. For a space to have Hopf Invariant One, it must have Hopf Invariant equivalent to 1 (mod 2). Consider $X = S^{2n} \cup_f e^{4n}$, and let η and μ be the two generators of $\mathbb{F}_2$-cohomology. Then if the Hopf Invariant of f is 1 (mod 2), $Sq^{2n}(\eta) = \mu$. Now, suppose that we can decompose the operation Sq^{2n} (considered as a natural transformation, or equivalently, as an element of the Steenrod Algebra) as a sum of products of Steenrod Squares Sq^i with $i < 2n$. This means that $Sq^{2n}(\eta) = 0$, because for all $0 < i < 2n$ we have $H^{i+2n}(X) = 0$, since there are no cells in these dimensions. But it is an easy exercise in using the Adem relation to show Sq^i is decomposable if i is not a power of 2. Thus we can conclude

Corollary 2.5.2. *There is no element of $\pi_{4n-1}(S^{2n})$ of Hopf Invariant 1 except possibly if n is a power of 2.*

Corollary 2.5.3. *$\mathbb{R}^n$ is not a division algebra except possibly if n is a power of 2.*

3 Spectra and the Stable Homotopy Category

Using the Freudenthal Suspension Theorem 2.2.3, we can construct the topological invariant $\pi_*^s(X)$ of stable homotopy groups. The functor π_*^s itself is somewhat awkward. It would be nice if we could create a category in which it was representable, that is, where stable maps were simply maps. Stable homotopy in the category of spaces is in fact a rather awkward game, because one constantly must suspend maps to stay in the stable range. To save ourselves from this annoyance, we will work in a "stable homotopy category", that is, a category of "spectra".

Spectra are a sort of dimension-less generalization of topological spaces. There is a functor from spaces to spectra called Σ^∞, with the homotopy classes of maps between $\Sigma^\infty S^0$ and $\Sigma^\infty Y$ being exactly $\pi_*^s(Y)$. Another somewhat strange thing happens; cohomology theories end up begin objects in the stable homotopy category. Cohomology theories end up on the same footing as Σ^∞ of spaces, and these "cohomology spectra" represent the cohomology functors on spectra.

The category has a number of improved formal properties over topological spaces. First of all, there is a "desuspension" functor Σ^{-1} which is an inverse to the suspension functor Σ. This means for any two spectra X and Y, $[X, Y]$ is an abelian group (see 2.1). Better yet, cofibrations and fiberations are the same, so when you form the Puppe-Barratt Sequence (discussed in [Hat01, Ch 4.3]) of a map, you can, for any spectrum X, apply the functor $[?, X]$ or $[X, ?]$, and get a long exact sequence. Of course, cohomology and stable homotopy are functors of that form in this category.

Without further ado, let us define the stable homotopy category, which we do following [Ada95, Ch 2]

3.1 Spectra

We will now define the objects in the stable homotopy category.

Definition 3.1.1. *A Spectrum is a sequence of topological spaces $X = \{X_n\}$ for $n \in \mathbb{Z}$, along with structure maps*

$$\epsilon_n : \Sigma X_n \to X_{n+1}$$

You can always assume the structure maps are inclusions of subcomplexes. A "cell" of a Spectrum is an equivalence class of cells e^m in X_n, where two cells e^m and $e^{m'}$ ($m < m'$) are equivalent if e^m becomes $e^{m'}$ after $m' - m$ applications of the structure map. The dimension of a cell represented by e^m in X_n is $m - n$.

We promised a functor from spaces to spectra, which will be an important source of examples.

Definition 3.1.2. *Let X be a topological space. Let $\Sigma^\infty X$ be a topological space with*

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ \{*\} & n < 0 \end{cases}$$

We call this the "suspension spectrum" of X. We define

$$S = \Sigma^\infty S^0$$

Sometimes we will write

$$S^n = \Sigma^n S$$

when no confusion is possible.

Let K^n be a cohomology theory, and let it be represented

$$K^n(?) = [X, K_n]$$

for some space K_n (for instance K_n could be $K(G, n)$ for some G). We have

$$[K_n, K_n] \cong K^n(K_n) \cong K^{n+1}(\Sigma K_n) \cong [\Sigma K_n, K_{n+1}]$$

Define the image of the identity in $[\Sigma K_n, K_{n+1}]$ to be the structure map, making a generalized cohomology theory into a spectrum. We call the spectrum associated with $K(G, n)$ by the name HG.

3.2 Functions, Maps and Morphisms

We now have two rich sources of examples of spectra, but objects alone do not a category make. We also need to define the morphisms of our category. We do this first by defining "functions".

Definition 3.2.1. *A degree i function between two spectra $f : X \to Y$ is a series of maps of topological spaces*

$$f_n : X_n \to Y_{n-i}$$

that commute with suspension, that is, for all n,

$$\epsilon_{n-i}(\Sigma f_n) = f_{n+1}\epsilon_n$$

The problem with functions is that there are not enough of them. The requirement they be defined on every single space keeps them from being useful in a stable sense. For instance, let $\eta : S^3 \to S^2$ be the Hopf Fibration (see 2.4). We would want this to define a degree 1 function $\eta : S \to S$ with $\eta_3 = \eta$. This means that would have to be $\eta_n = \Sigma^{n-3}\eta$ for $n \geq 3$. But what can η_2 be? Since is a map $\eta_2 : S^2 \to S^1$, η_2 must be null-homotopic, so the suspension of it would have to be null-homotopic, which the Hopf Fibration is not. We want it to be the case that only what goes on after many suspensions matter. Thus we must weaken this notion.

Definition 3.2.2. *A "subspectrum" $K \subset X$ is a sequence of spaces $K_i \subset X_i$ such that $\epsilon_i|_{\Sigma K_i} \subset K_{i+1}$. A "cofinal" subspectra $K \subset X$ is a subspectra such that for each n and each cell $e \in X_n$ there is an i such that applying the structure map to e i times will land e in K_{i+n}.*

Note that the intersection of two cofinal subspectra is again cofinal.

Definition 3.2.3. *A "map" $f : X \to Y$ between two spectra X and Y is an equivalence class of functions defined on any cofinal subspectra $K \subset X$. Two maps are considered equal if they are equal on the intersection of their domains.*

Now we can make $\eta : S \to S$ a map, since it is defined on the cofinal subspectrum $K \subset S$ with

$$K_n = \begin{cases} S^n & n \geq 3 \\ \{*\} & n < 3 \end{cases}$$

Finally, we can define morphisms

Definition 3.2.4. *Let X and Y be spectra. Then* $\mathrm{Cyl}(X)$ *is a spectra with*

$$(\mathrm{Cyl}(X))_n = I^+ \wedge X_n$$

where I^+ is the unit interval with disjoint basepoint. Note that the obvious structure map $1 \wedge \epsilon_n$ works as a structure map. There are two natural injections $i_1, i_2 : X \to \mathrm{Cyl}(X)$. Two maps, f and g, are homotopic if there is a map

$$H : \mathrm{Cyl}(X) \to Y$$

with $Hi_1 = f, Hi_2 = g$. A "morphism" $f : X \to Y$ is a homotopy class of maps. We let $[X, Y]_n$ be the set up homotopy classes of degree n maps between spectra X and Y.

We can now make the definition

Definition 3.2.5. *Let X be a spectrum. Define the homotopy group*

$$\pi_n(X) = [S, X]_n$$

Here is the thing to notice. The homotopy groups in the stable category are exactly the stable homotopy groups, that is

Theorem 3.2.6. *Let X be a topological space. Then*

$$\pi_n^s(X) = \pi_n(\Sigma^\infty(X))$$

Make sure to convince yourself of this before moving on.

We can define cohomology, but it is somewhat different than for spaces. A cohomology theory is always a spectrum, but in fact any spectrum is sufficient to define a cohomology functor on spectra.

Definition 3.2.7. *Let E and X be spectra. The E-cohomology of X is given*

$$E^k(X) = [X, E]_k$$

Notice that

$$(HG)^k(\Sigma^\infty X) = H^k(X; G)$$

and

$$(H\mathbb{F}_2)^*(H\mathbb{F}_2) = \mathcal{A}$$

by Cor 2.3.4. Notice also that negative dimensional cohomology can be nonzero, for instance

$$(H\mathbb{Z})^{-1}(\Sigma^{-1}S) = \mathbb{Z}$$

3.3 Additive Category of Spectra

We have the following self-evident consequence:

Lemma 3.3.1. *If X is a spectra and $K \subset X$ is cofinal, then the inclusion $K \to X$ is an isomorphism.*

Notice that since the subspectrum of a spectrum defined by collapsing negative-indexed spaces to a point is cofinal, so it doesn't matter whether we consider spaces indexed by all integers or positive integers.

Definition 3.3.2. *Let X be a spectra. Define ΣX by*

$$(\Sigma X)_n = \Sigma X_n$$

and use the obvious structure maps.

Obviously this is functorial. Define shift_+ and shift_- to be the obvious functors with $(\text{shift}_+X)_n = X_{n+1}$ and $(\text{shift}_-X)_n = X_{n-1}$. The structure maps define a degree 0 map

$$\epsilon : \Sigma X \to \text{shift}_+X$$

The image of ϵ is obviously cofinal and ΣX is isomorphic to its image, so ϵ is an isomorphism. But shift_- is an obvious inverse to shift_+, so there is a functor Σ^{-1} inverting Σ. Thus any spectrum $X \cong \Sigma^2 X'$ for some spectrum X', so X is isomorphic to a spectrum where each X_n is a double-suspension.

Let X and Y be spectra and $K \subset X$ be a cofinal subspectrum on which $f, g : K \to Y$ be representing functions for maps f and g. We can assume that for each n, K_n is a double suspension, so we can form

$(f+g)_n = f_n + g_n$. While the sum is only defined up to homotopy, representing maps can be picked to commute so that $f+g$ is a bonafide morphism in the stable homotopy category. This makes $[X,Y]_n$ a graded abelian group. Obviously composition in either direction is a bilinear group homomorphism, that is, function composition gives abelian group-maps

$$[X,Y]_* \otimes [Y,Z]_* \to [X,Z]_*$$

We can construct wedge products in the obvious way

Definition 3.3.3. *If X and Y are spectra, we can form $X \vee Y$ with*

$$(X \vee Y)_n = X_n \vee Y_n$$

Using that

$$\Sigma(X \vee Y) = (\Sigma X) \vee (\Sigma Y)$$

we define the structure maps by

$$\epsilon_i^X \vee \epsilon_i^Y : \Sigma X_i \vee \Sigma Y_i \to X_{i+1} \vee Y_{i+1}$$

for reduced suspensions. The same construction works for infinite wedge products.

This is obviously a coproduct, that is, for any spaces X, Y, W, we naturally have

$$[X \vee Y, W]_* \cong [X,W]_* \oplus [Y,W]_*$$

Consider the sequence of spectra

$$X \to X \vee Y \to Y$$

given by inclusion and then projection. The sequence

$$[W,X]_* \to [W, X \vee Y]_* \to [W,Y]_*$$

is clearly a short exact sequence, naturally split by the map induced from the inclusion $Y \to X \vee Y$. Thus we have, for any space W

$$[W, X \vee Y]_* \cong [W,X]_* \oplus [W,Y]_*$$

This is the universal property of products, so $X \vee Y$ is a product as well as a coproduct.

Finally, since $\{*\}$ has the property that $\Sigma\{*\} = \{*\}$ (again, recall we have been using reduced suspension), there is a zero spectrum $\{*\} = \Sigma^\infty\{*\}$.

3.4 Fibrations and Cofibrations

Let $i : A \to X$ be an inclusion of subcomplexes for (unstable) spaces. We can then form the mapping cylinder and cofibration sequence

$$A \xrightarrow{i} X \xrightarrow{j} X \cup_i CA$$

Let W be any space, and consider

$$[X \cup_i CA, W] \xrightarrow{j^*} [X,W] \xrightarrow{i^*} [A,W]$$

Obviously $(ij)^* = 0$, since it includes A into its contractible cone. Also, if a map $g : X \to W$ has $i^*(g) = 0$, then g restricted to A is nullhomotopic, but a nullhomotopy is just a map $CA \to W$ extending g, so g can be extended to $CA \cup_i X$. Thus the sequence above is exact, in the sense of pointed sets.

Now, j is an inclusion of a subcomplex, so we can continue the sequence

$$A \xrightarrow{i} X \xrightarrow{j} X \cup_i CA \xrightarrow{k} X \cup_i CA \cup_j CX \cong \Sigma A$$

if we continued the sequence we would get a map homotopic to $-\Sigma(i)$ going to ΣX, and so on. Each three terms is the inclusion of a subcomplex and a mapping cone, so is exact. Finally, there is a natural homotopy equivalence $CA \cup_i X \to X/A$ and isomorphism $[\Sigma X, Y]_n \cong [X, Y]_{n-1}$ We summarize in a lemma

Lemma 3.4.1 (Cofibration Sequence of Spaces)**.** *If $A \to X$ is an inclusion of a subcomplex of (unstable) spaces, then there is a long exact sequence of pointed sets for any space W*

$$... \to [X/A, W] \to [X, W] \to [A, W] \to [\Sigma(X/A), W] \to ...$$

Quotients by subcomplexes commute with suspenison, so given spectra $A \subset X$ which is an inclusion of subcomplexes, we can form X/A in the obvious way, and of course this is equivalent to a spectra $X \cup CA$ with $(X \cup CA)_n = X_n \cup CA_n$. Now, once again, a nullhomotpy from $g : X \to Y$ is a map $CX \to Y$ with which restricts to g. Thus, the argument above yields

Lemma 3.4.2 (Cofibration Sequence of Spectra)**.** *If $A \to X$ is an inclusion of a spectrum which is an inclusion of subcomplexes on each space, then Lemma 3.4.1 holds, giving a long exact sequence of abelian groups. This is known as a cofibration sequence of spectra.*

Lemma 3.4.3 (Fibration Sequences of Spectra)**.** *If $A \to X$ is a cofibration, then*

$$... \to [W, A]_n \to [W, X]_n \to [W, X/A]_n \to [W, A]_{n-1} \to ...$$

Proof. Obviously $A \to A/X$ is null homotopic, so let $g : W \to X$ be a map which becomes nullhomotopic in X/A. Consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "i"] & X \arrow[r, "j"] & CA \cup X \arrow[r, "k"] & \Sigma A \arrow[r, "-\Sigma i"] & \Sigma X \\
W \arrow[r, "1"] \arrow[u, dashed, "l'"] & W \arrow[r] \arrow[u, "g"] & CW \arrow[r] \arrow[u, "h"] & \Sigma W \arrow[r, "-1"] \arrow[u, "l"] & \Sigma W \arrow[u, "\Sigma g"]
\end{tikzcd}$$

We get h from the nullhomotpy jg and l from attaching another copy of h. Let $l = \Sigma l'$. Then we have $\Sigma(il') = \Sigma g$, so $il' = g$, so the g can be compressed to A and the first point is exact. The other exactness points follow from the symmetry in the sequence. □

Definition 3.4.4. *A triple (A, X, C) where $A \to X$ is a cofibration and C is the cofiber is called a"distinguished triangle".*

3.5 Classical Theorems

We state some classical theorems that hold in the category of spectra. The details are omitted, since they are similar to the situation on spaces and will play no roll in what is to follow. The full proofs can be found in [Ada95, Ch 2].

Theorem 3.5.1. *Homology and cohomology of spectra can be computed as the homology of (stable) cellular chains. The differentials are given by taking differentials on any cofinal subspectrum.*

This follows from the space level theorem, and remains a useful way to compute homology.

Theorem 3.5.2. *The Whitehead Theorem (2.2.4) holds in the category of spectra.*

The proof is almost exactly the same. The only subtlety is that Whitehead requires isomorphisms in all negative degrees.

Theorem 3.5.3. *The Hurewicz Theorem (2.2.2) holds in the category of spectra.*

The proof follows from the space level theorem. The subtlety this time is that connectivity can be negative or not at all.

3.6 Smash Products

Our category is equipped with a smash product. The construction is notorious for being as confusing as it is unnecessary. Frank Adams himself said of them, “In order to operate the machine, it is not necessary to raise the bonnet”, and we will take this approach as well. A construction can be found in [Ada95, Ch 4] The smash product should be seen as a generalization of the smash product of spaces and similar to a tensor product on modules.

Theorem 3.6.1. *Let X, Y be spectra. Then there is a spectrum $X \wedge Y$ with the properties that*

1. $S \wedge X = X$
2. *If X and Y are spaces,* $\Sigma^\infty(X \wedge Y) = (\Sigma^\infty X) \wedge (\Sigma^\infty Y)$
3. $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$*. Thus we just write* $X \wedge Y \wedge Z$
4. $X \wedge Y \cong Y \wedge X$
5. $\Sigma(X \wedge Y) \cong (\Sigma X) \wedge Y \cong X \wedge (\Sigma Y)$
6. $[W, Y \wedge Z]_* = [W, Y]_* \otimes [W, Z]_*$
7. $[Y \wedge Z, W]_* = [Y, W]_* \otimes [Z, W]_*$

This completes our brief tour of spectra, as these are all of the results and constructions we will need to construct the Adams Spectral Sequence.

