THE RICCI FLOW ON SURFACES AND THE UNIFORMIZATION THEOREM

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1. Introduction to the Ricci Flow

1.1. **Introduction.** The *Ricci flow* is a geometric evolution equation for the metric tensor on a general Riemannian manifold. First introduced by Richard Hamilton in [3], the (suitably normalized) flow has the property that any fixed point metric for the evolution equation is an Einstein metric. Hamilton was able to show in that first paper that in 3-manifolds, a nice curvature condition on the initial metric (positive Ricci curvature) implies that the flow exists for all time and rapidly converges to a constant-curvature metric, thus providing the first proof of the fact that all 3-manifolds with positive Ricci curvature carry a metric of constant positive curvature.

This process has been generalized to other types of curvature conditions in other dimensions. An interesting case to consider is dimension two, where C^∞ convergence of a metric under the normalized flow implies that the limit metric has constant curvature. The flow reduces to a scalar evolution equation in this case, so one might hope that the situation might be better-behaved and we could get convergence for any starting metric, at least on closed surfaces. The flow in 2-d is conformal, as we shall see, so the preceding fact would then imply part the famous *Uniformization Theorem* from complex analysis: any closed Riemannian surface has a conformal metric of constant curvature. Fortunately, these hopes turn out to be reasonable, and the Ricci flow on surfaces is extremely well-behaved compared to its higher-dimensional counterparts. Unfortunately, the early methods used to prove the convergence of the normalized flow required the Kazdan-Warner identity (see section 4.2 of [2]), which utilizes the Uniformization Theorem. Thus the Ricci flow technique failed to provide an alternate proof of the Uniformization Theorem.

In 2009, however, Ben Andrews and Paul Bryan managed to prove the most difficult case of existence and convergence of the flow on surfaces without using any results from the Uniformization Theorem. In its place, they used a clever argument based on comparing boundary lengths of domains in two manifolds evolving under the flow. With this argument in place, the Ricci flow proof of the Uniformization Theorem was complete.

The goal of this paper is to present all of the results necessary for the proof of convergence of the Ricci flow on closed surfaces, encorporating the technique of Andrews and Bryan to treat the most difficult case. Thus this paper should provide a mostly self-contained proof of the Uniformization Theorem for closed surfaces. For the sake of brevity, the proofs of some technical results on parabolic PDE and some lengthy flow evolution calculations will be omitted, but I will try to be clear about all of the results I am using.

1.2. **Notational Preliminaries.** Throughout this paper I will assume that the reader has an acquaintance with basic Riemannian geometry. We will be considering n-dimensional Riemannian manifolds (M^n, g), often abbreviated M^n or simply M. ∇ will denote the Levi-Civita connection on M which sends (k,l)-tensors to $(k,l{+}1)$-tensors. $\nabla_X T$ will then denote the covariant derivative of a tensor T by a vector field X. If we have coordinate vector fields ∂_i, we can define tensor components in the usual way: if T is a (k,l)-tensor then we define components $T^{i_1\ldots i_k}_{j_1\ldots j_l}$ by $T(\partial_{j_1},\ldots,\partial_{j_l}) = T^{i_1\ldots i_k}_{j_1\ldots j_l}\partial_{i_1}\otimes\cdots\otimes\partial_{i_k}$, with the Einstein summation convention in place. The connection has components given by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, with

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

I will use repeated ∇ without parentheses to denote higher-order tensor covariant derivatives, thus for example given a function f , I will denote the Hessian of f evaluated on vectors X, Y by $\nabla_X\nabla_Y f = \nabla_X(\nabla_Y f) - \nabla_{\nabla_X Y} f$.

Using the components of the $(0,2)$ metric tensor $g = (g_{ij})$ and its $(2,0)$ inverse $g^{-1} = (g^{ij})$ we can dualize tensors, i.e. raise and lower indices: in particular, given a vector field X and a 1-form field θ we will use the coordinate-free musical notation to write their duals:

$$\begin{aligned} X^\flat &= g(X,\cdot) = X^i g_{ij}dx^j \ \text{(a 1-form field)} \\ \theta^\sharp &= \theta_i g^{ij}\partial_j \ \text{(a vector field with } g(X,\theta^\sharp) = \theta(X)\text{)} \end{aligned}$$

We can also use the metric to take traces of tensors in the usual manner; given a tensor h of any type, I define the Laplacian of h (denoted Δh) to be the metric trace of the second covariant derivative: i.e.

$$\Delta h = g^{ij}\nabla_{\partial_i}\nabla_{\partial_j} h.$$

We now define the Riemannian curvature tensor; the $(1,3)$ version is given by

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \\ \text{or } R(\partial_i,\partial_j)\partial_k &= R^l_{ijk}\partial_l = (\partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp})\partial_l \text{ in local coordinates.} \end{aligned}$$

The (0,4) curvature tensor is then defined by

$$\begin{aligned} R(X,Y,Z,W) &= g(R(X,Y)Z,W) \\ R_{ijkl} &= g_{pl}R^p_{ijk} \end{aligned}$$

The (0,2) Ricci tensor is then formed by taking the trace of the curvature tensor in the following manner:

$$\begin{aligned} Ric(X,Y) &= tr(R(\cdot,X)Y) = tr_g(R(\cdot,X,Y,\cdot)) \\ R_{ij} &= R^l_{lij} = g^{kl}R_{kijl} \end{aligned}$$

Finally, the scalar curvature function $R : M^n \to \mathbb{R}$ is given by the metric trace of the Ricci tensor:

$$R = tr_g(Ric(\cdot,\cdot)) = g^{ij}R_{ij}.$$

When working with our geometric evolution equations it will sometimes be convenient to ignore the finer details of some tensor calculations for estimation purposes. We therefore introduce the following notation: given two tensors A, B of any type, we use the notation $A * B$ to refer to a tensor whose components are some linear combination of contractions, metric contractions, and tensor products of a component of A and a component of B. As examples, the metric inner product of two (k, l)-tensors could be written as $A * B$; also, a standard formula for commuting derivatives shows that for any tensor A

$$[\nabla, \Delta]A = Rm * \nabla A + \nabla Rc * A.$$

1.3. **The Ricci Flow Equations.** The *Ricci flow* (often abbreviated RF in this paper) is a geometric evolution equation defined on Riemannian manifolds (M^n, g) that was first defined by Richard Hamilton in his 1982 paper [1]. The geometry of a Riemannian manifold (M^n, g) is altered by changing its metric via a second-order nonlinear partial differential equation on symmetric $(0, 2)$-tensors:

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.1}$$

As we shall see later, on closed manifolds this evolution equation does not preserve volume. On the n-sphere of radius σ, for example, we have $g = \sigma^2 g_{S^n}$ and $Ric = Ric_{s^n} = (n-1)g_{S^2}$, so we see that the Ricci flow equation is satisfied if $\dot{\sigma}\sigma = -(n-1)$, i.e. if $\sigma = \sqrt{\sigma_0 - 2(n-1)t}$. Thus a round sphere will decrease in radius more and more rapidly until vanishing in finite time.

In many cases we will want to preserve volume during the evolution process; we will see later that this is possible by using the modified *normalized Ricci flow* (NRF) given by

$$\frac{\partial g}{\partial t} = -2Ric + \frac{2}{n} rg \tag{1.2}$$

where $r = \frac{\int_M R d\mu}{\int_M d\mu}$ is the average scalar curvature of M.

We will soon prove short-time existence and uniqueness results for the RF equation, and it would be nice if we could transfer these results to the NRF as well. Fortunately, the following proposition will immediately allow us to do so:

Proposition 1. *(Section 3,* [1]*) There is a bijection between solutions of the RF and NRF equations given by a rescaling of space and time; i.e. given a solution $g(t)$ of the RF on some time interval $[0, \epsilon)$ there exist real-valued functions $\psi(t)$, $t(\tilde{t})$ such that $\psi(t(\tilde{t}))g(t(\tilde{t}))$ is a solution in $\tilde{t}$ of the NRF, and conversely there is an inverse time and space scaling to transform a solution of the NRF to a solution of the unnormalized equation.*

Proof. Let $g(t)$ be a solution of the Ricci flow. The first thing we want to do is choose $\psi(t)$ such that $\psi(t)g(t)$ has constant volume. Let $\tilde{g}(t) = \psi(t)g(t)$ for some yet-unknown function ψ, and let tilde variables represent all of the geometric quantities associated with this metric. By examining the coordinate expressions for the volume form $d\mu = \sqrt{det(g_{ij})}dx^1 \dots dx^n$and the curvatures we see immediately that

$$d\tilde{\mu} = \psi(t)^{n/2} d\mu \quad \tilde{Ric} = Ric \quad \tilde{R} = \frac{1}{\psi} R \quad \tilde{r} = \frac{1}{\psi} r$$

so setting $\psi(t) = (\int_M d\mu(t))^{-2/n}$ yields $\int_M d\tilde{\mu}(t) = 1$. We now choose the function $\tilde{t}(t) = \int \psi(t)dt$; since this is a strictly increasing function of t, we can invert it to $t(\tilde{t})$ and change time variables to $\tilde{t}$ on the entire interval of existence for our original solution. Assuming the soon-to-be-proved fact that $\frac{d}{dt}d\mu = -Rd\mu$ for solutions of the RF, we calculate that $\frac{d}{dt}log\psi = \frac{2}{n}\frac{\int Rd\mu}{\int d\mu} = \frac{2}{n}r$, so

$$\begin{aligned}\frac{\partial \tilde{g}}{\partial \tilde{t}} &= \frac{dt}{d\tilde{t}}\psi(t)\frac{\partial g}{\partial t} + \frac{dt}{d\tilde{t}}\frac{d\psi}{dt}g(t) = \frac{\partial g}{\partial t} + \frac{d}{dt}log\psi \cdot g(t) = -2Ric + \frac{2r}{n}g \\ &= -2\tilde{Ric} + \frac{2\tilde{r}}{n}\tilde{g}\end{aligned}$$

Conversely, there is an inverse transformation that turns solutions of the NRF into solutions of the RF by multiplying the metric by $\frac{1}{\psi}$ and changing time variables from $\tilde{t}$ to t. □

2. Evolution of Geometric Quantities

It will be very important in our further analysis of the Ricci Flow to understand how fundamental geometric objects like the curvature tensor and connection change as the metric evolves. To that end, we will study the variation in these quantities under the general evolution equation

$$\frac{\partial}{\partial t}g_{ij} = h_{ij}.$$

Differentiating the identity $g_{ij}g^{jk} = \delta_i^k$, it immediately follows that we also have a simple evolution equation for the inverse metric:

$$\frac{\partial}{\partial t}g^{ij} = -g^{ik}g^{jl}h_{il}.$$

Before examining the evolution of curvatures, let's look at how the volume form $d\mu = \sqrt{det(g)}dx^1 \wedge \cdots \wedge dx^n$ changes with time. Using the well-known identity $\frac{\partial}{\partial t}det(g) = det(g)g^{ij}\frac{\partial}{\partial t}g_{ij}$, which can be proved by using the cofactor expansion of the determinant, we get

$$\begin{aligned}\frac{\partial}{\partial t}d\mu &= \frac{1}{2}\sqrt{det(g)}(g^{ij}h_{ij})dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{2}tr_g(h)d\mu\end{aligned}$$