4 Setting up the Adams Spectral Sequence

The Adams Spectral Sequence is a rather heavy-duty machine for computing 2-component of the homotopy groups $[Y, X]_*$ for spectra X and Y. If $Y = S$ and X is a suspension spectrum, this is the 2-component of the stable homotopy groups $\pi_*^s(X)$, which, if you'd like, can be written $\pi^s(X) \otimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the 2-adic integers. While this is twice removed from the original goal of computing $\pi_*(X)$, at least in this case we have a fighting chance at doing the computations. Since there are p-component analogues, these methods can be combined to compute all of $\pi_*^s(X)$. However, as we are about to find, even once the Adam's Spectral Sequence is set up, actually using it is still not easy.

Morally, the Adams Spectral Sequence works as follows. We start with a "geometric" resolution of the space X

$$X \to K_1 \to K_2 \to K_3 \to ...$$

such that each K_i is a wedge sum of Eilenberg-Maclane spaces and the sequence induced in mod-2 cohomology is exact. Thus we get an $\mathcal{A}$-free resolution of $H^*(Y)$, and we consider

$$[Y, X] \longleftarrow [Y, K_1] \longleftarrow [Y, K_2] \longleftarrow ...$$

Since K_i is a wedge sum of $H\mathbb{F}_2$, we have, for H^* being mod-2 cohomology,

$$[Y, K_i] \cong \bigoplus_i [Y, H\mathbb{F}_2] \cong \bigoplus_i H^*(Y) \cong \bigoplus_i \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, H^*(Y)) \cong \mathrm{Hom}_{\mathcal{A}}(H^*(K_i), H^*(Y))$$

One reason we needed to develop spectra is now clear: A space X cannot have $\mathcal{A}$-free cohomology. We can take homology of this chain complex and get $\mathrm{Ext}_{\mathcal{A}}(H^*(X), H^*(Y))$, the starting point or so-called "E_2" term of the Adams Spectral Sequence. We will show this is a sort of over-approximation of $[Y, X]_* \otimes \mathbb{Z}_2$, and the Adams Spectral Sequence "converges" to this. In the literature of spectral sequences, one might write

$$\mathrm{Ext}_{\mathcal{A}}(H^*(X), H^*(Y)) \implies [Y, X] \otimes \mathbb{Z}_2$$

Computing this Ext group can often be tough, but is algebraic in nature and usually more-or-less mechanical. However, we are not done yet.

The second step in running the Adams Spectral Sequence is to figure out exactly how the convergence works. When we construct $\mathrm{Ext}_{\mathcal{A}}(H^*(X), H^*(Y))$, we construct in it a way such that it is still a chain complex. Thus we can take homology and get what we call the E_3 page. Some elements of E_2 will not be cycles, and thus will disappear forever. Others will be related by boundaries and become equal. Thus E_3 is a sub-quotient of E_2, just as E_2 is a sub-quotient of $\mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(Y))$. We can continue this process until all the differentials become zero, which we call E_∞. It can be shown that, under certain hypothesis, the process does stabilize and the final answer, $[Y, X] \otimes \mathbb{Z}_2$, can be read off the E_∞ page. However, computing the differentials on each E_r is notoriously difficult, in fact, it will be the goal of most of the rest of this paper. The differential computations are often geometric in nature. This is not surprising since the algebra of Ext cannot possibly be enough to determine all the homotopy groups.

4.1 The Adams Resolution

Definition 4.1.1. *We say that a spectrum is "connective" if the dimensions in which H^i is nonzero is bounded below. We say that a spectrum is of "finite type" if it has only finitely many cells in each dimension.*

Let X and Y be spectra, and let X be connective and of finite type, and let $H^*(?) = (H\mathbb{F}_2)^*(?)$ be mod-2 cohomology. We want to start by creating the geometric resolution of Y described above. We construct the spaces of the resolution as follows. Consider $H^*(X)$, and which is finitely generated over $\mathcal{A}$. Call those generators $u_i \in [X, H\mathbb{F}_2]_{n_i}$. We can then form

$$\bigvee_i u_i : X \to \bigvee_i \Sigma^{n_i} H\mathbb{F}_2$$

to be a degree 0 map to a wedge sum of suspensions of Eilenberg-Maclane spaces. Call the codomain above K_0. We can then take the fiber of this map and call it X_1. Repeating this process, we get the following diagram.

$$\begin{tikzcd}
X \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & X_3 \arrow[l] \arrow[d] & \dots \arrow[l] \\
K_0 & K_1 & K_2 & K_3 &
\end{tikzcd}$$

This is one specific construction of the following definition:

Definition 4.1.2. *An Adams Complex of a spectrum X is a diagram, as above, where K_0 is a wedge sum of suspensions of $H\mathbb{F}_2$'s, X_{i+1} is the fiber of $X_i \to K_i$.*

An Adams Resolution of a spectrum X is an Adams Complex of X with $X_i \to K_i$ inducing a surjection in cohomology.

Consider an Adams Resolution of X. Now, notice that the fiber of $X_i \to X_{i-1}$ is $\Sigma^{-1}K_{i-1}$. Thus we get a diagram of spectra

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & \Sigma K_1 \arrow[rr] \arrow[dr] & & \Sigma^2 K_2 \arrow[rr] \arrow[dr] & & \Sigma^3 K_3 \arrow[r] & \dots \\
& & \Sigma X_1 \arrow[ur] & & \Sigma^2 X_2 \arrow[ur] & & \Sigma^3 X_3 \arrow[ur] & &
\end{tikzcd}$$

In cohomology we get:

$$\begin{tikzcd}
H^*X & H^*K_0 \arrow[l] & & H^*\Sigma K_1 \arrow[ll] \arrow[dl] & & H^*\Sigma^2 K_2 \arrow[ll] \arrow[dl] & & H^*\Sigma^3 K_3 \arrow[ll] \arrow[dl] & \dots \arrow[l] \\
& & H^*\Sigma X_1 \arrow[ul] & & H^*\Sigma^2 X_2 \arrow[ul] & & H^*\Sigma^3 X_3 \arrow[ul] & &
\end{tikzcd}$$

By exactness of

$$H^*\Sigma^i X_i \to H^*\Sigma^{i-1} K_{i-1} \to H^*\Sigma^{i-1} X_{i-1}$$

this gives an $\mathcal{A}$-free resolution of $H^*(X)$.

(Notice that if this was just a complex, the sequence of maps in cohomology would still be a complex).

Next, we construct what is known in the world of spectral sequences as an "exact couple". Let Y be a spectra and

$$E_1^{s,t} = [Y, K_s]_t$$

$$E_1 = \bigoplus_{s,t} E_1^{s,t}$$

and

$$A_1^{s,t} = \bigoplus [Y, X_i]_t$$

$$A_1 = \bigoplus_{s,t} A_1^{s,t}$$

We call s the filtration level, t the internal degree and $t-s$ the toplogical degree.

Consider the cofibration sequence

$$X_{i+1} \xrightarrow{i} X_i \xrightarrow{p} K_i \xrightarrow{\partial} \Sigma X_{i+1}$$

This induces a diagram

$$\begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "p_1"] \\ & E_1 \arrow[ul, "\partial_1"] & \end{tikzcd}$$

which is exact at each node.

Now, consider

$$d_1 = \partial_1 p_1$$

and note that this is a differential on E_1 since $d_1^2 = \partial_1(p_1\partial_1)p_1 = 0$. Thus, we can let E_2 be the cohomology of (E_1, d_1) and A_2 be the image of i_1. Let i_2 be the restriction, ∂_2 be the quotient of ∂_1 and define p_2 to the image of i_1, p_2 defined by

$$p_2(ia) = [p_1(a)]$$

It is an easy exercise to check that this all makes sense, is well defined and that the diagram

$$\begin{tikzcd} A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "p_2"] \\ & E_2 \arrow[ul, "\partial_2"] & \end{tikzcd}$$

is exact at each point. This is done in, for instance [Hat, Ch 1].

Thus we can iterate this construction and get a sequence of modules E_r and A_r (when X and Y are ambiguous, we write $E_r(Y, X)$ and $A_r(Y, X)$)

Definition 4.1.3. *We call the sequence of bigraded groups E_r the Adams Spectral Sequence*

We will see later that the limit, which we call E_∞, is closely related to $[Y, X] \otimes \mathbb{Z}_2$

This construction, at first, can be rather disorienting. The issue is that E_{r+1} is a sub-quotient (a quotient of a subgroup) of E_r. That means when going from E_r to E_{r+1}, some elements will become equal to others, while some elements will cease to exist. If $x \in E_r$ has $d_r(x) = 0$, then x has some image in E_{r+1}, so we say that x survives to the $r+1$ page, and we use the same symbol to denote it on the E_{r+1} page. This is not as confusing as it might sound, because geometrically x is represented by the same map on the E_r page and the E_{r+1} page. If $d_r(x) \neq 0$, we say the differential kills x. Notice that we say the x survives even if there is some other element y with $d_r(y) = x$, that is, even if x is a boundary and the image of $x = 0$ in E_{r+1}.

We recall

$$E_1^{s,t} = [Y, K_s]_t \cong \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_s), H^*(Y))$$

where Hom^t means maps of graded groups lowering degree by t and where the second equivalence comes from the argument in the introduction. Since the $H^*(K_s)$ make an $\mathcal{A}$-free resolution of $H^*(X)$, we have

$$E_2^{s,t} = \mathrm{Hom}^{s,t}_{\mathcal{A}}(H^*(X), H^*(Y))$$

Let us unravel what this sequence means. Our exact couple unravels into the following diagram where the “staircases” are exact.

Diagram 4.1.4.

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] \\
{[Y, X_s]_{t-s+1}} \arrow[r] \arrow[d] & {[Y, K_s]_{t-s+1}} \arrow[r] & {[Y, X_{s+1}]_{t-s}} \arrow[r] \arrow[d] & {[Y, K_{s+1}]_{t-s}} \arrow[r] & {[Y, X_{s+2}]_{t-s-1}} \arrow[d] \\
{[Y, X_{s-1}]_{t-s+1}} \arrow[r] \arrow[d] & {[Y, K_{s-1}]_{t-s+1}} \arrow[r] & {[Y, X_s]_{t-s}} \arrow[r] \arrow[d] & {[Y, K_s]_{t-s}} \arrow[r] & {[Y, X_{s+1}]_{t-s-1}} \arrow[d] \\
{[Y, X_{s-2}]_{t-s+1}} \arrow[r] \arrow[d] & {[Y, K_{s-2}]_{t-s+1}} \arrow[r] & {[Y, X_{s-1}]_{t-s}} \arrow[r] \arrow[d] & {[Y, K_{s-1}]_{t-s}} \arrow[r] & {[Y, X_s]_{t-s-1}} \arrow[d] \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

How do we calculate the differentials? Put your pencil on the module in the center row: $[Y, K_{s-1}]_{t-s+1}$. Pretend your pencil tip is $x \in [Y, K_{s-1}]_{t-s+1}$. By definition, the map d_1 is obtained by going straight across. If $d_1(x) = 0$, then x represents an element in E_2 and so d_2 is expected to be defined. To calculate d_2, move your pencil to $\partial_1(x) \in [Y, X_s]_{t-s}$. By exactness, since x is zero in $[Y, K_s]_{t-s}$, there is a preimage of x above in $[Y, X_{s+1}]_{t-s}$. Applying p_1 to this will give an element of $[Y, K_{s+1}]_{t-s}$, and this is $d_2([x])$. In general, a differential is calculated by pushing $\partial_1(x)$ "up" as far as you can before applying p_1.

For spheres, we can do a bit better. Exactly like with spaces (you should check this), there is a relative homotopy group

$$\pi_i(A, X) = [(D, S^{-1}), (X, A)]_i$$

where $D = \Sigma^\infty D^0$ and the homotopies are to leave the boundary of D in A. Rewriting our diagram like this for $Y = S$, we have

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] \\
\pi_{t-s+1} X_s \arrow[r] \arrow[d] & \pi_{t-s+1}(X_s, X_{s+1}) \arrow[r] & \pi_{t-s} X_{s+1} \arrow[r] \arrow[d] & \pi_{t-s}(X_{s+1}, X_{s+2}) \arrow[r] & \pi_{t-s-1} X_{s+2} \arrow[d] \\
\pi_{t-s+1} X_{s-1} \arrow[r] \arrow[d] & \pi_{t-s+1}(X_{s-1}, X_s) \arrow[r] & \pi_{t-s} X_s \arrow[r] \arrow[d] & \pi_{t-s}(X_s, X_{s+1}) \arrow[r] & \pi_{t-s-1} X_{s+1} \arrow[d] \\
\pi_{t-s+1} X_{s-2} \arrow[r] \arrow[d] & \pi_{t-s+1}(X_{s-2}, X_{s-1}) \arrow[r] & \pi_{t-s} X_{s-1} \arrow[r] \arrow[d] & \pi_{t-s}(X_{s-1}, X_s) \arrow[r] & \pi_{t-s-1} X_s \arrow[d] \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

You can, without loss of generality (using a mapping-cylinder construction) assume all the maps $X_i \to X_{i-1}$ are inclusions of subcomplexes, although the only difference this makes is to allow you to use language as if $X_i \subset X_{i-1}$. Thus, let

$$f : (D^{t-s+1}, S^{t-s}) \to (X_s, X_{s+1})$$

be an element of $\pi_{t-s+1}(X_{s-1}, X_s)$. Then $\partial_1(f)$ is the boundary of f, that is, $f|_{S^{t-s}}$. The image of $\partial_1(f)$ is in X_s, but you may be able to find a homotopy compressing the image to X_{s+r-1}. Thus $d_r(f)$ is the inclusion of this map into $\pi_{t-s}(X_{s+r-1}, X_{s+r})$. This is the strategy we will use to compute the vast majority of the differentials.

4.2 Convergence of the Adams Spectral Sequence

Before we go any further in discussing how to calculate E_∞, let us prove the following result

Theorem 4.2.1. *Use the notation of above. Let*

$$F^{s,t} = \operatorname{Im}([Y, X_s]_{t-s} \to [Y, X]_{t-s})$$

Then

$$\bigcap_n F^{s+n,t+n} = \mathrm{Torsion}_{p>2}[Y, X]_{t-s}$$

where $\mathrm{Torsion}_{p>2}$ *means the set of all elements annihilated by a power of an odd prime. Finally, for each* (s, t) *there is an* R *such that for all* $r \geq R$

$$E_r^{s,t} = \frac{F^{s,t}}{F^{s+1,t+1}}$$

We call such $E_r^{s,t}$ *by* $E_\infty^{s,t}$*. We can write this compactly as*

$$E_2 = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y)) \implies [Y, X]_{t-s} \otimes \mathbb{Z}_2$$

We need a lemma to prove the rest. First of all.

Lemma 4.2.2. *Adams Resolutions are "comparable", that is, given spectra* X *and* Y*,* $f : X \to Y$ *and Adams Resolutions* X_i *and a complex* Y_i*, you can find* $f_i : X_i \to Y_i$ *making the following diagram commutative up to homotopy*

$$\begin{tikzcd}
X \arrow[d, "f"'] & X_1 \arrow[l] \arrow[d, "f_1"'] & X_2 \arrow[l] \arrow[d, "f_2"'] & \ldots \arrow[l] \\
Y & Y_1 \arrow[l] & Y_2 \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

Proof. Let K_i be the cofiber of $X_{i+1} \to X_i$, L_i the cofiber of $Y_{i+1} \to Y_i$, and recall that the suspensions of the K_i and L_i give free resolutions of X and Y in cohomology. By Theorem 4.2.2 for cohomology, we can find $\hat{f}_i^*$ in cohomology such that the following diagram commutes

$$\begin{tikzcd}
H^*X & H^*K_0 \arrow[l] & H^*\Sigma K_1 \arrow[l] & \ldots \arrow[l] \\
H^*Y \arrow[u, "f^*"] & H^*L_0 \arrow[l] \arrow[u, "\hat{f}_1^*"] & H^*\Sigma L_1 \arrow[l] \arrow[u, "\hat{f}_2^*"] & \ldots \arrow[l]
\end{tikzcd}$$

But recall that that

$$[?, K_i] \cong \mathrm{Hom}_{\mathcal{A}}(H^*(K_i), H^*(?))$$

And thus $\hat{f}_i^*$ is induced by $\hat{f}_i : K_i \to L_i$. But K_i, K_{i+1} and X_i form one distinguished triangle, L_i, L_{i+1} and Y_i form another, and we have maps from the K's to the L's, so we automatically get a map $f_i : X_i \to Y_i$ so that everything commutes. $\square$

The proof from here on out will proceed much like in [Hat, Ch 2].

Proof of 4.2.1: Now, let us focus our attention on $\bigcap_i F^{s+i,t+i}$. Recall from Diagram 4.1.4 that $[Y, K_i]_l$ is a $\mathbb{F}_2$ vector space, and thus it has no odd prime torsion. The staircase is exact, so the vertical maps must be isomorphisms on the odd prime torsion. Thus the odd prime torsion in $[X, Y]_{t-s}$ is passed all the way down from $[Y, X_s]_{t-s}$ to $[Y, X]_{t-s}$ and so $F^{s,t}$ contains it for each (s, t).