At all times during the following geometric calculations I will be assuming that at the time t in question we have chosen geodesic coordinates centered at the point p we are examining, so $\partial_i g_{jk}(p) = 0$ for all i, j, k. Note that the variation of the Christoffel symbols with respect to time is in fact a tensor quantity.

Lemma. $\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\nabla_i h_{lj} + \nabla_j h_{il} - \nabla_l h_{ij})$.

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_{ij}^k &= \frac{\partial}{\partial t}(g^{kl}(\partial_i g_{lk} + \partial_j g_{il} - \partial_l g_{ij})) \\ &= g^{kl}(\partial_i \frac{\partial}{\partial t}g_{lk} + \partial_j \frac{\partial}{\partial t}g_{il} - \partial_l \frac{\partial}{\partial t}g_{ij})\end{aligned}$$

Now note that partial differential is covariant differentiation at p. □

Lemma.

$$\frac{\partial}{\partial t}R^l_{ijk} = \frac{1}{2}g^{lm}[-R^p_{ijm}h_{pk} - R^p_{ijk}h_{pm} + \nabla_i\nabla_k h_{jm} - \nabla_j\nabla_k h_{im} + \nabla_j\nabla_m h_{ik} - \nabla_i\nabla_m h_{jk}]$$

Proof. Use the previous lemma to differentiate the formula

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}$$

with respect to time, and use geodesic coordinates to find that

$$\frac{\partial}{\partial t}R^i_{jkl} = \frac{1}{2}g^{lm}[\nabla_i\nabla_j h_{mk} - \nabla_j\nabla_i h_{mk} + \nabla_i\nabla_k h_{jm} - \nabla_j\nabla_k h_{im} + \nabla_j\nabla_m h_{ik} - \nabla_i\nabla_m h_{jk}].$$

Now use the standard identities for commuting covariant derivatives on tensors. □

Lemma.

$$\frac{\partial}{\partial t}R_{jk} = \frac{1}{2}g^{ab}[\nabla_a\nabla_k h_{jb} - \nabla_j\nabla_k h_{ab} + \nabla_j\nabla_b h_{ak} - \nabla_a\nabla_b h_{jk}]$$

Proof. Contract on $i = l$ in the equation immediately preceding this lemma. □

Since we will be focusing on surfaces in this paper, these calculations will mostly be used in the form of this last result:

Theorem.

$$\begin{aligned}\frac{\partial}{\partial t}R &= -\Delta(tr_g(h)) + g^{aj}g^{bk}(\nabla_a\nabla_b h_{jk} - R_{ab}h_{jk})\\ &= -\Delta(tr_g(h)) + g^{aj}g^{bk}\nabla_a\nabla_b h_{jk} - \langle Ric, h\rangle_g\end{aligned}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}R &= \frac{\partial}{\partial t}(g^{jk}R_{jk})\\ &= -g^{jm}g^{kl}h_{kl}R_{jk} + g^{jk}(\frac{1}{2}g^{ab}[\nabla_a\nabla_k h_{jb} - \nabla_j\nabla_k h_{ab} + \nabla_j\nabla_b h_{ak} - \nabla_a\nabla_b h_{jk}])\\ &= -g^{jk}\nabla_j\nabla_k(g^{ab}h_{ab}) + g^{jk}g^{ab}\nabla_j\nabla_a h_{kb} - g^{jm}g^{kl}h_{kl}R_{jk}\end{aligned}$$

□

So far we have considered a general evolution of the metric tensor, both for the sake of completeness and the fact that the general variation formulae will be needed in the following section. In our examination of the Ricci flow on surfaces, however, we will very frequently be using the evolution equation for the scalar curvature, which we now derive:

Corollary 2. *Under the Normalized Ricci flow on closed surfaces given by* $\frac{\partial}{\partial t}g = -2Ric + rg = -(R-r)g$*, the scalar curvature* R *evolves under the PDE*

$$\frac{\partial}{\partial t}R = \Delta R + R(R-r)$$

where $r = \frac{\int R d\mu}{\int d\mu} = 2\pi\chi(M)$*, the average scalar curvature, is a constant.*

Proof. The stated formula for r is just the Gauss-Bonnet theorem.

In the notation of our above calculations, for the NRF on surfaces we have $h_{ij} = -2R_{ij} + rg_{ij} = -(R-r)g_{ij}$ since $Ric = \frac{1}{2}Rg$ in two dimensions. Thus $tr_g(h) = -(R-r)tr_g(g) = -2(R-r)$, so $-\Delta(tr_g(h)) = -2\Delta R$. We further compute that

$$\begin{aligned} g^{jk}g^{ab}\nabla_j\nabla_a h_{kb} &= g^{jk}(\nabla_j\nabla_k R) \text{ (metric compatibility)} \\ &= \Delta R \end{aligned}$$

and also that

$$\begin{aligned} -\langle Ric, h\rangle &= \frac{1}{2}R(R-r)\langle g, g\rangle \\ &= R(R-r). \end{aligned}$$

Combining these results yields the evolution equation. □

3. Short-time Existence for Closed Manifolds

As we will soon see, the Ricci Flow is a nonlinear weakly parabolic partial differential equation for the metric. The first thing we need to understand is what it means for an evolution equation on a vector bundle (in this case the bundle S_2T*M of symmetric $(0,2)$-tensors over M) to be parabolic. To be concise we will stick with $(0,2)$-tensors, so suppose we are looking at the vector bundle PDE

$$\begin{cases} \frac{\partial}{\partial t}h_{ij} = P(h_{ij}) \\ h_{ij}(0) = \alpha_{ij} \end{cases} \tag{3.1}$$

where P is a k-th order differential operator mapping $C^\infty(S_2T*M)$ into itself. The *linearization* of P at a section h_{ij} is then the linear bundle map $DP : C^\infty(S_2T*M) \to C^\infty(S_2T*M)$ given by

$$\begin{aligned} DP(h)[b]_{ij} &= \frac{d}{ds}|_{s=0}[P(h+sb)]_{ij} \\ &= P_{ij}^{p,lm}(h)\partial_p b_{lm} \end{aligned}$$

where p is summed over all multi-indices with norm up to k. The *principal symbol* of P at h in the direction ξ (a one-form) is then a linear map $\sigma[DP(h)](\xi) : C^\infty(S_2T*M) \to C^\infty(S_2T*M)$ defined by taking the highest-order terms from this linearization (those with $|p| = k$) and replacing the multi-index derivatives $\partial_p$with one-form coefficients ξ_p; thus

$$\sigma[DP(h)](\xi)(b)_{ij} = \sum_{|p|=k} P_{ij}^{p,lm}(h)\xi_p b_{lm}.$$

If the principal symbol is an isomorphism *for every one-form ξ and some section h*, then we say that the nonlinear operator P is *elliptic* at h. An evolution equation of the form 3.1 is said to be *parabolic* if the right side of the equation is an elliptic operator.

PDE theory guarantees the short-time existence of a solution to 3.1 on a closed (compact) manifold if the equation is parabolic at the initial data. Thus to prove short-time existence for the Ricci Flow (and hence the normalized flow) we would like to show that Ric, a second-order nonlinear differential operator on the metric tensor, is elliptic on any initial metric.

We already calculated the linearization of the Ricci tensor in the previous section:

$$DRic(g)[h]_{jk} = \frac{1}{2}g^{ab}[\nabla_a\nabla_k h_{jb} - \nabla_j\nabla_k h_{ab} + \nabla_j\nabla_b h_{ak} - \nabla_a\nabla_b h_{jk}]$$

(here we have switched the notational usage for h to that of the previous section). This is some linear combination of derivatives of h involving the Christoffel symbols and their derivatives, but the highest-order derivatives of h are simply partial derivatives, so the principal symbol for the Ricci tensor is

$$\sigma[DRic(g)](h)(\xi)_{jk} = \frac{1}{2}g^{ab}[\xi_a\xi_k h_{jb} - \xi_j\xi_k h_{ab} + \xi_j\xi_b h_{ak} - \xi_a\xi_b h_{jk}].$$

Unfortunately, this symbol is most certainly not elliptic, since for any ξ whatsoever we can define $h_{ij} = \xi_i\xi_j$ and see that the symbol evaluates to zero. The fact that the symbol of the Ricci curvature operator has a nontrivial nullspace is intimately related with the fact that the Ricci curvature tensor is a diffeomorphism invariant; see Chapter 3 of [2] for a detailed discussion of this fact and the rest of the short-time existence results in this section.

In some sense, though, this is the only thing that goes wrong with our approach: by modifying the flow on the metric by a time-dependent set of diffeomorphisms, we can form a strictly parabolic equation that allows us to apply short-time existence results. Transforming this solution by another set of diffeomorphisms, we can then get a short-time solution for the Ricci flow (and thus the NRF as well). The remainder of this section will consist of the development of this technique, which is called the “DeTurck trick” after its inventor Dennis DeTurck.

We will first fix a background metric $\tilde{g}$, to remain constant throughout this process. All geometric quantities related to this metric will be marked with a tilde. We then define a vector field W, a function of the current metric g and the background metric $\tilde{g}$, by the equation

$$W^i = g^{jk}(\Gamma^i_{jk} - \tilde{\Gamma^i_{jk}}).$$

We will also use the one-form which is metric-dual to W, as usual denoting its components with lowered indices.

We now define the *Ricci-DeTurck flow*:

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i.$$

Note that W appears in the equation through the term $\nabla_i W_j + \nabla_j W_i$, which is actually just $L_W g_{ij}$, the Lie derivative of the metric with respect to the vector field W, which deals with the evolution of the metric under diffeomorphism flows generated by W. It is this term that will allow us to modify the Ricci-DeTurck flow by a diffeomorphism to get back the Ricci flow.