For the other direction, pick some integer k. Since $[X, X]$ is an abelian group, we can take the identity map and multiply it by 2^k, which as a map we will denote $(*2^k)$. Let Q be the cofiber of this map, so that we have the long exact sequence

$$... \to [Y, X]_i \xrightarrow{(*2^k)_*} [Y, X]_i \to [Y, Q]_i \to ...$$

and note by exactness that the image of the map $[Y, Q]_i \to [Y, X]_{i-1}$ is all 2-torsion, as is the kernel of $[Y, X]_i \to [Y, Q]_i$. Thus $[Y, Q]_i$ is all 2-torsion. If $\alpha \in [Y, X]_i$ is either odd prime torsion or non-torsion, then our connective and finite-type hypothesis implies $[Y, X]_i$ is finitely generated, so there is a k such that α is not divisible by 2^k. Thus α is not in the image of $(*2^k)$, and so has nonzero image in $[Y, Q]_i$. By Theorem 4.2.2, if α has a preimage in $[Y, X_j]_i$ for all j, then the image of α in $[Y, Q]_i$ will have a similar property for the Adams resolution of Q. Thus it is sufficient to prove that if $[Y, X]_i$ is all 2-torsion then the Adams resolution eventually has $[Y, X]_i = 0$.

To show this, assume $X = Z_0$ is all 2-torsion. Note that all the $[Y, X]_i$ are finite. We inductively build an Adams Complex

$$\begin{tikzcd}
Z_0 \arrow[d] & Z_1 \arrow[l] \arrow[d] & Z_2 \arrow[l] \arrow[d] & Z_3 \arrow[l] \arrow[d] & ... \arrow[l] \\
L_0 & L_1 & L_2 & L_3 &
\end{tikzcd}$$

let n_i be the smallest number with $[Y, Z_i]_{n_i} \neq 0$, and let L_i be a wedge sum of $H\mathbb{F}_2$ on a basis for $H^{n_i}(Z_i)$. Let the map from $X_i \to L_i$ be the obvious one and let Z_{i+1} be the cofiber. Notice that in H^{n_i} the map $Z_i \to L_i$ is an isomorphism, so also in H_{n_i}, so in $[Y, ?]_k$ for $k < n_i$ it is an isomorphism and we have

$$[Y, L_i]_{n_i} = [Y, Z_i]_{n_i} \otimes \mathbb{Z}_2$$

This is a surjection, so by the cofiber sequence we have for $k < n_i$ the group $[Y, Z_{i+1}]_k = 0$ and $[Y, Z_{i+1}]_{n_i}$ is smaller than $[Y, Z_i]_{n_i}$. Since these groups are finite, we must eventually get $[Y, Z_{i+1}]_{n_i} = 0$. This means for each i, there is an n such that $[Y, Z_{i+1}]_i = 0$ for $i \geq n$. Applying Theorem 4.2.2 over the identity map $X \to X$, we find if any element in $[Y, X]_{t-s}$ has preimage in $[Y, X_s]_{t-s}$ for all s (recall to make this sequence we take suspensions, which is equivalent to a grading shift), that element would have nonzero image in $[Y, Z_s]_{t-s}$ for all s, which we just said is impossible. Thus the intersection of the $F^{s+n,t+n}$ must be only odd prime torsion.

We can finally prove the convergence result. Recall that $A_r^{s,t}$ is all the elements of $[Y, X_s]_{t-s}$ with vertical preimages in $[Y, X_{s+r}]_{t-s}$. By that which has been proved thus far, for sufficiently large r this contains no 2-torsion. Also, the map $A_r^{s,t} \to A_r^{s-1,t-1}$ is an isomorphism on the non torsion and odd prime torsion, so this map is injective. Thus, recalling the definition of the differential d_r, since the map $E_r^{s,t} \to A_r^{s,t}$ is zero by exactness of the staircase, for large r the differentials originating at $E_r^{s,t}$ are zero. Also for large enough r there are no differentials into $E_r^{s,t}$, since such differentials would come from $E_r^{s-r,t-r-1}$, which is nothing for $r > s$. Thus for all r greater than some R, we have that projection gives an isomorphism $E_r^{s,t} \to E_{r+1}^{s,t}$. Notice that, in fact, $E_\infty^{s,t}$, by exactness of the staircase, is isomorphic to the cokernel of the previous vertical maps, which for large r is exactly the inclusion $F^{s+1,t+1} \to F^{s,t}$, which is the theorem.

□

Given an actual homotopy class, we can tell what filtration it will be detected in in the Adams Spectral Sequence. Notice that if $f \in [Y, X]_*$ is detected by an element in filtration s, we can factor it as the composite of s maps which induce 0 in cohomology. To see this, see that f is detected by a map $[Y, K_s]_*$, which can be pushed to a map $[Y, X_s]_*$ which is a lift of f. Thus f factors as the $s-1$ maps in $[X_l, X_{l-1}]_*$ for $l < s$ (these are 0 in cohomology) and the composite of $Y \to X_s \to X_{s-1}$. We state this easy fact as a lemma.

Lemma 4.2.3. *If $f \in [Y, X]_*$ is detected by an element in filtration s, then it can factor as the composite of s maps, each of which induce the zero map in cohomology.*

4.2.1 Some Remarks

There are a few ways to generalize this process or just make it a bit nicer. First of all, we can use homology instead of cohomology. The difference here is that we end up using smash products instead of wedge products of $H\mathbb{F}_2$, but in the end we get a spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{A}_*}(H_*(Y), H_*(X)) \implies [Y, X] \otimes \mathbb{Z}_2$$

where the $\implies$ symbol means that there is an E_∞ page and it is isomorphic successive quotients of the right hand of the $\implies$ arrow, and

$$\mathcal{A}_* = \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}, \mathbb{F}_2) = (H\mathbb{F}_2)_*(H\mathbb{F}_2)$$

We can also replace $H\mathbb{F}_2$ with any generalized cohomology theory E, like cohomology mod odd primes, your favorite flavor of K-Theory (for instance $E = BU, BO, KO$) or cobordism ($E = MU, MSU, BP$). The zoo of spectra and cohomology theories are discussed in great detail in Ravenel's famous "Green Book" [Rav86]. In this case, under certain hypothesis about E, X and Y, we have

$$E_2 = \mathrm{Ext}_{E_*E}(E_*(Y), E_*(X)) \implies [Y, X]^E$$

where $[Y, X]^E$ is, roughly, maps f whose equivalence is detected by the induced map $E_*(f)$. More details on this can be found in, for instance [Rav86], [Bru86, Ch IV]. When $E = BP$, the spectral sequence is has many many fewer nonzero differentials, and earns the name "Adams-Novikov Spectral Sequence".

The reader is invited to look at Appendix B at this point, containing a diagram of the E_2 page of the Adams Spectral Sequence for Spheres.

4.3 Hopf Invariant One Maps in $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$

Let $f \in \pi_{4n-1}(S^{2n})$ have odd Hopf Invariant. Since the group $\pi_{4n}(S^{2n+1})$ is stable, there is an $\hat{f} = \Sigma f \in \pi^s_{2n-1}(S)$ coming from f. Notice that $\hat{f} \neq 0$, since the Steenrod Squares commute with suspensions, so in the cohomology ring of $S^{2n+1} \cup_{\Sigma f} D^{4n+1}$, the Steenrod Square Sq^{2n} is not zero. You might notice that in $S^{2n+1} \cup_g D^{4n+1}$ for some $g \in \pi_{4n}(S^{2n+1})$, we know that the cup product square on the $2n+1$ cohomology class is zero. However, we can still apply Sq^{2n} and ask if it is zero or not. If g is the suspension of a map with a Hopf invariant, this will detect the parity of the Hopf Invariant. Since Hopf Invariant is a homomorphism and nullhomotpic maps have Hopf Invariant zero, we know that Σf has even order. This means that $\hat{f}$ is detected by an element in the Adams Spectral Sequence.

We can do better however. We are able to say exactly what elements of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ correspond to possible maps of Hopf Invariant One. We need the following lemma

Lemma 4.3.1. *Suppose that $\hat{f} : S^{2n-1} \to S$ comes from an element of odd Hopf Invariant. Suppose there is a spectrum X such that the following diagram commutes*

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_1"] \arrow[dr, "\hat{f}"'] & X \arrow[d, "f_2"] \\ & S \end{tikzcd}$$

Then either f_1^* or f_2^* is nonzero.

Proof. Suppose both maps are zero in cohomology. We have the following diagram of spectra

$$\begin{tikzcd}
 & S^{2n-1} \arrow[d, "f_1"] \\
S \arrow[d] & X \arrow[l, "f_2"'] \arrow[d, "j"] \\
S \cup_{\hat{f}} D^{2n} \arrow[d] & X \cup_{f_1} D^{2n} \arrow[l, "i"'] \arrow[d] \\
S^{2n} \arrow[r, equal] & S^{2n}
\end{tikzcd}$$

where the two columns are cofibrations sequences. Applying cohomology, we have that the two columns are exact. In fact, since $f_1^* = 0$, the right column is short-exact.

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[r, "0"] & H^*X \\
\mathbb{F}_2 \oplus \Sigma^{2n}\mathbb{F}_2 \arrow[u] \arrow[r, "i^*"] & H^*(X \cup_{f_1} D^{2n}) \arrow[u, two heads, "j^*"'] \\
\Sigma^{2n}\mathbb{F}_2 \arrow[u] \arrow[r, equal] & \Sigma^{2n}\mathbb{F}_2 \arrow[u, hook]
\end{tikzcd}$$

Let α be the generator of $H^0(S \cup_{\hat{f}} D^{2n})$ and β be the generator of $H^{2n}(S \cup_{\hat{f}} D^{2n})$. Since we know β has a preimage in $H^*(S^{2n})$ and since the bottom right vertical map is an injection (by exactness), $i^*\beta$ generate the kernel of j^* and is nonzero. But $\beta = Sq^{2n}\alpha$, so $Sq^{2n}i^*\alpha = i^*\beta$, so $i^*\alpha \neq 0$ and thus is not in the kernel of j^*. This means $j^*i^*\alpha \neq 0$. But by commutativity of the square, it should be, so we get a contradiction. □

Corollary 4.3.2. *If there is a map of odd Hopf Invariant in $\pi_{4n-1}(S^{2n})$, there is an element of $\mathrm{Ext}^{1,2n}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ which survives to the E_∞ page.*

Proof. By Lemma 4.2.3. □

Since the kernel of $\mathcal{A} \to \mathbb{F}_2$ is generated over $\mathbb{F}_2$ by the indecomposable elements, we can conclude that $\mathrm{Ext}^{1,t}(\mathbb{F}_2, \mathbb{F}_2)$ is nonzero when and only when t is a power of 2.

Definition 4.3.3. *We will refer to the generator of $\mathrm{Ext}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ as h_i.*

The element h_0 detects twice the identity map in $\pi_0(S)$, h_1 detects the Hopf Fibration and h_2 and h_3 detect similar maps for the Quaternions and Octonions.

5 Products and Steenrod Operations in the Adams Spectral Sequence

The Adams Spectral Sequence has a ton of structure, inherited from the structure of Ext on one side and from the structure of $[Y, X]_*$ on the other. For instance it is easy to see that the Adams Spectral Sequence is functorial in Y, and, by Theorem 4.2.2, is functorial in X as well. The functorial maps between spectral sequences commute with the differentials, that is, a map $f : X \to Z$ induces, for each r, a map $f_r : E_r(Y, X) \to E_r(Y, Z)$ and $d_r f_r = f_r d_r$, and a similar contravariant thing for Y (this is easy to see, but it requires some thinking about it. Simply use Theorem 4.2.2 to draw maps between exact couples and trace the differentials).

5.1 The Smash Product Pairing

Consider this product

$$[X_1, X_2]_i \otimes [Y_1, Y_2]_j \to [X_1 \wedge Y_1, X_2 \wedge Y_2]_{i+j}$$

Coming from the functoriality of the wedge sum. Consider Adams Resolutions X_2^i of X_2 and Y_2^i of Y_2. Notice that we can replace the first k maps in the Adams resolutions with injections, and define,

$$Z_k = \bigcup_{i+j=k} X_2^i \wedge Y_2^j$$

with the evident maps between them, and notice that the cofibers are wedge sums of $H\mathbb{F}_2$ inducing surjections on cohomology, so this is an Adams resolution for $Z_0 = X_2 \wedge Y_2$ Note that this is not true for an arbitrary cohomology theory. For a general proof, see [Rav86, Ch 2.3] or [Bru86, Ch IV]. This follows directly from the easy to show isomorphism of spectra

$$\frac{X}{A} \wedge \frac{Y}{B} \cong \frac{X \wedge Y}{X \wedge B \cup A \wedge Y}$$

Let L_i be the cofiber $Z_{i+1} \to Z_i$, K_i the cofiber of $X_2^{i+1} \to X_2^i$ and J_i the cofibers of $Y_2^{i+1} \to Y_2^i$. Note that elements in the E_1 page of the Adams Spectral Sequence are represented by maps $x \in [X_1, K_{s_x}]_{s_x - t_x}$, $y \in [Y_1, J_{s_y}]_{s_y - t_y}$ and $z \in [X_1 \wedge Y_1, L_s]_{s-t}$ where $s = s_x + x_y$ and $t = t_x + t_y$. Setting $z = x \wedge y$, we have defined a product

$$E_1(X_1, X_2) \otimes E_1(Y_1, Y_2) \to E_1(X_1 \wedge Y_1, X_2 \wedge Y_2)$$

An element $a \in [Y, K_s]_{t-s}$ lives to the E_r page if a can represent a map $[Y, X_{s+r-1}]_{t-s-1}$. It is easy to see then that if x and y survive to page r, then, so does $x \wedge y$, so the product is defined on every page,

$$E_r(X_1, X_2) \otimes E_r(Y_1, Y_2) \to E_r(X_1 \wedge Y_1, X_2 \wedge Y_2)$$

We also have

$$H^* L_k = \bigoplus_{i+j=k} H^* K_i \otimes H^* J_j \cong \bigoplus_{i+j=k} H^*(K_i \wedge J_j)$$

where the second map is Künneth. Thus, again using Künneth and the fact that it is an isomorphism, the product on the E_2 page is given by the Ext product derived from the functoriality of $\otimes$:

$$\mathrm{Ext}_{\mathcal{A}}(H^* X_2, H^* X_1) \otimes \mathrm{Ext}_{\mathcal{A}}(H^* Y_2, H^* Y_1) \to \mathrm{Ext}_{\mathcal{A}}(H^* X_2 \otimes H^* Y_2, H^* X_1 \otimes H^* Y_1)$$

(see A.1 for details on the homological algebra). Finally, from the fact that the Adams Resolution for the smash product is the smash of the Adams Resolutions, this falls to the tensor of the maps in cohomology, the differential on $E_r(X_1, X_2) \otimes E_r(Y_1, Y_2)$ is given $d_r \otimes 1 + 1 \otimes d_r$, or equivalently, for all $x \in E_r(X_1, X_2), y \in E_r(Y_1, Y_2)$, we have

$$d_r(xy) = x d_r(y) + d_r(x) y$$

We summarize in a theorem

Theorem 5.1.1. *There is a product*

$$E_r(X_1, X_2) \otimes E_r(Y_1, Y_2) \to E_r(X_1 \wedge Y_1, X_2 \wedge Y_2)$$

such that

1. *The product on E_{r+1} is induced by the product on E_r.*
2. *The product on E_∞ is induced by the smash product of maps.*
3. *The product on E_2 is the tensor product pairing on* Ext
4. *The product obeys the Leibniz Rule with respect to the differentials*

A very nice thing about this is the isomorphism $S \wedge X \to X$ for any spectrum X. This means two things. First of all the spectral sequence for homotopy groups of spheres has a map

$$E_r(S, S) \otimes E_r(S, S) \to E_r(S, S)$$

meaning that each page of the spectral sequence is a ring. Secondly, for any pair of spectra, we have

$$E_r(S, S) \otimes E_r(X, Y) \to E_r(X, Y)$$

meaning that any such spectral sequence is a module over this ring. This is extremely powerful, because it means if you know the differentials in $E_r(S, S)$ and the module structure of $E_r(X, Y)$, you automatically learn a ton about the differentials in $E_r(X, Y)$ by the Leibniz Rule.

5.2 The Composition Product Pairing

Recall the Yoneda Product (see Appendix A.1) There is an obvious paring in map groups

$$[X, Y]_i \otimes [Y, Z]_j \to [X, Z]_{i+j}$$

by just composing maps. It turns out that, given $f : X \to Y$ and $g : Y \to Z$, if you find elements in $\mathrm{Ext}_{\mathcal{A}}(H^*Y, H^*X)$, $\mathrm{Ext}_{\mathcal{A}}(H^*Z, H^*Y)$ which detect f and g, their Yoneda composite will detect $g \circ f$. This may seem rather obvious, and surely you should expect something like this to be true, but the proof is unexpectedly long and unenlightening, so it will be omitted, but it can be found in [Mos68].

Theorem 5.2.1. *There exists a natural pairing of Spectral Sequences*

$$E_r^{s_1,t_1}(X, Y) \otimes E_r^{s_2,t_2}(Y, Z) \to E_r^{s_1+s_2,t_1+t_2}(X, Z)$$

which is the Yoneda product in E_2, obeys the Leibniz Rule, is induced at each page by the previous page and is induced by composition of maps at the E_∞ page.