The above evolution equation is actually strictly parabolic; to see this, we calculate the linearization of the extra term on the right-hand side, which we will write as $A(g)_{ij} = \nabla_i W_j + \nabla_j W_i = \nabla_i(g_{jm}g^{ab}(\Gamma^m_{ab} - \tilde{\Gamma^m_{ab}})) + \nabla_j(g_{im}g^{ab}(\Gamma^m_{ab} - \tilde{\Gamma^m_{ab}}))$. Using geodesic coordinates we again see that we can regard the covariant derivatives as partial derivatives when linearizing, so using our results from last chapter we get that

$$\begin{aligned} DA(g)(h)_{ij} &= \frac{1}{2}g_{jm}g^{ab}\nabla_i[g^{ml}(\nabla_a h_{lb}+\nabla_b h_{al}-\nabla_l h_{ab})]+\frac{1}{2}g_{im}g^{ab}\nabla_j[g^{ml}(\nabla_a h_{lb}+\nabla_b h_{al}-\nabla_l h_{ab})] \\ &\quad +(\text{lower-order derivatives of } h) \\ &= g^{ab}(\nabla_i\nabla_a h_{jb}+\nabla_j\nabla_a h_{ib}-\nabla_i\nabla_j h_{ab})+\text{l.o.d.} \end{aligned}$$

Comparing this to the linearized Ricci tensor above, we immediately see that

$$D[-2Ric+A](g)(h)_{ij} = g^{ab}\nabla_a\nabla_b h_{ij}+\text{l.o.d} = \Delta h_{ij}+\text{l.o.d.}$$

so $-2Ric+A$ is an elliptic operator! The parabolic existence results from PDE therefore apply and yield a short-time solution to the Ricci-DeTurck flow starting from any initial metric.

We now want to modify this solution to obtain a Ricci flow solution. As suggested above, we can try to do this by a "backwards" diffeomorphism under the vector field W: we construct a family of maps $\psi_t : M \to M$ by solving the ODE

$$\begin{aligned} \frac{\partial}{\partial t}\psi_t(x) &= -W(\psi_t(x),t) \\ &= id \end{aligned}$$

The vector field W exists as long as our solution $g(t)$ to the Ricci-DeTurck flow does. A basic compactness argument (see [2] section 3.1) demonstrates that the ψ_t also exist and are diffeomorphisms for as long as $g(t)$ exists. We now have everything we need to construct our short-time solution:

Theorem. *The time-dependent metric $\bar{g}(t) = \psi_t^*(g(t))$ is a solution to the Ricci flow.*

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}|_{t=T}\bar{g}(t) &= \frac{\partial}{\partial t}|_{t=0}\psi_t^*(\psi_T^*(g(t+T))) \\ &= -L_{W(T)}(\psi_T^* g(T))+\psi_T^*(-2Ric(g(T))+L_{W(T)}(g(T)) \\ &= -2Ric(\psi_T^* g(T)) \end{aligned}$$

In the last line we have used the diffeomorphism invariance of the Ricci tensor along with the fact that ψ_t^* commutes with $L_{W(t)}$ since $-W$ generates the diffeomorphisms ψ. □

Remarks on Uniqueness of the Ricci Flow:

While the Ricci-DeTurck flow is strictly parabolic and hence satisfies the usual uniqueness properties, we cannot use that fact alone to demonstrate that solutions to the Ricci flow are unique. The reason for this is simple: if we start with two solutions to the RF with identical initial conditions, we can modify them by diffeomorphisms to get two solutions of the Ricci-DeTurck flow with identical initial conditions, so the modified solutions are the same; however, the diffeomorphisms used depend on the solutions themselves, hence may be different, so we can't conclude anything about the original solutions.

The way around this is to find an alternate characterization of the diffeomorphisms ψ_t in terms of a strictly parabolic evolution equation called the *harmonic map flow.* In the interest of brevity I omit this procedure, but you can find a thorough discussion of the issue in [2] Section 3.4.

4. Long-time Existence Under Bounded Curvature

Given an initial metric g_0 on a closed manifold, we know from our short-time existence and uniqueness results that there is a *maximal* solution to the Ricci flow starting from this metric, i.e. there is a solution metric $g(t)$ defined on some time interval $[0, T)$ such that there is no solution on a larger time interval that restricts to g on $[0, T)$. Our goal in this section will be to outline the proof of the the following result:

Theorem 3. *Let $g(t)$ be a maximal solution to the Ricci flow (normalized or unnormalized) on a closed manifold, defined on the time interval $[0, T)$. If there is some $K < \infty$ such that for all $t \in [0, T)$ we have the uniform curvature bound*

$$|Rm(t)|_{g(t)} \leq K$$

then $T = \infty$, i.e. the solution exists for all time.

Remark: In three dimensions the required curvature bound is equivalent to bounded Ricci curvature, and on surfaces it is equivalent to bounded scalar curvature.

The first step in proving the theorem is to demonstrate that uniform bounds on the Riemann curvature tensor lead to uniform bounds on all higher derivatives of the curvature. This is the content of the following result:

Proposition 4. *Suppose we have a solution of the Ricci flow $g(t)$ on a closed manifold defined on $[0, T)$, along with a uniform curvature bound*

$$|Rm(t)|_{g(t)} \leq K$$

for all $t \in [0, \frac{\alpha}{K}]$. Then there are bounds for all $t \in (0, \frac{\alpha}{K}]$ of the form

$$|\nabla^m Rm(t)|_{g(t)} \leq \frac{C_m K}{t^{m/2}}$$

where the constants C_m depend only on m, n, and $max\{\alpha, 1\}$.

Proof. We will prove the proposition by induction on m. Only the case $m = 1$ will be treated in full since the higher cases follow an entirely similar strategy that is notationally far more tedious.