However, we don't care about this product in and of itself. We care a lot more about the smash product pairing. The nice thing is that, in terms of the module structure of $E_r(S, X)$, these two products are the same! To see this, let $f \in [X, Y]_i$ and $g \in [Y, Z]_j$. Then $g \circ f$ can be computed as the composite

$$X \wedge S^{i+j} \to X \wedge S^i \wedge S^j \xrightarrow{f \wedge 1} Y \wedge S^j \xrightarrow{g} Z$$

For $X = Y = S$, this is the smash product paring on $[S, S] \otimes [S, Z]$ (for odd primes there are rather annoying sign issues, but we have enough to worry about). This means that we can compute the products using the readily computable Yoneda Product without having to pay the memory cost of tensoring our resolution with itself to compute the tensor product pairing. On a computer this is the difference between easily computing products and choking on an out-of-memory exception.

The reader is invited to consider again Appendix B, specifically with $t - s < 14$. There are only a few possible non-zero differentials here, and the source of each be decomposed as a product. Using the Leibniz formula, we see immediately that, in fact, these differentials much all be zero. Thus the first 13 stable stems can be read off from that diagram.

By being a bit more clever, we prove a nonzero differential.

$$d_2 h_4 = h_0 h_3^2$$

To see this, note that h_3 and h_3^2 both live to become stable homotopy classes. Since the multiplication in π_*^s, which the composition product converges to, is skew-commutative, we have that $\hat{h}_3^2 = -\hat{h}_3^2$, where $\hat{h}_3$ is a choice of element in π_*^s detected by h_3. This means that $2\hat{h}_3^2 = 0$, so $h_0 h_3^2$ cannot detect a map in π_*^s. All differentials on $h_0 h_3^2$ are zero for dimension reasons, so the only other possibility is $h_0 h_3^2$ is a boundary, and the only possibility is $d_2 h_4$.

5.3 Steenrod Squares in the Adams Spectral Sequence

Through pure homological algebra, if M is an coalgebra and N is a algebra over a cocommutative Hopf algebra A, then $\mathcal{A}$ acts on $\mathrm{Ext}_A(M, N)$, and this action is computable (see A.2). Luckily, $\mathcal{A}$ is a cocommutative Hopf Algebra itself, the cohomology of a suspension spectrum is an algebra, and there is a notion of a ring spectrum, which is a spectrum whose cohomology is a co-algebra.

Definition 5.3.1. *A "ring spectrum" is a spectrum X with a multiplication map*

$$\mu : X \wedge X \to X$$

which is associative in the sence that

$$\mu(\mu \wedge 1) = \mu(1 \wedge \mu) : X \wedge X \wedge X \to X$$

and a unit map

$$e : S \to X$$

with the property that the isomorphisms

$$S \wedge X \cong X \qquad \textit{and} \qquad X \wedge S \cong X$$

factor as

$$\mu(1 \wedge e) \qquad \textit{and} \qquad \mu(e \wedge 1)$$

A ring spectrum is commutative if the switching map

$$\rho : X \wedge X \to X \wedge X$$

has

$$\mu\rho = \mu$$

Lemma 5.3.2. *If X is a ring spectrum then H^*X is an $\mathcal{A}$-coalgebra. If X is a commutative ring spectrum, then H^*X is a cocommutative $\mathcal{A}$-coalgebra.*

Thus if X is a commutative ring spectrum and Y is a suspension spectrum we can define these operations on the E_2 page. If we force $Y = S$, we have a geometric way of realizing these operations, first published by Kahn [Kah70]. The idea is that, given geometric ways of realizing the algebraic operations, we can use the geometric description of the differentials to construct algebraic laws constraining the interaction of the Steenrod Squares and the differentials. I find the interplay between the topological and algebraic construction of these operations striking and beautiful, in a sort of delirious way.

In giving this construction, we follow the proof and notation of [Kah70] and [Mil72], however, we will work stably. Kahn and Milgram work unstably, which has the effect of obscuring things behind explicit suspensions and unnecessary extra indexing. The other extreme is [Bru86, Ch IV.4], in which Bruner works in far greater generality and more modern language. I will work in only slightly greater generality than Kahn and Milgram, but using the language of spectra developed earlier.

5.3.1 The Quadratic Construction

Let A and B be topological spaces with basepoints a_0 and b_0. We define the half smash product

$$A \ltimes B = (A \times B)/(A \times b_0)$$

For A a space and B a spectrum, the definition is not quite so simple. May says in [Bru86] "The pragmatist is invited to accept our word that everything one might naively hope to be true about [the half smash product for spectra] is in fact true", and we will take this approach as well. If B is a suspension spectrum, the resulting thing should be the suspension spectrum of the half-smash product on spaces.

If X is any spectrum, consider the spectrum

$$S^n \ltimes (X \wedge X)$$

We can define the $\mathbb{Z}/2$ action by

$$\tau(y, x, x') = (-y, x', x)$$

that is, the antipodal action on the sphere and the twisting map on $X \wedge X$ (notice this makes perfect sense for spaces and smash products of spectra have a twisting map, so this should make sense for spectra). Define, for a spectrum X,

$$Q^n(X) = \frac{S^n \ltimes (X \wedge X)}{\mathbb{Z}/2}$$

This is called the quadratic construction on X. In [Bru86], the functor Q^∞ is denoted D_2 or $D_{\mathbb{Z}/2}$ and is referred to as the extended power construction. Notice that Q^n is a functor and $Q^n(X)/Q^{n-1}(X) = S^n \wedge X \wedge X$. Also notice that, as spaces (and thus as suspension spectra)

$$Q^n(S^m) = \Sigma^m \frac{\mathbb{R}P^{m+n}}{\mathbb{R}P^{m-1}}$$

This will become important, so we define

$$P_m^{m+n} = \frac{\mathbb{R}P^{m+n}}{\mathbb{R}P^{m-1}}$$

Other properties that May's pragmatist would take for granted include (although on the space level one can see these work), are

$$\frac{S^n \ltimes_{\mathbb{Z}/2} X}{S^{n-1} \ltimes_{\mathbb{Z}/2} X} = P_n^n \wedge X$$

and for $A \subset X$

$$\frac{S^n \ltimes_{\mathbb{Z}/2} X}{S^{n-1} \ltimes_{\mathbb{Z}/2} X \cup S^n \ltimes_{\mathbb{Z}/2} A} = P_n^n \wedge \frac{X}{A}$$

and finally

$$S^n \ltimes_{\mathbb{Z}/2} (A \wedge B) \cong (S^n \ltimes_{\mathbb{Z}/2} A) \wedge B$$

5.3.2 Geometric Realization of the Steenrod Squares

Definition 5.3.3. *Let X be a commutative ring spectrum equipped with a map $\Theta : Q(X) \to X$ which extends the ring map $X \wedge X \to X$. Then X is called an H_2 ring spectrum.*

Remark 5.3.4. *The condition for H_2 structure is a necessary but not sufficient condition for what some authors call H_∞ structure. This notation is nonstandard.*

For spheres, $Q(S) = (\mathbb{R}P^\infty)^+$, that is $\mathbb{R}P^\infty$ with a disjoint basepoint, and the map $Q(S) \to S$ is given by collapsing the $\mathbb{R}P^\infty$ component to a point.

Let $X \longleftarrow \{X_i\}$ be an Adams Resolution. Then if

$$Z_i = \bigcup_{i=j+k} X_j \wedge X_k$$

we have that $\mathbb{Z}/2$ acts on $Z_0 = X \wedge X$ by twisting, and that action is inherited by the Z_i, and $S^n \ltimes_{\mathbb{Z}/2} Z_{i+1}$ is a subspectrum of $S^n \ltimes_{\mathbb{Z}/2} Z_i$ and $S^{n-1} \ltimes_{\mathbb{Z}/2} Z_i$ is a subspectrum of $S^n \ltimes_{\mathbb{Z}/2} Z_i$.

Here is the construction of Kahn which allows for a geometric realization of the Steenrod Squares.

Theorem 5.3.5. *There exist maps*

$$\Theta_{n,s} : S^n \ltimes_{\mathbb{Z}/2} Z_s \to X_{s-i}$$

coming from a lift of

$$\Theta : Q(X) \to X$$

In other words, the following diagrams need to commute for all n, s:

$$\begin{tikzcd} Q(X) \arrow[d, "\Theta"'] & S^n \ltimes_{\mathbb{Z}/2} Z_s \arrow[l] \arrow[d, "\Theta_{n,s}"'] \\ X & X_{s-n} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} S^n \ltimes_{\mathbb{Z}/2} Z_{s-1} \arrow[d, "\Theta_{n,s-1}"'] & S^n \ltimes_{\mathbb{Z}/2} Z_s \arrow[l] \arrow[d, "\Theta_{n,s}"'] & S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s \arrow[l] \arrow[d, "\Theta_{n-1,s}"'] \\ X_{s-n-1} & X_{s-n} \arrow[l] & X_{s-n+1} \arrow[l] \end{tikzcd}$$

Proof. Obviously $\Theta_{0,s}$ exists, since $S^0 \ltimes Z_s = Z_s$ and so $\Theta_{0,s}$ is just the map of Adams Resolutions $Z_s \to X_s$ coming from the ring spectrum map $X \wedge X \to X$, so the leftmost square commutes. By the definition of H_2 structure, the right square commutes as well. By induction, assume we have defined $\Theta_{l,s}$ for $l < n$ and $\Theta_{n,t}$ for $t < s$.

We want to lift $\Theta_{k,s-1}$ to get $\Theta_{k,s}$, and the obstruction to doing so is a map in

$$\left[\frac{S^n \ltimes_{\mathbb{Z}/2} Z_s}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s}, \frac{X_{s-k-1}}{X_{s-k}}\right] \cong \mathrm{Hom}_{\mathcal{A}}\left(F, H^*\left(\frac{S^n \ltimes_{\mathbb{Z}/2} Z_s}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s}\right)\right)$$

where we used that the X_* is an Adams Resolution so the cofiber is a wedge sum of Elienberg Maclane spectra whose cohomology, which we call F above, is $\mathcal{A}$-free.

Now, the following diagram commutes

$$\begin{tikzcd} S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s \arrow[r] \arrow[d] & S^{n-1} \ltimes_{\mathbb{Z}/2} Z_{s-1} \arrow[r] \arrow[d] & X_{s-n} \arrow[d] \\ S^n \ltimes_{\mathbb{Z}/2} Z_s \arrow[r] & S^n \ltimes_{\mathbb{Z}/2} Z_{s-1} \arrow[r] & X_{s-n-1} \end{tikzcd}$$

and thus any possible obstruction comes from

$$\left[\frac{S^n \ltimes_{\mathbb{Z}/2} Z_{s-1}}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_{s-1}}, \frac{X_{s-k-1}}{X_{s-k}}\right] \cong \mathrm{Hom}_{\mathcal{A}}\left(F, H^*\left(\frac{S^n \ltimes_{\mathbb{Z}/2} Z_{s-1}}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_{s-1}}\right)\right)$$

Of course, by a remark above,

$$\frac{S^n \ltimes_{\mathbb{Z}/2} Z_s}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s} = P_n^n \wedge Z_s$$

and the map

$$P_n^n \wedge Z_s \to P_n^n \wedge Z_{s-1}$$

induces zero in cohomology, since $Z_s \to Z_{s-1}$ is part of an Adams Resolution, and thus the obstruction is zero. $\square$

Let $K_j = \frac{X_j}{X_{j+1}}$, and $C_j = H^*\Sigma^j K_j$. To geometrically define the squaring operations, notice the following fact about the half-smash product before passing to the orbit space

$$S^n \wedge \frac{Z_s}{Z_{s+1}} \cong \frac{D^n \ltimes Z_s}{S^{n-1} \ltimes Z_s \cup D^n \ltimes Z_{s+1}}$$

Thus the cohomology is $(C \otimes C)_n$ by Künneth. Also, note that the action of $\mathbb{Z}/2$ on the above spectrum induces the tensor product switching map in cohomology. Letting D^n_+ and D^n_- be the two caps of S^n, there are two inclusions

$$\psi^n_\pm : \frac{D^n_\pm \ltimes Z_s}{S^{n-1} \ltimes Z_s \cup D^n_\pm \ltimes Z_{s+1}} \to \frac{S^n \ltimes Z_s}{S^{n-1} \ltimes Z_s \cup S^n \ltimes Z_{s+1}}$$

The two spectra in the denominator of the left hand side pull back to map to X_{s-n+1}, while the numerator pulls back to map to X_{s-n}. Thus, if we fix n we get two coherent systems of maps n maps

$$\varphi^{n,s}_\pm = \frac{\Theta_{n,s}}{\Theta_{n,s+1} \cup \Theta_{n-1,s}} \circ \overline{\psi}^n_\pm : S^n \wedge \frac{Z_s}{Z_{s+1}} \to K_{s-n}$$

where $\overline{\psi}^n_\pm$ is $\psi^n_\pm$ composed with the projection to orbits mod $\mathbb{Z}/2$. If ρ is the $\mathbb{Z}/2$ action, then $\varphi^{n,s}_+\rho = \varphi^{n,s}_-$. Finally, when splicing the sequences together, everything must commute with everything else, in the sense that the following will work out. Define chain maps of degree n

$$\Delta_n : C \to C \otimes C$$

by

$$\Delta_n = (\varphi^{n,*}_+)^*$$

Theorem 5.3.6.

$$\Delta_n \partial + \partial \Delta_n = \Delta_{n-1} + \rho \Delta_{n-1}$$

The proof is long and boring, but straightforward. Simply splice together the suspensions of the spaces Z_s/Z_{s+1} and use the diagrams relating the Θ's to check that φ's obey the right laws.

Corollary 5.3.7. *Let Y be a spectrum with a diagonal $d : Y \to Y \wedge Y$ (for instance, a suspension spectrum). If $u \in [Y, K_s]_{t-s}$ then the following diagram commutes (and thus we can calculate $Sq^{2s-i}u$)*

$$\begin{tikzcd}
\Sigma^{2t-2s+i} Y \arrow[r, "Sq^{2s-i}u"] \arrow[d, "d"'] & K_{2s-i} \\
S^i \wedge \Sigma^{t-s} Y \wedge \Sigma^{t-s} Y \arrow[r, "\Sigma^i(u\wedge u)"] & S^i \wedge K_s \wedge K_s \arrow[u, "\varphi^{i,2s}_+"']
\end{tikzcd}$$

where the right map makes sense since $K_s \wedge K_s \subset \frac{Z_{2s}}{Z_{2s+1}}$

Proof. As a map in cohomology, this basically asserts the definition of $Sq^{2s-i}u$.

$$Sq^{2s-i}u = d^* \circ (u \otimes u) \circ \Delta_i$$

where we abuse notation by using the same symbols for maps between spaces and maps between cohomology modules. □

Now assume $Y = S$, so that we can discuss relative homotopy groups. The above construction can be made relative as follows: Define for spaces

$$(X, A) \ltimes (Y, B) = (X \ltimes Y, A \ltimes Y \cup X \ltimes B)$$

Then $\Theta_{n,s}$ becomes

$$\Theta_{n,s} : (S^n, S^{n-1}) \ltimes_{\mathbb{Z}/2} (Z_s, Z_{s+1}) \to (X_{s-i}, X_{s-i+1})$$

and $\psi^n_\pm$ is a

$$\psi^n_\pm : (D^n, S^{n-1}) \ltimes (Z_{2s}, Z_{2s+1}) \to (S^n, S^{n-1}) \ltimes (Z_{2s}, Z_{2s+1})$$

and Δ_n is

$$\Delta_n = (\Theta_{n,s}\psi^n_\pm)^*$$

Thus we have the following

Corollary 5.3.8. *If $u \in \pi_{t-s}(X_s, X_{s+1})$ then the following diagram commutes*

$$\begin{tikzcd}
(D^{2t-2s+i}, S^{2t-2s+i-1}) \arrow[rr, "Sq^{2s-i}u"] \arrow[d, "\approx"'] & & (X_{2s-i}, X_{2s-i+1}) \\
(D^i, S^{i-1}) \ltimes (D^{2t-2s}, S^{2t-2s-1}) \arrow[r, "1\ltimes u\wedge u"] \arrow[d, "i_+"'] & (D^i, S^{i-1}) \ltimes (Z_{2s}, Z_{2s+1}) \arrow[r, "\overline{\psi}^i_+"] & (S^i, S^{i-1}) \ltimes_{\mathbb{Z}/2} (Z_{2s}, Z_{2s+1}) \arrow[u, "\Theta_{i,s}"'] \\
(S^i, S^{i-1}) \ltimes (D^{2t-2s}, S^{2t-2s-1}) \arrow[rr] & & (S^i, S^{i-1}) \ltimes_{\mathbb{Z}/2} (D^{2t-2s}, S^{2t-2s-1}) \arrow[u, "1\ltimes u\wedge u"']
\end{tikzcd}$$

where we consider $D^{2t-2s} = D^{t-s} \wedge D^{t-s}$, i_+ is induced by the inclusion of the upper hemisphere $D^i \to S^i$ and the unlabeled map is passage to orbits mod $\mathbb{Z}/2$.

Proof. The top cycle is exact the relative version of Corollary 5.3.7 with $\Theta_{i,s}\overline{\phi}^i_+$ factored. It is clear that the bottom cycle commutes. □

Now that we have a geometric realization of the Steenrod Squares, we can reason about the differentials $d_r Sq^i x$ for some $x \in E_2$. This will be our main source of differential computations in the Adams Spectral Sequence.

6 Some Differentials on Steenrod Squares in the Adams Spectral Sequence

For the rest of this paper we will set $Y = S$, so that the spectral sequence in question will converge to the 2-component of π_* (for X a suspension spectrum, this is π_*^s). In this special case, we have that if $X \longleftarrow \{X_i\}$ is an Adams Resolution with K_i the cofibers, then

$$\pi_{t-s}(K_s) \cong \pi_{t-s}(X_s, X_{s+1}) = [(D^{t-s}, S^{t-s-1}), (X_s, X_{s+1})]_0$$

The differential is of a map $f \in \pi_{t-s}(K_s)$ is given by restricting

$$f|_{S^{t-s-1}} : S^{t-s-1} \to X_{s+1}$$

and compressing the image to X_{s+r-1}, and consider this as an element of $\pi_{t-s-1}(X_{i+r-1}, X_{i+r})$.

6.1 Topology of Stunted Projective Spectra and the J Homomorphism

It will happen that the elements of the form $Sq^i x$ in the Adams Spectral Sequence will factor through various quadratic constructions, and these are stunted projective spectra, so we need to know a bit about their topology. A stunted projective spectra, $\Sigma^n P_n^{n+k}$, has one cell in each dimension from $2n$ to $2n + k$. We say that such a spectrum is reducible if

$$\Sigma^n P_n^{n+k} \cong \Sigma^n (P_n^{n+k-1} \wedge S^{n+k})$$

Definition 6.1.1.

$$v(n) = \max\{v | P^n_{n-v+1} \textit{ is reducible}\}$$

Let $a(n) \in \pi_{n-1}(S^{n-v(n)}) = \pi_{v(n)-1}(S)$ *be the composite*

$$S^{n-1} \to P_0^{n-v(n)} \to P_{n-v(n)}^{n-v(n)} \cong S^{n-v(n)}$$

where the first map is the attaching map of the top cell of P_0^n*. By the definition of* $v(n)$ *the attaching map of this cell lies in* $P_0^{n-v(n)}$*, since* $P^n_{n-v(n)+1}$ *is reducible.*

There is a map, known as $J : \pi_r(SO) \to \pi_r^s(S)$, from the unstable homotopy of the Lie Group SO to the stable homotopy groups of spheres. This map was studied by Mahowald in [Mah70]. We state the following theorem without proof.

Theorem 6.1.2. *There is a map*

$$J : \pi_r(SO) \to \pi_r^s(S)$$

such that

1. *The image of* J *in* $\pi_r(S)$ *is a cyclic summand.*
2. *The image is trivial if* $r \not\equiv$ *0,1,3 or 7 (mod 8).*
3. *The order of the image is 2 if* $r > 0$ *is 0 or 1 mod 8.*
4. *If* $r \equiv 3$ *(mod 4), the order is given by the denominator of* $B_{2n}/4n$*, where* B_{2n} *is a Bernoulli number. If* $r \equiv 3$ *(mod 8), the two component of the image of* J *is* 2^3*, so this image is detected by three elements in the Adams Spectral Sequence of spheres.*
5. *In the Adams Spectral Sequence in topological degree* $t - s \equiv 0, 3$ *or 7 (mod 8), the image of the* J *is detected by the "tower" of the appropriate size ending in as high a filtration as possible. In topological degree* $t - s \equiv 1$*, the image of* J *is detected by* h_1 *times the image of* J *when* $t - s \equiv 0$ *(mod 8). There is never any ambiguity in determining which elements detect the* Im J.

One might consider this cheating, but we can actually infer some differentials from this. In particular, we know that all differentials of elements detecting $\operatorname{Im} J$ and no differential can hit them. However, we can also infer some non-zero differentials in the Adams Spectral Sequence for spheres when $r \equiv 7 \pmod 8$. If $h_0 x$ detects the image of J but x does not, we know x must either not be a cycle or be a boundary. Thus we can infer, for instance, in $t - s = 15$, that

$$d_3(h_0^2 h_4) = h_0 d_3(h_0 h_4) \neq 0$$

We will need he following theorem, which we will state without proof. It can be found in [Mil72] and [Bru86, V.2.14-V.2.17].

Theorem 6.1.3. *Let $\varphi(k)$ be the number of integers congruent to $0, 1, 2$ or 4 mod 8 less than or equal to k. The following are true of $\Sigma^n P_n^{n+k}$*

1. *P_n^{n+k} is reducible iff $n + k + 1 \equiv 0 \pmod{2^{\varphi(k)}}$. Equivalently, if ϵ is the number of factors of 2 in $n + 1$ and $\epsilon = 4a + b$ for $0 \leq b < 4$, then*

$$\nu(n) = 8a + 2^b$$

2. *If $n \equiv n' \pmod{2^{\varphi(k)}}$, then $P_n^{n+k} \equiv \Sigma^{n-n'} P_{n'}^{n'+k}$*
3. *If $\nu(n) = 1$, then $a(n) = 2 \in \pi_0(S)$, that is, $a(n)$ is a degree two map of spheres.*
4. *If $\nu(n) > 1$, then $a(n)$ generates $\operatorname{Im} J$ in $\pi_{\nu-1}(S)$.*

6.2 Setting Up The Cells

Fix n, s, k, r and an element $x \in E_r^{s,n}(S, X)$, represented by a map

$$x : (D^n, S^{n-1}) \to (X_s, X_{s+r})$$

We are now ready to begin the long hard computation of $d_r Sq^{k+n} x$.

Give D^n the cell structure of one $(n-1)$-cell and one n-cell, respectively called dx and x. This should be suggestive: we want to mentally connect these cells with the map x and its boundary. Let the following spaces inherit this cell structure

$$\Gamma_0 := D^{2n} = D^n \wedge D^n$$

$$\Gamma_1 := S^{2n-1} = D^n \wedge S^{n-1} \cup S^{n-1} \wedge D^n$$

$$\Gamma_2 := S^{2n-2} = S^{n-1} \wedge S^{n-1}$$

$\mathbb{Z}/2$ acts on coherently on the Γ_i by swapping the wedge factors.

The cellular chains in each of these spaces are the obvious subspaces of

$$C_* D^{2n} = \langle x \otimes x, x \otimes dx, dx \otimes x, dx \otimes dx \rangle$$

Give S^∞ the cell structure with two k-cells in dimension k, called e_k and ρe_k. Let $\mathbb{Z}/2 = \langle \rho \rangle$ act on this CW complex antipodally, by swapping the cells in each dimension.

Similar to Q, we define the functor P^j as

$$P^j(\Gamma_i) = S^j \ltimes_{\mathbb{Z}/2} \Gamma_i \subset Q^j D^n = P^j \Gamma_0$$

so that We define this so that we can more easily refer to $P^j \Gamma_i$ as a filtration of $Q^j D^n$.

Consider the inclusion

$$P^{k-1} \Gamma_1 \to P^k \Gamma_1$$

The cofiber is

$$P^k\Gamma_1 \cup C(P^{k-1}\Gamma_1) \cong P^k\Gamma_1 \cup P^{k-1}\Gamma_0$$

where we recall the reduced cone of a spectrum X is $CX = I \wedge X$ and use the properties mentioned in 5.3.1. However, the inclusion is equivalent to the inclusion of the $2n+k-2$ skeleton (that is, all but the top dimension cell) of

$$\Sigma^{n-1}P_{n-1}^{n+k-2} \to \Sigma^{n-1}P_{n-1}^{n+k-1}$$

and so the cofiber is isomorphic to S^{2n+k-1}. Of course, $P^k\Gamma_0 \cong D^{2n+k}$ with boundary $P^k\Gamma_1 \cup P^{k-1}\Gamma_0$. Thus we have that maps in $\pi_{t-s}(Z_s, Z_{s+1})$ (for some Adams resolution Z_*) are represented by maps of pairs

$$(P^k\Gamma_0, P^k\Gamma_1 \cup P^{k-1}\Gamma_0) \to (Z_s, Z_{s+1})$$

Our strategy for computing d_*Sq^jx will be as follows. We will show that the boundary of the map $Sq^{k+n}x$, as a relative homotopy class, factors through $P^k\Gamma_1 \cup P^{k-1}\Gamma_0$, which are stunted projective spaces. This boundary can be recognized as the top cell of $P^{k+1}\Gamma_2$, which we shoehorn into $P^k\Gamma_1$, and glue together in $P^k\Gamma_1 \cap P^{k-1}\Gamma_0$ with some other homotopy class which depends on $\nu(k+n)$.

6.3 Identification of the Boundary and some Chain Calculations

Recall the diagram in Corollary 5.3.8 constructing $Sq^{k+n}x$. Using the notation of this section, it looks like

Diagram 6.3.1.

$$\begin{tikzcd}
(D^{2n+k}, S^{2n+k-1}) \arrow[rr, "Sq^{n+k}x"] \arrow[d, "\approx"'] & & (X_{2s-k}, X_{2s-k+1}) \\
(D^k, S^{k-1}) \ltimes (\Gamma_0, \Gamma_1) \arrow[r, "1\ltimes x\wedge x"] \arrow[d, "i_+"'] & (D^k, S^{k-1}) \ltimes (Z_{2s}, Z_{2s+1}) \arrow[r, "\overline{\psi}_+^k"] & (P^kZ_{2s}, P^{k-1}Z_{2s} \cup P^kZ_{2s+r}) \arrow[u, "\Theta_{k,s}"'] \\
(S^k, S^{k-1}) \ltimes (\Gamma_0, \Gamma_1) \arrow[rr] & & (P^k\Gamma_0, P^k\Gamma_1 \cup P^{k-1}\Gamma_0) \arrow[u, "1\ltimes x\wedge x"']
\end{tikzcd}$$

There are a few things to note. First note that we lifted the boundary of x, which by abuse of notation we call dx, to X_{s+r}, so $d(1 \ltimes x \wedge x)$ is lifted to $P^{k-1}Z_{2s} \cup P^kZ_{2s+r}$. Second, note that the outside composite

$$(D^{2n+k}, S^{2n+k-1}) \to (P^k\Gamma_0, P^k\Gamma_1 \cup P^{k-1}\Gamma_0)$$

is a homotopy equivalence, by the argument in Section 6.2. Finally, call the right side vertical composite

$$\xi : (P^k\Gamma_0, P^k\Gamma_1 \cup P^{k-1}\Gamma_0) \to (X_{2s-k}, X_{2s-k+1})$$

More generally, we will use ξ to represent any composite of $\Theta_{*,*}$ and $1 \ltimes x \wedge x$, that is, ξ is a map

$$\xi : P^k\Gamma_i \to X_{2s+ir-k}$$

This indexing makes sense if we recall the definition of Γ_i and that dx lands in X_{s+r}, so

$$x \wedge x : (\Gamma_0, \Gamma_1, \Gamma_2) \to (Z_{2s}, Z_{2s+r}, Z_{2s+2r})$$

Remark 6.3.2. *Notice that ξ depends on x. The map induced on homotopy should be thought of as a conversion between the world of cells of stunted projective spectra and the world of homotopy classes in the Adams Spectral Sequence. For instance, if $f \in \pi_{k+2n}(P^k\Gamma_0, P^k\Gamma_1 \cup P^{k-1}\Gamma_0)$ is the inclusion of the top cell, then clearly ξ_*f represents $Sq^{k+n}x$, and the Hurewicz image of f is $e_k \otimes x \otimes x$. Similarly, the inclusion of the cell $e_k \otimes dx \otimes dx$ is sent to $Sq^{n+k-1}d_rx$. The inclusion of $e_0 \otimes x \otimes dx$ goes to xd_rx for some r.*

Let

$$\partial : S^{2n+k-1} \cong P^k\Gamma_1 \cup P^{k-1}\Gamma_0 \to X_{2s-k+1}$$

denote the boundary. We want to find conditions on an element in $\pi_{2n+k-1}X_{2s-k+1}$ to recognize it.

Recall that

$$C^*(P^k\Gamma_i) = C^*(S^k) \otimes_{\mathbb{Z}/2} C^*\Gamma_i$$

Here is an easy but important calculation

Lemma 6.3.3. *In the integral homology group, $H_*(P^{i+1}\Gamma_1)$, if $n \not\equiv i$ (mod 2), then*

$$e_{i+1} \otimes dx \otimes dx = (-1)^i e_i \otimes d(x \otimes x)$$

and if $n \equiv i$ (mod 2)

$$e_{i+1} \otimes dx \otimes dx = (-1)^i e_i \otimes d(x \otimes x) - 2e_i \otimes x \otimes dx$$

Proof.

$$d(e_{i+1} \otimes x \otimes dx) \;\; = \;\; (\rho + (-1)^{i+1})e_i \otimes x \otimes dx + (-1)^{i+1}e_{i+1} \otimes dx \otimes dx$$

But since the tensor product is over $\mathbb{Z}/2$, we have that this is equal to

$$e_i \otimes dx \otimes x + (-1)^{i+1}e_i \otimes x \otimes dx + (-1)^{i+1}e_{i+1} \otimes dx \otimes dx$$

Using that $d(x \otimes x) = dx \otimes x + (-1)^n x \otimes dx$, we have in homology that

$$e_{i+1} \otimes dx \otimes dx = (-1)^i e_i \otimes d(x \otimes x) - (1 + (-1)^{i+n} e_i \otimes x \otimes dx$$

□

Remark 6.3.4. *The introduction of integral homology, non unit coefficients and signs may be confusing. Just recall that the situation in the Adams Spectral Sequence is actually an associated-graded version of the situation in actual homotopy or homology. Multiplication by 2 in homotopy corresponds to a filtration shift in the Adams Spectral Sequence.*

Consider the inclusion

$$P^{i+1}\Gamma_2 \to P^{i+1}\Gamma_1$$

Since $P^{i+1}\Gamma_1 = \Sigma^{n-1}P^{n+i}_{n-1}$ is $2n + i - 1$ dimensional and $P^{i+1}\Gamma_1/P^i\Gamma_1 \cong S^{2n+i}$ is $2n + i$ dimensional, the inclusion factors through $P^i\Gamma_1$. Define that map to be

$$e : P^{i+1}\Gamma_2 \to P^i\Gamma_1$$

Lemma 6.3.5. *if $i \not\equiv n$ (mod 2), then*

$$e_*(e_{i+1} \otimes dx \otimes dx) = (-1)^i e_i \otimes d(x \otimes x)$$

Proof. The proof is given by chasing through the various homology groups and using the equivalences previously given. □

We can now recognize ∂.

Lemma 6.3.6. *Let $\mathrm{i} \in \pi_{2n+k-1}(P^k\Gamma_1 \cup P^{k-1}\Gamma_0)$ be the Hurewicz preimage of the homology class*

$$(-1)^k e_k \otimes d(x \otimes x) + \begin{cases} 0 & k \not\equiv n \text{ (mod 2)} \\ (-1)^k 2e_{k-1} \otimes x \otimes x & k \equiv n \text{ (mod 2)} \end{cases}$$

Since Hurewicz is an isomorphism in this dimension, this defines i. We have

$$\xi_*(\mathrm{i}) = \partial$$

Proof. Hurewicz is an isomorphism in this dimension since $P^k\Gamma_1 \cup P^{k-1}\Gamma_0 \cong S^{2n+k-1}$, so we need only notice that this is the boundary of $e_k \otimes x \otimes x$, which is the Hurewicz image of that which ξ_* sends to $Sq^{k+n}x$. □

6.4 Differentials When v is Large

Theorem 6.4.1. *If $v(k+n) > k+1$ then*

$$d_{2r-1} Sq^{k+n} x = Sq^{k+n} d_r x$$

Proof. Consider the diagram

$$\begin{tikzcd}
& & S^{2n+k-1} \arrow[dll, dashed, "C"'] \arrow[d, "\mathfrak{i}"] \\
P^{k+1}\Gamma_2 \arrow[r, "e"] \arrow[d, "\xi"] & P^{k}\Gamma_1 \arrow[r, "i"] \arrow[d, "\xi"'] & P^{k}\Gamma_1 \cup P^{k-1}\Gamma_0 \arrow[d, "\xi"'] \\
X_{2s+2r-k+1} \arrow[r] & X_{2s+r-k+1} \arrow[r] & X_{2s-k+1}
\end{tikzcd}$$

Note that the middle ξ technically has codomain X_{2s+r-k}, but we want to diagram to still make sense when $r = 2$, so we include $X_{2s+r-k} \subset X_{2s+r-k+1}$. We have

$$P^{k+1}\Gamma_2 \cong \Sigma^{n-1} P^{n+k}_{n-1} \cong S^{2n+k-1} \vee \Sigma^{n-1} P^{n+k-1}_{n-1}$$

where the second equivalence comes from the fact that $P^{n+k}_{n+k-m+1}$ is reducible for $m \leq k+2 \leq v(k+n)$. Thus the homology of $P^{k+1}\Gamma_2$ is generated by $e_{k+1} \otimes dx \otimes dx$, so there is an $C \in \pi_{2n+k-1}(P^{k+1}\Gamma_2)$ (for instance the inclusion of the top cell) which has this as Hurewicz image. If the diagram commutes, ξC is a lift of $\xi\mathfrak{i} = \partial$ and we are done, since, by Diagram 6.3.1 and Remark 6.3.2,

$$\xi C = Sq^{k+n} d_r x$$

By Theorem 6.1.3, $k+n$ is odd, so ieC into has, by Lemma 6.3.5, Hurewicz image $(-1)^k e_k \otimes d(x \otimes x)$, which is the Hurewicz image of $\mathfrak{i}$, and since Hurewicz is an isomorphism the relevant dimensions, this means $ieC = \mathfrak{i}$, so the diagram commutes and we are done. □

Theorem 6.4.2. *Let $v(k+n) = k+1$ and let f be the (readily computable by Theorem 6.1.3) filtration of the image of J in $\pi_k S$. Let a detect the generator of $im(J)$ in π_k. Then*

$$\begin{array}{rcccl}
d_{2r-1} Sq^{n+k} x & = & Sq^{n+k} d_r x & \text{if} & 2r-1 < r+f+k \\
d_{2r-1} Sq^{n+k} x & = & Sq^{n+k} d_r x + \overline{a} x d_r x & \text{if} & 2r-1 = r+f+k \\
d_{r+f+k} Sq^{n+k} x & = & \overline{a} x d_r x & \text{if} & 2r-1 > r+f+k
\end{array}$$

where $\overline{a}$ is the detector of a in the Ext.