We begin by recalling several commutator formulas: for any tensor A, possibly time-dependent, under the Ricci Flow we have

$$\begin{aligned}
[\nabla, \Delta]A &= Rm * \nabla A + \nabla Rc * A = Rm * \nabla A + \nabla Rm * A \\
[\frac{\partial}{\partial t}, \nabla]A &= \nabla Rc * A = \nabla Rm * A \\
[\frac{\partial}{\partial t}, \Delta]A &= Rc * \nabla^2 A + \nabla Rc * \nabla A + \nabla^2 Rc * A \\
&= Rm * \nabla^2 A + \nabla Rm * \nabla A + \nabla^2 Rm * A
\end{aligned}$$

We also recall the general evolution formula for the curvature tensor under the unnormalized flow:

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm$$

Using these formulae, it is simple to derive the following evolution formula for the square of the derivative of the curvature tensor:

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla Rm|^2 &= \frac{\partial}{\partial t}\langle \nabla Rm, \nabla Rm\rangle_g \\
&= 2\langle \nabla \frac{\partial}{\partial t} Rm, \nabla Rm\rangle + \nabla Rm * \nabla Rm * Rm \\
&= 2\langle \nabla(\Delta Rm + Rm * Rm), \nabla Rm\rangle + \nabla Rm * \nabla Rm * Rm \\
&= 2\langle \Delta \nabla Rm, \nabla Rm\rangle + \nabla Rm * \nabla Rm * Rm \\
&= \Delta|\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + Rm * (\nabla Rm)^2
\end{aligned}$$

Now define the function $F = t|\nabla Rm|^2 + \beta|Rm|^2$ with β a constant to be chosen soon. F satisfies the evolution equation

$$\frac{\partial}{\partial t}F = \Delta F + (1 + c_1 t|Rm| - 2\beta)|\nabla Rm|^2 + c_2\beta|Rm|^2$$

where the c depend only on n. By assumption we have $|Rm| \leq K$ for $t \in [0, \frac{\alpha}{K}]$, so in this time interval we choose $\beta \geq (1 + c_1\alpha)/2$ and see that

$$\frac{\partial}{\partial t}F \leq \Delta F + c_2\beta K^3$$

Applying the maximum principle then tells us that in this time interval

$$F \leq \beta K^2 + c_2\beta K^3 t \leq C_1^2 K^2$$

where C_1 depends only on n and $max\{\alpha, 1\}$. Looking at the definition of F, this implies that

$$|\nabla Rm| \leq \frac{C_1 K}{t^{1/2}} \text{ for all } t \in (0, \tfrac{\alpha}{K}]$$

The higher-derivative cases are completely analogous: the first step is to prove evolution equations of the form

$$\frac{\partial}{\partial t}|\nabla^k Rm|^2 = \Delta|\nabla^k Rm|^2 - 2|\nabla^{k+1}Rm|^2 + \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j}Rm * \nabla^k Rm.$$

After defining the function

$$G = t^k|\nabla^k Rm|^2 + \beta_k \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!} t^{k-j}|\nabla^{k-j}Rm|^2,$$

a tedious calculation reveals that by choosing β_k appropriately and applying the inductive hypothesis, we can derive the inequality

$$\frac{\partial}{\partial t}G \leq \Delta G + \bar{C}_k K^3 \ (\bar{C}_k \text{depending only on } k, n, max\{\alpha, 1\})$$

for all times $t \in (0, \frac{\alpha}{K}]$. As before, applying the maximum principle yields the desired bound

$$|\nabla^k Rm| \leq \sqrt{\frac{G}{t^k}} \leq \frac{C_k K}{t^{k/2}}$$

□

There is a simpler yet still fundamental result of bounded curvature on some finite time interval: the metrics at all times are uniformly equivalent. More specifically, if $|Rc| \leq K$ for $t \in [0, T)$ with T finite, then for all such t we have the bound

$$e^{-KT} g(0) \leq g(t) \leq e^{KT} g(0).$$

This fact follows immediately from a quick lemma:

Lemma 5. *Let M^n be closed, with $g(t)$ a smooth 1-parameter family of metrics on M. If there is a constant $C < \infty$ with*

$$\int_0^T |\frac{\partial}{\partial t} g(x,t)| dt < C \text{ for all } x \in M,$$

then

$$e^{-C} g(0) \leq g(t) \leq e^{C} g(0)$$

and the metrics $g(t)$ converge uniformly to a limit metric g.

Proof. Let $v \in T_x M$, and let $\tau \in [0, T)$. Then

$$\begin{aligned}
|log(\frac{|v|^2_{g(\tau)}}{|v|^2_{g(0)}})| &= |\int_0^\tau \frac{\frac{\partial}{\partial t}|v|^2_{g(t)}}{|v|^2_{g(t)}} dt| \\
&\leq \int_0^\tau |\frac{\partial}{\partial t}| \frac{v}{|v|_{g(t)}}|^2_{g(t)}| dt \\
&\leq \int_0^\tau |g(t)|_{g(t)} dt \\
&\leq C
\end{aligned}$$

Exponentiate this bound to yield the uniform bounds on g. The compactness of M along with the time derivative bound on g yields uniform convergence to a continuous symmetric 2-tensor which must be positive-definite by the uniform equivalence above. □

The uniform equivalence of all metrics $\{g(t)\}_{t \in [0,T)}$ is essential because it demonstrates that we can bound the size of any quantity in a fixed coordinate system by bounding its metric norm; as we know from our previous calculations, metric norms of geometric quantities satisfy nice evolution equations.