Proof. Let $C \in \pi_{2n+k-1}(P^{k+1}\Gamma_2)$ be the attaching map of the top cell. Since $v(k+n) = k+1$, ∂C can be compressed into the $(2n-2)$-skeleton of $P^{k+1}\Gamma_2$, which is equivalent to $P^0\Gamma_2 \cong \Gamma_2$. By Theorem 6.1.3 $a = \partial C$, and C has Hurewicz image $e_{k+1} \otimes dx \otimes dx$. Let

$$R : (D^{2n-1}, S^{2n-2}) \to (P^0\Gamma_1, P^0\Gamma_2) \cong (\Gamma_1, \Gamma_2)$$

bet the inclusion of the upper hemisphere of Γ_1, with Hurewicz image $x \otimes dx$ (which we can call $e_0 \otimes x \otimes dx$). Of course, ∂R has Hurewicz image $e_0 \otimes dx \otimes dx$ and is an isomorphism.

With this notation, there are two cases we need to check: $k = 0$ and $k \neq 0$. Let Ca be the cone on a. Assume first that $k \neq 0$ and get the following diagram:

$$\begin{tikzcd}
S^{2n+k-2} \arrow[rr] \arrow[dr, "(\partial R)a"] \arrow[dd] & & D^{2n+k-1} \arrow[dr, "R(Ca)"] \arrow[dd] & & \\
& \Gamma_2 \arrow[rr] \arrow[dd] & & \Gamma_1 \arrow[dr, "i"] \arrow[dd] & \\
D^{2n+k-1} \arrow[rr] \arrow[dr, "C"'] & & S^{2n+k-1} \arrow[dr, "R(Ca)\cup C"] & & P^{k-1}\Gamma_0 \arrow[dd] \\
& P^{k+1}\Gamma_2 \arrow[rr] \arrow[dr, "e"'] & & \Gamma_1 \cup P^{k+1}\Gamma_2 \arrow[dr, "i\cup e"] & \\
& & P^k\Gamma_1 \arrow[rr] & & P^{k-1}\Gamma_0 \cup P^k\Gamma_1
\end{tikzcd}$$

Recall that $P^{k-1}\Gamma_0$ is the cone on $P^{k-1}\Gamma_1$, so there is a homotopy equivalence

$$\pi : P^{k-1}\Gamma_0 \cup P^k\Gamma_1 \to P^k\Gamma_1/P^{k-1}\Gamma_1$$

and, since $[eC \cap iR(Ca)] = [a] = [0] \bmod P^{k-1}\Gamma_1 = P^{k-1}\Gamma_0 \cap P^k\Gamma_1$, we have

$$\pi(iR(Ca) \cup eC) = [eC] - [iR(Ca)]$$

where $[\alpha]$ denote the equivalence class of such a map α mod $P^{k-1}\Gamma_1$. Of course, by the diagram, $iR(Ca) \equiv 0 \bmod P^{k-1}\Gamma_1$. Thus the Hurewicz image of $iR(Ca) \cup eC$ is the Hurewicz image of eC, which by Lemma 6.3.5 is

$$(-1)^k e_k \otimes d(x \otimes x)$$

and so, by Lemma 6.3.6

$$iR(Ca) \cup eC = \mathfrak{i}$$

Of course, we have, by Diagram 6.3.1, we have

$$\xi C = Sq^{n+k} d_r x \in \pi_*(X_{2s-k+2r-1}, X_{2s+2r})$$

for r small enough for this to make sense. Of course, recalling from Theorem 5.3.5 the base case in the inductive definition of

$$\xi_* : \pi_*(\Gamma_0, \Gamma_1) \to \pi_*(X_{2s+r}, X_{2s+2r})$$

we see that ξ_* sends the Hurewicz preimage of $x \otimes dx (= e_0 \otimes x \otimes dx)$ to $x d_r x$. We thus have in $\pi_*(X_{2s-k+1}, X_{2s+2r})$, using that ξ_* commutes with e and i,

$$\begin{aligned}
\partial &= \xi_*(eC \cup iR(Ca)) \\
&= \xi_*(eC) - \xi_*(iR(Ca)) \\
&= \xi_*(C) - \bar{a}\xi_*(R)
\end{aligned}$$

Noting that this represents $d_{2r-1}x$ and sorting through the filtrations, the theorem is proved for $k > 0$

If $k = 0$, one gets the diagram

$$\begin{tikzcd}
S^{2n-2} \arrow[rr] \arrow[dr, "(\partial R)a"] \arrow[dd] & & D^{2n-1} \arrow[dr, "R(Ca)"] \arrow[dd] & & \\
& \Gamma_2 \arrow[rr] \arrow[dd] & & \Gamma_1 \arrow[dr] \arrow[dd] & \\
D^{2n-1} \arrow[rr] \arrow[dr, "C"'] & & S^{2n-1} \arrow[dr, "R(Ca)\cup C"'] & & \Gamma_1 \arrow[dd] \\
& P^1\Gamma_2 \arrow[rr] \arrow[dr, "e"'] & & \Gamma_1 \cup P^1\Gamma_2 \arrow[dr, "1\cup e"] & \\
& & \Gamma_1 \arrow[rr] & & \Gamma_1
\end{tikzcd}$$

Note that $\pi : \Gamma_1 \to \Gamma_1/\Gamma_2$ is an injection on cohomology (this is just a quotient of S^{2n-1} by the equator, so the map in H_{2n-1} is the diagonal $\mathbb{F}_2 \to \mathbb{F}_2^2$). The intersection $eC \cap R(Ca)$ is zero mod Γ_2, so the Hurewicz image of $\pi(eC \cap R(Ca)) = [eC] - [R(Ca)]$ is, using $k + n = n$ is even,

$$\begin{aligned}
e_*(e_1 \otimes dx \otimes dx + 2e_0 \otimes x \otimes dx) &= e_0 \otimes dx \otimes x - e_0 \otimes x \otimes dx + 2e_0 \otimes x \otimes dx \\
&= e_0 \otimes dx \otimes x + e_0 \otimes x \otimes dx \\
&= e_0 \otimes d(x \otimes x)
\end{aligned}$$

by Lemma 6.3.3and thus

$$eC \cap R(Ca) = \imath$$

Like above, we have

$$\begin{aligned}
\partial &= \xi(eC \cup R(Ca)) \\
&= \xi(eC) - \xi(R(Ca)) \\
&= \xi(C) - \bar{a}\xi(R)
\end{aligned}$$

and the theorem is proved.

□

6.5 Differentials on the Hopf Invariant One Elements

We would now like to give a differential formula which puts the Hopf Invariant question to rest once and for all.

Theorem 6.5.1. *If $k > 0$ and $k + n$ is even (so $\nu(k + n) = 1$)*

$$d_2 Sq^{k+n} x = h_0 Sq^{n+k-1} x$$

Proof. Like before, let C be the top cell in $\pi_{k+2n-1}(P^k\Gamma_1, P^{k-1}\Gamma_1)$, so that it has Hurewicz image $(-1)^k e_k \otimes d(x \otimes x)$ (we can pick either sign, so pick this one). Since $P^{k-1}\Gamma_0$ is contractible, there is an isomorphism

$$\partial : \pi_{2n+k-1}(P^{k-1}\Gamma_0, P^{k-1}\Gamma_1) \to \pi_{2n+k-2}(P^{k-1}\Gamma_1)$$

coming from the fibration long exact sequence. Of course the boundary of C, abusively denoted ∂C, is in the codomain, so define $A \in \pi_{2n+k-1}(P^{k-1}\Gamma_0, P^{k-1}\Gamma_1)$ by

$$A = \partial^{i-1}(\partial C)$$

and once can easily check, using the corresponding isomorphism in integral homology and the fact that the Hurewicz map on $\pi_{k+2n-1}(P^k\Gamma_1, P^{k-1}\Gamma_1)$ is an isomorphism, that the Hurewicz image of A is

$$(-1)^{k-1}2e_{k-1} \otimes x \otimes x$$

Consider now the map

$$C \cup A \in \pi_{2n+k-1}(P^k\Gamma_1 \cup P^{k-1}\Gamma_0)$$

Since the boundary is only $2n+k-2$ dimensional (and thus has zero $2n+k-1$ homology), we can use the same trick as before to see that this map actually splits as

$$(-1)^k(e_k \otimes d(x \otimes x) + 2e_{k-1} \otimes x \otimes x) \in H_{2n+k-1}(P^k\Gamma_1 \cup P^{k-1}\Gamma_0, P^{k-1}\Gamma_1)$$

Thus we have that, by Lemma 6.3.6,

$$\imath \equiv C \cup A \pmod{P^{k-1}\Gamma_1}$$

However, since $H_{2n+k-1}P^{k-1}\Gamma_1$ is zero for dimension reasons, the long exact sequence in homology says that the inclusion

$$H_{2n+k-1}(P^k\Gamma_1 \cup P^{k-1}\Gamma_0) \to H_{2n+k-1}(P^k\Gamma_1 \cup P^{k-1}\Gamma_0, P^{k-1}\Gamma_1)$$

is an injection so in fact,

$$\imath = C \cup A$$

Now, A is divisible by 2, since it is divisible by 2 in homology and Hurewicz is a surjection. We have

$$\xi_*\left(\frac{(-1)^{k-1}}{2}A\right) = Sq^{n+k-1}x \in \pi_*(X_{2s-k+1}, X_{2s+2r-k+1})$$

Since multiplying by 2 in homology is the same as pre-composing with the map detected by h_0, we have

$$\xi_*A = (-1)^{k-1}h_0Sq^{n+k-1}x \in \pi_*(X_{2s-k+2}, X_{2s+2r-k+1})$$

Note that we can push the image of A from X_{2s-k+1} to X_{2s-k+2} since it is twice a map into $(X_{2s-k+1}, X_{2s+2r-k+1})$. Thus we have, mod $X_{2s+2r-k+1}$ (where ξ sends $P^{k-1}\Gamma_1$, the domain of $C \cap A$), that

$$\partial = \xi(C \cup A) = \xi C - \xi A = \xi C - (-1)^{k-1}h_0Sq^{n+k-1}x$$

so if we can just push ξC up farther than X_{2s-k+2}, we will be done.

We have the following relative diagram

$$\begin{tikzcd}
(X_{2s-k+2}, X_{2s+r-k+1}) & (X_{2s-k+1}, X_{2s+r-k+1}) \arrow[l] & (P^k\Gamma_1 \cup P^{k-1}\Gamma_0, P^{k-1}\Gamma_1) \arrow[l, "\xi"'] \\
(X_{2s+r-k}, X_{2s+r-k+1}) \arrow[u] \arrow[d, equal] & (P^k\Gamma_1, P^{k-1}\Gamma_1) \arrow[l, "\xi"'] \arrow[r] \arrow[d] \arrow[ur] & (P^{k+1}\Gamma_1, P^{k-1}\Gamma_1 \cup P^{k+1}\Gamma_2) \\
(X_{2s+r-k}, X_{2s+r-k+1}) & (P^k\Gamma_1, P^{k-1}\Gamma_1 \cup P^k\Gamma_2) \arrow[l, "\xi"'] \arrow[ur] &
\end{tikzcd}$$

Now, consider the image of C in $\pi_{2n+k-1}(P^{k+1}\Gamma_1, P^{k-1}\Gamma_1 \cup P^{k+1}\Gamma_2)$. By Lemma 6.3.3, C has Hurewicz image $2e_k \otimes x \otimes dx$ here. Since Hurewicz is still an isomorphism, this means there is an element

$$\frac{1}{2}C \in \pi_{2n+k-1}(P^{k+1}\Gamma_1, P^{k-1}\Gamma_1 \cup P^{k+1}\Gamma_2)$$

We can uniquely (Hurewicz is an isomorphism) pull this back to

$$\frac{1}{2}C \in \pi_{2n+k-1}(P^k\Gamma_1, P^{k-1}\Gamma_1 \cup P^k\Gamma_2)$$

since $\partial(e_k \otimes x \otimes dx) \in H_{2n+k-2}(P^{k-1}\Gamma_1 \cup P^k\Gamma_2)$, making $\frac{1}{2}C$ still a cycle here. Thus $2 \cdot \frac{1}{2}C$ is the image of C in $\pi_{2n+k-1}(P^k\Gamma_1, P^{k-1}\Gamma_1 \cup P^k\Gamma_2)$. But $\pi_*(X_{2s+r-k}, X_{2s+r-k+1})$ is an $\mathbb{F}_2$-vector space, so since ξC is divisible by 2 there, it must be zero, and so zero in $\pi_*(X_{2s-k+2}, X_{2s+r-k+1})$. Thus, in the actual Adams Spectral Sequence, we get

$$\partial = h_0 Sq^{n+k-1}x$$

□

To see that this answers the Hopf Invariant question, consider this formula when $x = h_i$ and $k = 1$, so that $n = 2^i - 1$ and $k + n = 2^i$. Since $Sq^{2^i}h_i = h_{i+1}$, this formula says

$$d_2 h_{i+1} = h_0 Sq^{2^i-1}h_i = h_0 h_i^2$$

Thus the formula says that whenever $h_0 h_i^2 \neq 0$ in E_2, there is no element of Hopf Invariant 1 in $\pi^s_{2^i-1}(S^{2^{i-1}})$.

Lemma 6.5.2 ([Wan67])**.** *In* $\mathrm{Ext}^3_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2)$*, the only relationships in elements of the form $h_i h_j h_k$ are*

$$h_i h_{i+1} = 0 \quad h_1 h_{i+2}^2 = 0 \quad h_i^2 h_{i+2} = h_{i+1}^3$$

In particular, for $i \geq 3$,

$$h_0 h_i \neq 0$$

Thus h_i does not make it past E_2 for $i > 3$.

Corollary 6.5.3. *The only n such that $\pi_{2n-1}(S^n)$ contains a map of Hopf Invariant One are $n = 1, 2, 4$ and 8.*

Corollary 6.5.4. *The only finite dimensional division algebras over $\mathbb{R}$ are the complex numbers, quaternions and octonions.*

7 Computational Methods

We implemented computer algorithms in the programming language Haskell to do the computations described in this paper. The computations include computing free resolutions, $\mathrm{Ext}_{\mathcal{A}}$ modules, Yoneda Products and Steenrod Operations on $\mathrm{Ext}_{\mathcal{A}}$. Many of the algorithms in this section are similar, if not identical, to the ones in [Bru89], although the author discovered them independently.

7.1 Representing Modules

All modules in this program are free. For a ring R and (possibly infinite) totally ordered set B, let $R(B)$ be the free R-module with basis B. Any vector in this module can be written uniquely in the form

$$\sum_{i=1}^{n} r_i b_i$$

where the b_i are unique and the $r_i \neq 0$ for all i. We represent this vector as a map data structure built on a balanced binary search tree, with keys b_i and values r_i. It is apparent, then, that we can efficiently do scalar multiplication, addition and coefficient look-up. While operations are never destructive in Haskell, much memory can be shared between vectors involved in the same computations; if $x + y = z$, then z may share internal pointers with x and y. Tricks are used to prevent unnecessary copying of vector data-structures, for instance, $1 \cdot x$ returns x in $O(1)$ time and memory. We found this to be a substantial bonus, especially in the case of interest $R = \mathbb{F}_2$.

Suppose that $R = k(B)$ is itself a ring, which is true if there is a specified map of sets from $B \times B \to R$. Consider the free R-module $R(C)$. This is represented by a binary search tree whose nodes are labeled with binary search trees. However, it is often convenient to consider $R(C)$ as a k-vector space. We use the isomorphism

$$R(C) \cong R \otimes_R R(C) \cong k(C \times B)$$

We denote the tuples in the basis $C \times B$ by $c \otimes b$, and switch between these two representations when necessary.

7.2 Representing Subspaces

Definition 7.2.1. *Let $v_1, ..., v_n \in R(B)$, all nonzero, and write*

$$v_{i,j} = \sum_{j=1}^{n_i} r_{i,j} b_{i,j}$$

ordered so that for all i, if $j < j'$ then $b_{i,j} < b_{i,j'}$ and with all $r_{i,j} \neq 0$. We say the set is lower triangular if, $i < i'$ implies

$$b_{i,1} < b_{i',1}$$

It is clear that the v_i are linearly independent.

We represent n-dimensional subspaces of $R(B)$ by a set of n lower triangular vectors, where these vectors are stored in a set data structure built on a balanced binary search tree.

Let $S = \langle v_1, .., v_n \rangle \subset k(B)$ be a subspace represented in this way, with k a field. Any vector v can be written as

$$v = u + \sum a_i v_i$$

with $u \notin S$. Recovering the r_i is as easy as solving an over-constrained lower-triangular system of linear equations. If the v_i and v are dense, this can be expensive, but if they are sparse (as is often the case), this is relatively cheap. If the v_i have on average l nonzero coefficients then the run time of this operation is $O(nl)$.

Definition 7.2.2. *We say we have reduced v by S if we have written it in the form above*

This has many applications. One application is to see if $x = y$ in $k(B)/S$, which specializes to seeing if $x \in S$ when $y = 0$. Another is creating the subspace $S + \langle x \rangle$: simply let x' be x reduced by S and, if $x' \neq 0$, insert it into to the set. Notice that x' has minimal nonzero entry distinct from any such in S, so this algorithm allows us to build S from the zero subspace by adding one vector at a time.

Finally, let $X \oplus Y \subset k(B)$ with X and Y subspaces (in the mathematical sense) and consider a map

$$f : X \to Y$$

Let $x_1, ..., x_n$ be vectors in X with $f(x_1), ..., f(x_n)$ linearly independent in Y, and let $y \in \langle f(x_1), ..., f(x_n) \rangle$. To compute a preimage of y in X, reduce

$$y = \sum a_i f(x_i)$$

and conclude that

$$f^{-1}(y) = \sum a_i x_i$$

is a preimage. Better yet, this process is linear in y, which is algebraically convenient.

7.3 Calculating $\mathrm{Ext}_A(X, k)$

There is a relatively straightforward way to calculate $\mathrm{Ext}_A(X, k)$ if A is a graded k-algebra with the property that $A^0 = k$ (we say in this case that A is connective). To do this, we use a "minimal" resolution of A, that is, a resolution constructed inductively with the least number of generators in each degree in each dimension and a preferred generator set. Let me describe the algorithm. We want a resolution

$$X \longleftarrow F_1 \longleftarrow F_2 \longleftarrow F_3 \longleftarrow ...$$

The generators of X as an A module are the generators of F_1, mapping in the obvious way into A. Construct F_i inductively as follows. Let F_i^j be the degree j elements in F_i, and A^i be elements of A of degree i. The algorithm will work by adding generators where needed, so let G_i^j be the generators added in F_i^j. Define

$$(F_i^j)' = \langle sg | g \in G_i^k, k < j, s \in A^{j-k} \rangle$$

$$I = d((F_i^j)') \subseteq F_{i-1}^j$$

$$K = \ker(d : (F_{i-1}^j)' \to (F_{i-2}^j)')$$

$$F_i^j = (F_i^j)' \oplus \langle G_i^j \rangle$$

The set G_i^j is formed by adding generators to $F_i^{j'}$ until $d|_I : I \to K$ is an isomorphism. Notice that we can use $(F_{i-1}^j)'$ instead of F_{i-1}^j since the kernel of d is the same on both modules. It is easy to see that, by construction, these are free-A modules and the differentials are exact.

To make this explicit, when computing F_i^j, we compute K and I, stored as k-subspaces as described in Section 7.2. Then we simply reduce the basis vectors in $u \in K$ against I and, if we find a $u \notin I$, we add an element to G_i^j whose differential is the reduction of u and add u to I. What was I during the computation of F_i^j is reused as K during the computation of F_{i+1}^j, and K is freed.

Lemma 7.3.1.

$$\mathrm{Ext}_A^s(X, k) \cong k(G_s^*)$$

That is, $\mathrm{Ext}_A^*(X, k)$ *is k-free on the A-generators of a minimal resolution* F_*.

Proof. Let $\sum_i s_i g_i$ be a homogeneous boundary, with the $g_i \in G_*^*$ and $s_i \in A$. Then all the s_i are homogeneous, and cannot be 1 by the construction. But any $\phi \in \mathrm{Hom}_A(F_*, k)$ will have

$$\phi\left(\sum s_i g_i\right) = s_i\left(\sum \phi(g_i)\right) = 0$$

since $s_i \cdot 1 = 0$ for $s_i \in A^j$, for $j > 0$, for grading reasons. Thus $d^*\phi = \phi \circ d = 0$. Thus all differentials are zero, so

$$\mathrm{Ext}_A(X, k) = \mathrm{Hom}_A(F_*, k)$$

This is isomorphic to the k-vector space on the same basis as F_*. □

7.4 Yoneda Products

We want to compute the module structure

$$\mathrm{Ext}_A(M, k) \otimes \mathrm{Ext}_A(k, k) \to \mathrm{Ext}_A(M, k)$$

given by Yoneda products (see A.1). It is clear that, to do this, we need only compute the isomorphism between $\mathrm{Ext}_A(M, k)$ and the homotopy classes of chain maps between A-free resolutions of of M and k. Let $F_* \to M$ and $K_* \to k$ be minimal A-resolutions, computed as above (in particular, with specified A-basis), and g a A-basis vector of F_s. Let g^* denote the cocycle which sends $g \mapsto 1$ and all other basis vectors of F_* to zero. Let $g_i^* : F_{s+t} \to K_i$ be the a chain map associated to g^* and the object of our computation. Then g_0^* sends g to $1 \in F_0 = A$ and the rest of the A-basis of F_s to 0. We can compute g_i^* inductively. We have the law that for each basis vector $u \in F_{s+i}$ for $i > 0$,

$$dg_i^* u = g_{i-1}^* du$$

So to compute $g_i^* u$, merely pick a preimage of the differential on $g_{i-1}^* du$ by reducing it against an easily computable k-basis for the image of the differential in the proper grading.

Once the cocycles are extended to chain maps, it is easy to compute $g^* \times h^*$ for h^* dual to an A-basis vector of $K_{s'}$. The formula is

$$g^* \times h^* = \sum_l h^*(g^*(l))l^*$$

where l runs over all A-basis vectors in the appropriate grading of $F_{s+s'}$.

7.5 Steenrod Operations

While the Steenrod Operations give us much information about the differentials, it turns out they are massively difficult to compute. The formula

$$\Delta_n \partial + \partial \Delta_n = \rho \Delta_{n-1} + \Delta_{n-1}$$

from Appendix A.2 gives (using the same trick to invert differentials as above) an algorithm for computing Steenrod Operations, but it requires tensoring already huge resolutions together, which is computationally infeasible. Instead, there is a trick, communicated to me by Bob Bruner, who claimed it was due to Christian Nassau.

Recall that the classes in $\mathrm{Ext}_A^s(k, k)$ are in bijection with equivalences classes of extensions of length s from k to k. Let F_* be an A-free resolution of k and let

$$\mathcal{X}_* = 0 \to k \to X_{s-1} \to \ldots \to X_0 \to k \to 0$$

be such an extension. If there is a chain map $F_* \to \mathcal{X}_*$ such that in dimension s the map $F_s \to k$ is a cocycle u and in dimension -1 is identity $k \to k$, then this extension represents u. Tensoring this extension with itself, we get a length $2s$ extension, and using the free resolution comparison lemma we can find a chain map

$$\Delta_0 : F_* \to \mathcal{X}_* \otimes \mathcal{X}_*$$

over the isomorphism $k \to k \otimes k$, and this cocycle will represent u^2. Of course, the switching map ρ is realizable in $\mathcal{X}_* \otimes \mathcal{X}_*$, so we can inductively compute

$$\Delta_n : F_* \to \mathcal{X}_* \otimes \mathcal{X}_*$$

by the formula

$$\Delta_n \partial + \partial \Delta_n = \rho \Delta_{n-1} + \Delta_{n-1}$$

And use this to compute Steenrod Operations.

The reason this helps is that the size of $\mathcal{X}_*$ can be, in principle, much smaller than F_*, since it need not be A-free. However, finding small extensions algorithmically is quite difficult. The obvious extension, given a nonzero cocycle $u : F_s \to k$, is to set $X_{s-1} = F_{s-1}/d_s(\ker u)$ and $X_i = F_i$ for $0 \leq i < s-1$. Clearly this is just as big as F_* itself, and of little use in and of itself. It does however provide a starting point.

Suppose we already have an extension $\mathcal{X}_*$ representing u, but we feel it is too big. If we can find a subcomplex $L_* \subset \mathcal{X}_*$ with $\mathcal{X}_*/L_*$ exact and the k at either end preserved, we will have succeeded in finding a smaller extension. Notice that property of $\mathcal{X}_*/L_*$ being exact is equivalent to, for each i

$$d_i(L_i) = \mathrm{Im}(d_i) \cap L_{i-1}$$

which is itself equivalent to

$$\frac{L_{i-1} \cap \mathrm{Im}(d_i)}{L_{i-1} \cap d_i(L_i)} = 0 \subset \frac{X_{i-1}}{d_i(L_i)}$$

Definition 7.5.1. *Finding $L_* \subset \mathcal{X}_*$ satisfying the properties above with L_* as large as possible is called the "extension slimming problem".*

One strategy for solving this problem is to use a greedy algorithm. That is, start with X_{s-1} and find $L_{s-1} \subset X_{s-1}$ away from the image of $\mathrm{Im}(d_s) = k$. Then, for each i from $s-2$ to 1, set

$$X'_i = X_i/d_{i+1}(L_{i+1})$$

And find $L'_i \subset X'_i$ such that

$$L'_i \cap \mathrm{Im}(\overline{d}_{i+1}) = 0$$

where

$$\overline{d}_{i+1} : X'_{i+1} \to X'$$

is the induced differential. Clearly L_i can be constructed by lifting the L'_i. To find the L'_i, we need to solve the following problem.

Definition 7.5.2. *Let X be an R-module and $I \subset X$ a submodule. The "Submodule Avoidance Problem" is to find a submodule $L \subset X$ as large as possible with $I \cap L = 0$*

We will reduce this to the following problem, which we will show is NP-hard.

Definition 7.5.3. *Let V be a k-vector space, and let $S_1, ..., S_n \subset V$ be subspaces. The "Subspace Union Avoidance Problem" is to find a subspace $W \subset V$ with $W \cap S_i = 0$ for each i*

Lemma 7.5.4. *The submodule avoidance problem is at least as hard as the subspace union avoidance problem if R is a k-algebra.*

Proof. Let $V, S_1, ..., S_n$ be a subspace union avoidance problem. Let V_i be a copy of V for each i. Adopt the notation that if $v \in V$, then v_i is the copy of the vector v in V_i. Let

$$R = k[x_1, ..., x_n]$$

be a polynomial ring Let

$$X = V \oplus \left(\bigoplus_{i=1}^{n} V_i \right)$$

and let R act on X by

$$x_j v_i = v_j$$

Let

$$\Phi : X \to V$$

be defined by $v_i \mapsto v$. Finally, denote the copy of $S_i \subset V_i$ as T_i, and let

$$I = \bigoplus_i T_i \subset X$$

It is clear that $W \subset V$ is a solution to the original subspace union avoidance problem if and only if the R-module generated by W is a solution to the submodule avoidance problem. Likewise, L is a solution to the submodule avoidance problem we created if and only if $\Phi(L)$ is a solution to the original subspace union avoidance problem, since if $v^{(0)}, v^{(1)}, ..., v^{(n)} \subset V$ and $v = \sum_{i=0}^{n} (v^{(i)})_i \in X$, then $\Phi(v) = \sum_{i=0}^{n} v^{(i)}$ is in some S_j if and only if $x_j v \in T_j$. Since both ways of going between solutions preserve relative k-dimensions, the optimal L has $\Phi(L)$ optimal as a solution to the original problem. Thus the submodule avoidance problem is at least as hard as the subspace union avoidance problem.

□

Lemma 7.5.5. *If $k = \mathbb{F}_p$, deciding if there is a solution to the subspace union avoidance problem of codimension l is NP-hard, even if the S_i are 1-dimensional, unless $l = 1$ and $p = 2$.*

Corollary 7.5.6. *The submodule avoidance problem is NP-hard.*

Proof of 7.5.5: Let (G, E) be a graph and $V = k(G)$ be the free k-space on G. For each edge $uv \in E$, let

$$S_i = \langle u - v \rangle \subset V$$

If there is a $W \subset V$ of codimension l, then there are l linearly independent functionals $f_i : V \to \mathbb{F}_p$ such that the kernel of f_i contains W and furthermore W is the intersection of these kernels. Notice that this means that for each edge uv there is an i with $f_i(u - v) \neq 0$, so $f_i(u) \neq f_i(v)$. Let

$$f = \bigoplus f_i : V \to \mathbb{F}_p^l$$

For each edge $uv \in E$, $f(u) \neq f(v)$, so, $f|_G$ is a $|\mathbb{F}_p^l| = p^l$-coloring on (G, V). To decide if such a thing exists is NP-complete for $l > 1$ for all p and when $l > 0$ for $p > 2$. □

7.6 Automated Differential Propagation

We automatically propagate information about known differentials using propositional logic. Consider the following context free grammar

$$
\begin{array}{rcl}
\text{BoolNode} & := & \text{True} \\
& | & \text{False} \\
& | & \text{BoolVar String} \\
& | & \text{And (Set BoolNode)} \\
& | & \text{Or (Set BoolNode)} \\
& | & \text{XOr (XOrSet BoolNode)} \\
& | & \text{Not BoolNode} \\
& | & \text{EqZero SSNode} \\
\text{SSNode}_r^{s,t} & := & \text{SSConst}_r^{s,t} \\
& | & \text{Differential}_r\ \text{SSNode}_r^{s-r,t-r+1} \\
& | & \text{Projection}_r\ \text{SSNode}_2^{s,t} \\
& | & \text{Sum (XOrSet SSNode}_r^{s,t}) \\
& | & \text{ScalerMult BoolNode SSNode}_r^{s,t}
\end{array}
$$

The notation "And (Set BoolNode)" means an And node which contains as its data an unordered set of BoolNodes. An XOrSet is a Set in which insertion is defined by the rule

```
insert x set = if set contains x then remove x set else insert x set
```

A BoolNode represents, equally, a Boolean value True or False or an element of $\mathbb{F}_2$. A BoolVar node represents a Boolean valued indeterminate. An $\text{SSNode}_r^{s,t}$ represents an element of $E_r^{s,t}$. An $\text{SSConst}_r^{s,t}$ represents a known literal element of the group $E_r^{s,t}$. A Projection_r node represents the projection of a node to the r page, and is not necessarily defined. A ScalerMult node represents "scalar" multiplication of an SSNode by a BoolNode, seen as an element of $\mathbb{F}_2$. A node "EqZero s" is true if the value of s is zero and false otherwise.

Our algorithm for computing differentials in $E_r^{s,t}(S, X)$ is as follows. We start out knowing the E_2 page of the Adams Spectral Sequence, the module structure over the spectral sequence for $X = S$, and the Steenrod Operations. If $X \neq S$, we also know the differentials in $E_r^{s,t}(S, S)$, meaning we must run this algorithm with $X = S$ first. We will maintain a BoolNode which is a quantifier free propositional formula representing all known constraints on the Adams Spectral Sequence. We will maintain a table of preferred bases for all known groups $E_r^{s,t}$, a table of all known differentials and all known projections, called the page, differential and projection table. Finally, we maintain a table of BoolVar substitutions, which we will populate as we learn more about the indeterminates we create, called the variable table. Our work-loop will run as follows

1. Simplify constraint formula using rewrite rules, and reduce linear systems using Gaussian Elimination
2. Populate projection and page table using new information learned
3. Add new constraints using updated tables.
4. Repeat if any tables modified.

7.6.1 Rewriting and Simplification

The rewrite rules we use are mostly standard: things like associativity, identity, nilpotence, etc. One important thing is the way we treat ScalarMult nodes. We have obvious rules like

$$\text{ScalarMult } b \text{ (Sum } x_1 \dots x_n) \mapsto \text{Sum (ScalarMult } b\ x_1) \dots \text{(ScalarMult } b\ x_n)$$

but also less obvious rules, like, if x is an SSConst, then

$$\text{Sum (ScalarMult } b_1\ x) \ldots (\text{ScalarMult } b_n\ x) \mapsto \text{ScalarMult (XOr } b_1 \ldots b_n)\ x$$

that is, we gather in the scalar and distribute in the module. The reason for this is so we can solve. If each x_i is a distinct SSConst, we add the rule

$$\text{EqZero (Sum (ScalarMult } b_1\ x_1) \ldots (\text{ScalarMult } b_n\ x_n)) \mapsto \text{And (Not } b_1) \ldots (\text{Not } b_n))$$

and if x is an SSConst,

$$\text{EqZero (ScalarMult } b\ x) \mapsto \text{Not } b$$

For technical reasons, we add,

$$\text{EqZero (Sum} \ldots x \ldots) \mapsto \text{EqZero (Sum} \ldots (\text{ScalarMult True } x) \ldots)$$

so that if $b \cdot x = x$, we can infer $b = True$.