We have therefore demonstrated that the components of all metric derivatives of the curvature remain bounded in a given coordinate system. In fact, we are in an even better situation:

Proposition 6. *If $g(t)$ is a solution to the Ricci flow on a closed manifold M for the finite time interval $0 \leq t < T < \infty$ with $|Rm| < C < \infty$, and $\{x^i\}$ is a fixed coordinate system, then for each multi-index α there exist constants C_α, D_α such that for all $t \in [0, T)$ and indices i, j we have the bounds*

$$\begin{aligned}
|\frac{\partial}{\partial x^\alpha} g_{ij}| &< C_\alpha \\
|\frac{\partial}{\partial x^\alpha} R_{ij}| &< D_\alpha
\end{aligned}$$

Proof. Omitted for brevity, but can be found in [2] Section 6.7. The main idea is that our bounds on $|\nabla^k Rm|$ can be used to bound the derivatives of g via the evolution equation and uniform equivalence. □

We now have all of the ingredients required to prove long-time existence for the unnormalized flow. Proceed by contradiction: suppose we have a curvature bound for $t \in [0,T)$ but the solution cannot be extended further. Fix a coordinate system around an arbitrary point. By our previous Lemma we know that a continuous limit metric $g(T)$ exists and is given in coordinates by

$$g_{ij}(T) = g_{ij}(\tau) - 2\int_\tau^T R_{ij}(t)dt.$$

By the preceding proposition, we can differentiate under the integral sign to see that $g(T)$ is smooth. Furthermore, for any multi-index α there is a constant C so that

$$|\partial_\alpha g_{ij}(T) - \partial_\alpha g_{ij}(\tau)| \leq 2\int_\tau^T |\partial_\alpha R_{ij}|dt \leq C(T-\tau)$$

so $g(t) \to g(T)$ uniformly in any C^k norm.

By using the evolution equation to eliminate time derivatives in favor of curvature and its spatial derivatives, it immediately follows that all spacetime derivatives of the components g_{ij} converge uniformly in all C^k norms. Thus our short-time existence results allow us to smoothly extend $g(t)$ to a solution of the flow on a time interval $[0, T+\epsilon)$, contradicting maximality.

We have demonstrated that for the unnormalized flow there is a maximal time $T > 0$ to which a solution can be extended and that $\limsup_{t\to T} |Rm| = \infty$. If we want to extend our results to the normalized flow, the first step is to demonstrate that in fact we have $\lim_{t\to T} |Rm| = \infty$ for the RF:

Theorem 7. *(Doubling-Time Estimate, Corollary 7.5 of* [2]*) There is a constant $c > 0$ depending only on n such that if $g(t)$ is a maximal solution to the RF defined on $[0,T)$ and $M(t) \doteqdot \sup_{x\in M} |Rm(x,t)|$ then*

$$M(t) \leq 2M(0) \textit{ whenever } 0 \leq t \leq \min\{T, \tfrac{c}{M(0)}\}.$$

Proof. Using our earlier calculations of the evolution of $\frac{\partial}{\partial t} R^l_{ijk}$ under the flow, it is not too difficult to calculate that

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 \leq \Delta|Rm|^2 + C|Rm|^3$$

(see Lemma 7.4 of [2]) where C depends only on n. This inequality implies that $M(t)$ is Lipschitz with

$$\frac{dM}{dt} \leq \frac{C}{2}M^2$$

where the time derivative is taken in the sense of a lim supof forward difference quotients. Thus for $t \in [0, \frac{2}{CM(0)})$ we have

$$M(t) \leq \frac{1}{\frac{1}{M(0)} - \frac{C}{2}t}.$$

If we now define $c = \frac{1}{C}$, then for all $t \in [0, \frac{c}{M(0)})$ we get the doubling-time estimate $M(t) \leq 2M(0)$. □

This theorem immediately implies that $\lim_{t\to T} |Rm| = \infty$ for the RF since otherwise we would have some $K < \infty$ and $t_0 < T$ satisfying $|Rm(\cdot, t_0)| \leq K$ and $t_0 \geq T - \frac{c}{K}$, from whence we see that $|Rm| \leq 2K < \infty$ for $t \in (t_0, T)$, bounding the curvature and allowing us to extend past $t = T$.

In fact, an even stronger fact about the scalar curvature blow-up for the RF is true:

Theorem 8. *(Theorem 15.3 of* [1]*) For a maximal solution of the unnormalized flow that exists on a finite time interval* $[0, T)$*, we have*

$$\int_0^T R_{max} dt = \infty.$$

Proof. Let $f(t)$ satisfy the ODE

$$\begin{cases} \frac{d}{dt} f = 2R_{max} f \\ f(0) = R_{max}(0) \end{cases}$$

Then you can check that

$$\frac{\partial}{\partial t}(R - f) \leq \Delta(R - f) + 2R_{max}(R - f)$$

so the maximum principle tells us that $R \leq f$ on $[0, T)$. Since $R \to \infty$ as $t \to T$, $f \to \infty$ also. But from the evolution equation for f we see that

$$\log \frac{f(t)}{f(0)} = 2 \int_0^t R_{max}(\tau) d\tau.$$

□

Theorem 9. *(Long-time existence for the normalized flow) Any maximal solution to the NRF on a compact manifold with bounded curvature must exist for all time.*