When building rewriting systems, one must chose how to deal with negations in XOr. For us, a negated XOr node is an XOr node with True as a child. To that end, we have the following rules

$$\text{Not (Xor} \ldots) \mapsto \text{XOr True} \ldots$$

$$\text{XOr} \ldots (\text{Not } b) \ldots \mapsto \text{XOr True} \ldots b \ldots$$

And, for "negated singletons"

$$\text{Xor True } b \mapsto \text{Not } b$$

We find this is helpful when we get to the solving phase.

More interesting are our rules regarding BoolVars. As mentioned above, we maintain a variable table, which assigns a BoolVar b to an arbitrary simplified BoolNode x which does not contain b as a descendant. In addition, each entry in this table has a "timestamp", which says the last time that entry was simplified. When the simplification algorithm reaches a b, it checks to see if b is in the table. If there is some x associated to b, we check the timestamp to see if x has been simplified since the last time something has changed, and if not we simplify it and update the timestamp. Either way we rewrite b to x.

BoolVars are created in various ways. For instance, consider the node Differential$_r x$ for some SSConst $x \in E_r^{s,t}$, and suppose nothing is known about this differential. If we know a basis of $E_r^{s+r,t+r-1}$, say $y_1, ..., y_n$, we have

$$d_r x = \sum_i c_i y_i$$

for some $c_i \in \mathbb{F}_2$. We then rewrite this to

$$\text{Sum (ScalarMult (BoolVar } c_1)\ y_1) \ldots (\text{ScalarMult (BoolVar } c_n)\ y_n)$$

Like the variable table, the differential table associates an SSConst x to arbitrary SSNode y, and we add the above to the differential table upon its discovery. Like the variable table, the differential table has timestamps and the entries within are simplified when they become out-of-date. Thus if later we find more information about the c_i, our differential table will be updated accordingly.

Often, our root is an And node, since it represents a conjunction of constraints. We say a BoolNode is "toplevel" if it is a direct child of the root And node, or if it is the root and not an And node. If either a BoolVar b is toplevel and not already in the variable table, we add an entry to the variable table simply associating b to True. Likewise for the negated situation, Not b, we associate b to False.

7.6.2 Linear Solving

When no more simplification is possible using the strategies above, we attempt linear solving. This entails gathering all the toplevel XOr nodes. Let $\mathcal{V}$ be the free $\mathbb{F}_2$-vector space on the set of all possible BoolNodes. These XOr nodes together determine a system of $\mathbb{F}_2$-linear equations over $\mathcal{V}$. We use Tarski's principle component algorithm to split this system up into a direct sum of often many much smaller system, and use Gaussian Elimination to simplify. The resulting system is equivalent to the first, but simpler. When doing the Gaussian Elimination, we order the basis vectors so that BoolVars come first, since they are preferred as pivots. For each equation of the form $b + x_1 + ... + x_n = 0$ in the simplified system where the pivot b is a BoolVar, add an entry to the variable table sending b to the BoolNode representing $x_1 + ... + x_n$. For the remaining nontrivial equations in the simplified system, simply put them back to the original formula. Note that sometimes it could be the case that $n = 1$ and $x_1 = 1$ or $n = 0$, in which case we sould just be setting b to a Bool literal. If anything is accomplished during this step (that is, anything is added to the variable table), we start the simplification step over. Otherwise, we go on to the next step.

7.6.3 Updating E_r Tables

We wish to update the tables containing preferred bases for each $E_r^{s,t}$. We first simplify everything that needs to be simplified in the differential and projection tables, and update the timestamps accordingly. In the best case, for a given $E_{r-1}^{s,t}$, we know all the differentials into and out of this summand exactly, in which case we can simply pick projections onto $E_r^{s,t}$ and basis for $E_r^{s,t}$. If, we know all of the cycles but not all of the differentials into $E_{r-1}^{s,t}$ (and therefor not all of the boundary relations), we can still often say something intelligent. First pick an ordered basis of literals for the cycles $x_1, .., , x_n$ to over approximate our basis. The actual basis for $E_r^{s,t}$ will be some a subset of these. If we know the differentials into $E_{r-1}^{s,t}$, the way we would pick relations is by saying if some y in our basis of $E_{r-1}^{s-r,t-r+1}$ has

$$dy = \sum_{i=1}^{i} c_i x_i \neq 0$$

then, for the minimum i with $c_i \neq 0$, add the relation $x_i = \sum_{j=i+1}^{n} c_i x_i$. We can implement logical "if-then-else" statements by

$$\text{If } c \text{ Then } x \text{ Else } y \ := \ \text{Sum(ScalarMult } c\ x\text{)(ScalarMult (Not } c\text{) } y\text{)}$$

Thus we can make our basis out of the cascading "if-then-else" statements on if the basis vectors in $E_{r-1}^{s-r,t-r+1}$ have a coefficient of x_i. Since most $E_r^{s,t}$ are low dimensional (usually 1-dim or 2-dim, occasionally 3-dim in as far as we can compute Ext), the terms stay reasonably small. Since often times we can know some but not all of the coefficients of the x_i, these statements will often simplify further to exactly the sort of reasoning a human might attempt: for instance, if either $dy = x$ or $dy = 0$ for the only nonzero y in $E_{r-1}^{s-r,t-r+1}$, we might say the basis element corresponding to x is

$$\text{If } c \text{ Then } 0 \text{ Else } x \ := \ \text{ScalarMult (Not } c\text{) } x$$

where c is some BoolVar which is true if and only if $dy = x$.

7.6.4 Generating Constraints

Our three main sources of constraints are the Leibniz Rule (Theorem 5.1.1), the Steenrod Operations (Theorem 6.4.1, Theorem 6.4.2, Theorem 6.5.1), and the J-homomorphism (Theorem). The Leibniz Rule constraints are easy to generate: for each new cycle $x \in E_r$ discovered in the previous round and each other known cycle y, write

$$d_r(x)y + d_r(y)x + d_r(xy) = 0$$

The Steenrod Operation constraints follow much the same way, provided you can evaluate $Sq^i x$, or read them from a table.

The $\mathrm{Im}(J)$ constraints are more interesting. Since we can calculate the $\mathrm{Im}(J)$ directly on the E_2 page, we can generate a few constraints. Since the $\mathrm{Im}(J)$ must last until E_∞, we know these elements are always cycles and never boundaries. Better yet, we know that if $x \notin \mathrm{Im}(J)$ and $h_0 x \in \mathrm{Im}(J)$, then x must at some point not be a cycle. Many differentials, especially near $t - s \equiv 7$, can be inferred just from these rules and Leibniz, so we take great advantage of this.

Appendices

A Homological Algebra and Steenrod Operations

A.1 Tensor and Yoneda Products in Ext

The Ext functors have quite a bit of structure, which we will discuss briefly in this section. A fuller treatment can be found in [CE99]. Fix some field k. Let A be a k-algebra and X,Y, Z and W be A-modules. Then there is a product, which we will call the tensor product,

$$\mathrm{Ext}^s_A(X, Z) \otimes \mathrm{Ext}^t_A(Y, W) \to \mathrm{Ext}^{s+t}_{A\otimes A}(X \otimes Y, Z \otimes W)$$

To construct the product, let K be an A-resolution of X and L be an A-resolution of Y. Then $K \otimes L$ is an $A \otimes A$-resolution of $X \otimes Y$. If $[x]$, $[y]$ are cocycles in $\mathrm{Ext}^s_A(X, Z)$ and $\mathrm{Ext}^t_A(Y, W)$ respectively, then consider

$$x \otimes y : K \otimes L \to Z \otimes W$$

This is clearly a cocycle, so represents a class in $\mathrm{Ext}^{s+t}_{A\otimes A}(X\otimes Y, Z\otimes W)$, and the class represented depends only in $[x]$ and $[y]$, not on choice of representatives. If A is a Hopf algebra, there is a map $A \to A \otimes A$, making $X \otimes Y$ an A-resolution. In this case, this tensor product becomes

$$\mathrm{Ext}^s_A(X, Z) \otimes \mathrm{Ext}^t_A(Y, W) \to \mathrm{Ext}^{s+t}_A(X \otimes Y, Z \otimes W)$$

In the case that X is a co-algebra, we can pull back by the co-multiplication, and in the case that Y is an algebra, we can compose with the multiplication, yielding.

$$\mathrm{Ext}^s_A(X, Y) \otimes \mathrm{Ext}^t_A(X, Y) \to \mathrm{Ext}^{s+t}_A(X, Y)$$

These are sometimes referred to as cup products.

We also have Yoneda products in Ext, which are often much easier to compute because you do not need to construct the resolution $K \otimes L$. The Yoneda product is a map

$$\mathrm{Ext}^s_A(X, Y) \otimes \mathrm{Ext}^t_A(Y, Z) \to \mathrm{Ext}^{s+t}_{A\otimes A}(X, Z)$$

extending the composition product

$$\mathrm{Hom}_A(X, Y) \otimes \mathrm{Hom}_A(Y, Z) \to \mathrm{Hom}_{A\otimes A}(X, Z)$$

It is computed by considering $\mathrm{Ext}^s_A(X, Y)$ as homotopy classes of degree s chain-maps between resolutions $K \to X$ and $L \to Y$. This can be done by the comparison theorem for projective resolutions. Since a cocycle in $\mathrm{Ext}^s_A(X, Y)$ is represented by a map $K_s \to Y$, we can extend that map to a chain map with component maps $K_{s+i} \to L_i$, and this is unique up to homotopy. Likewise, given a degree s chain map x between resolutions K and L, taking the map from $K_s \to L_0 \to Y$ gives a chain x_0. Since $x_0 d = d_{-1} x = 0$, x_0 is a cocycle, so indeed Ext is in bijection with homotopy classes of chain maps. But chain maps can be composed yielding the desired map. In particular, this means $\mathrm{Ext}_A(X, X)$ is a ring and $\mathrm{Ext}_A(X, M)$ is a left module and $\mathrm{Ext}_A(M, X)$ is a right module.

A.2 Chain Level Construction of Steenrod Operations

This construction is found most clearly in [Kah70], but also in [Mil72] and [Bru86].

Theorem A.2.1. *Let A be a cocommutative $\mathbb{F}_2$-Hopf Algebra, X an algebra over A and Y a coalgebra over A. Then there is a map of graded modules*

$$\mathcal{A} \otimes \mathrm{Ext}_A(Y, X) \to \mathrm{Ext}_A(Y, X)$$

Proof. Let K be a projective resolution of Y. We can view $K \otimes K$ as an A-module via the Hopf-comultiplication $A \to A \otimes A$ and a resolution of $Y \otimes Y$. We have an $\mathcal{A}$-map

$$\Delta_0 : K \to K \otimes K$$

lifting the comultiplication $Y \to Y \otimes Y$, and this is good for computing cup products. Let ρ be the switching map on $K \otimes K$, that is

$$\rho(x \otimes y) = y \otimes x$$

We have

$$\rho\Delta_0 : K \to K \otimes K$$

is another map lifting the comultiplication, and thus $rho\Delta_0$ is homotopic to Δ_0. Because K has a contracting homotopy since it is a resolution, we can compute Δ_1 which satisfies

$$\Delta_1\partial + \partial\Delta_1 = \rho\Delta_0 + \Delta_0$$

Likewise, Δ_1 and $\rho\Delta_1$ are homotopic, and so on, so we can find chain of degree n

$$\Delta_n : K \to K \otimes K$$

$$\Delta_n\partial + \partial\Delta_n = \rho\Delta_{n-1} + \Delta_{n-1}$$

Letting σ be our contracting homotopy on $K \otimes K \to Y \otimes Y$, we can write an explicit recursive formula

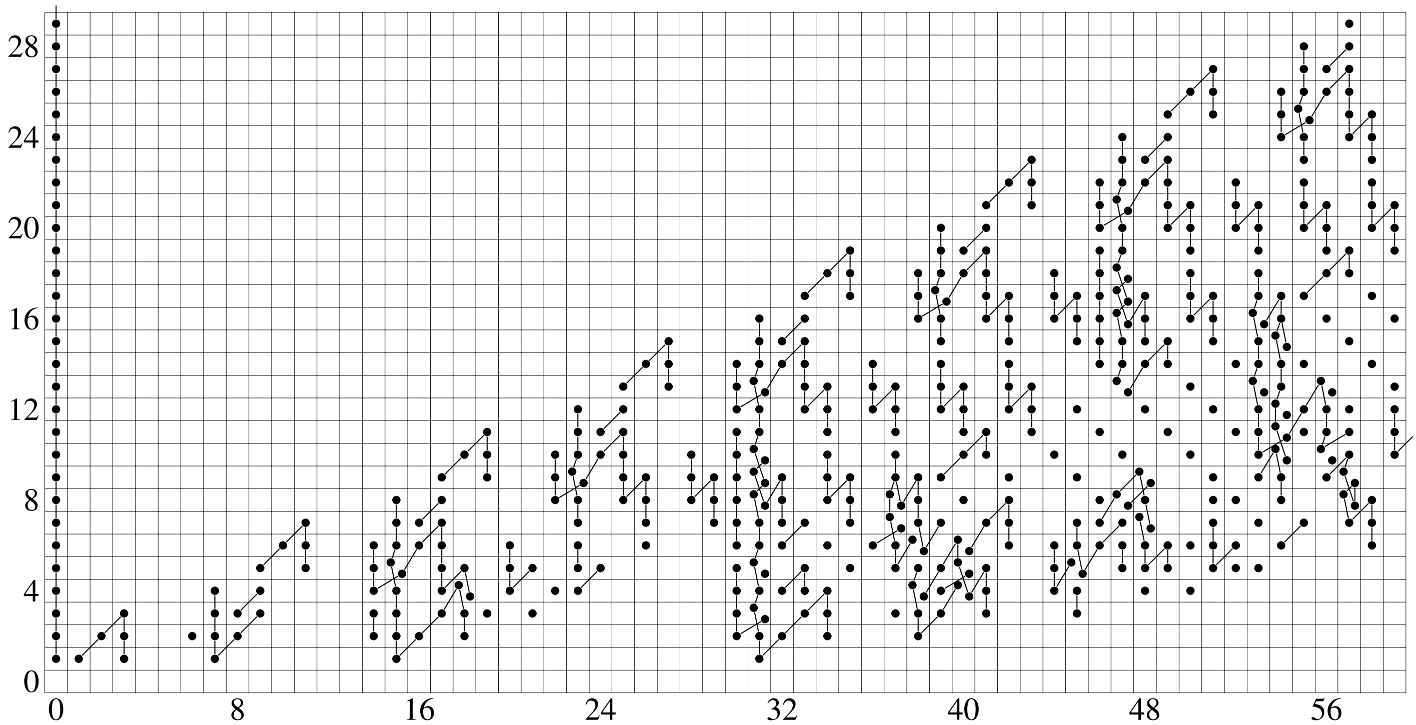
$$\Delta_n = \sigma(\partial\Delta_n + \rho\Delta_{n-1} + \Delta_{n-1})$$

Finally, define for $[u] \in \mathrm{Ext}_A^{s,t-s}(\mathbb{F}_2, M)$

$$Sq^i(u)(\sigma) = (u \otimes u)(\Delta_{2s-i}\sigma)$$

Once can check that these operation obey the Cartan formula and Adem relations, as in [May]. □

B Spectral Sequence Diagram



We have here the a diagram of the first 60 stems of the E_2 page of the Adams Spectral Sequence. The horizontal axis represents $t-s$ and the vertical axis represents s, so a dot in slot $(s, t-s)$ represents a $\mathbb{F}_2$ summand of $E_s^{s,t-s}$. The vertical lines represent multiplication (see Theorem 5.1.1) by h_0, which converges to multiplication by 2 in π_*^s (see Section 4.3 and 2.4 for a definition of h_i); that is, x and h_0x are connected by a vertical line. The diagonal lines represent multiplication by h_1, the element of the Adams Spectral Sequence detecting the Hopf Fibration. A differential, d_r, will go 1 square left and r squares up. Just from this picture and the Leibniz Rule, we can reason that all differentials before $t-s = 13$ are zero, as mentioned in Section .

One is forced to notice an imperfect pattern when staring at this picture. The patterns in the dots seem to have a mod-8 periodicity towards the top of the diagram. This is very deep and consequence of the so-called J-homomorphism and Bott Periodicity. Indeed, the three dots in the "triangles" at the top of the diagram ending when $t-s \equiv 3 \pmod 8$ are exactly the image of J. The long vertical "towers" when $t-s \equiv 7 \pmod 8$ are also the image of the J in those dimensions, and these converge to large cyclic summands of π_*^s, in part explaining the apparent jump in the size of π_i^s when $i \equiv 7 \pmod 8$. The reader is referred to [Rav86] and [Mah70].

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