Proof. Recalling our analysis of the scaling between RF and NRF solutions at the start of the paper, let $g(t)$ be a solution of the RF, with $\psi(\tilde{t})g(\tilde{t})$ the corresponding NRF solution. Then $\frac{d\tilde{t}}{dt} = \psi$ and $\psi \tilde{R}_{max} = R_{max}$, so

$$\int_0^{\tilde{T}} \tilde{R}_{max}(\tilde{t}) d\tilde{t} = \int_0^T R_{max}(t) dt = \infty$$

and the boundedness of the curvature implies that $\tilde{T} = \infty$. □

5. The Ricci Flow on Surfaces: Introductory Remarks

We will now shift our discussion from the general Ricci Flow to the two-dimensional case. Considerable simplifications happen when we make this shift; in particular, all of the curvature quantities are now given using solely the scalar curvature R and the metric. Specifically, we will be considering the normalized Ricci flow, which under the simplification $Ric = \frac{1}{2} Rg$ becomes the PDE

$$\begin{cases} \frac{\partial}{\partial t} g & = -(R - r)g \\ g(0) & = g(0) \end{cases}$$

where the average scalar curvature $r = 2\pi\chi(M)$ is constant for surfaces. We showed earlier that the scalar curvature then satisfies the PDE

$$\frac{\partial}{\partial t} R = \Delta R + R(R - r). \tag{5.1}$$

It turns out that in addition to being much easier to analyze in dimension 2, the Ricci flow also exhibits much better behavior than in higher dimensions. Given the long-time existence results of the previous section, our analysis of existence and convergence of the Ricci flow on surfaces completely reduces to the study of the scalar curvature under 5.1. Given the dependence of the evolution equation on Euler characteristic through r, we must then break down the analysis and prove that R remains bounded in three separate cases: $\chi < 0$, $\chi = 0$, and $\chi > 0$. The latter case, in which we are dealing with a metric on S^2 (we can assume orientability by taking a double cover if necessary), is by far the most difficult and will require special isoperimetric techniques used by Ben Andrews and Paul Bryan in[4].

The methods that we will use to bound the curvature in all three cases will also allow us to demonstrate the decay of all derivatives of the curvature with respect to the evolving metric. As shown in Hamilton's paper [3], this is sufficient information to prove that the metric tensor converges in C^∞ to a constant-curvature metric as $t \to \infty$.

Aside from the $\chi > 0$ isoperimetric results from [4], the following arguments and results on surfaces can be found in [2], though as noted above some convergence results will be drawn from [3].

To begin, we need a result from basic PDE theory.

6. A weak scalar maximum principle

The following result will be our main tool in the proof of long-time existence for all $\chi(M)$, and also convergence in the case $\chi \leq 0$:

Theorem 10. *(Weak Scalar Maximum Principle, ODE bound version)*

Suppose M is a closed Riemannian manifold and that $f : M \times [0, \epsilon) \to \mathbb{R}$ is a solution of the parabolic inequality

$$\frac{\partial}{\partial t} f \leq \Delta f + H(f, t)$$

where $H : \mathbb{R} \times [0, \epsilon) \to \mathbb{R}$ is smooth. Let $h : [0, \epsilon) \to \mathbb{R}$ be a solution of the ODE

$$\frac{d}{dt} h = H(h, t)$$

(assuming existence on this time interval) that satisfies the initial inequality $f \leq h$. Then $f(\cdot, t) \leq h(t)$ for all time.

Proof. Fix any time $\tau \in (0, T)$. Since H is smooth and f, h are differentiable, we have bounds on this time interval of the form $|f|, |h| < C$, implying that $|H(f(t), t) - H(h(t), t)| \leq L|f(t) - h(t)|$ for some constant L and all $t \in [0, \tau]$. We therefore have the differential inequality

$$\frac{\partial}{\partial t}(h - f) \geq \Delta(h - f) - L|h(t) - f(t)|.$$

If we now define $J(x, t) = e^{Lt}(h(t) - f(x, t))$, we see that on $[0, \tau]$ it satisfies

$$\begin{aligned} \frac{\partial}{\partial t} J &\geq \Delta J + LJ - LJ \cdot sgn(h - f) \\ &\geq \Delta J \end{aligned}$$

The standard weak minimum principle for supersolutions to the heat equation then tells us that $J \geq 0$ for all times in $[0, \tau]$, i.e. $h \geq f$. Since τ was arbitrary, we are done. □

7. Long-Time Existence on Surfaces

In order to demonstrate long-time existence for solutions to the NRF on surfaces, we know from previous chapters that it is enough to ensure that the scalar curvature does not go to infinity in finite time. Using the weak maximum principle, we can very easily derive lower bounds for the curvature in all cases using the evolution equation

$$\frac{\partial}{\partial t}R = \Delta R + R(R-r)$$

From the previous section we know that if we define f to be a solution of the ODE

$$\begin{cases} \frac{d}{dt}f = f(f-r) \\ f(0) = R_{min}(0) = R_{min} \end{cases}$$

then $g(t) \leq R(\cdot, t)$ for as long as f and R exist. But the above ODE has the solutions

$$\begin{cases} f(t) = \frac{r}{1-(1-\frac{r}{R_{min}})e^{rt}} & r \neq 0, R_{min} \neq 0 \\ f(t) = \frac{R_{min}}{1-R_{min}t} & r = 0 \\ f(t) = 0 & R_{min} = 0 \end{cases}$$

If we assume that $R_{min} < min\{r, 0\}$ then these solutions remain bounded as $t \to \infty$, and we prove the following:

Proposition 11. *If $g(t)$ is a solution to the NRF on a compact surface, then as long as the solution exists we have lower bounds on the scalar curvature:*

$$\begin{cases} \text{If } r < 0, \text{ then} & R - r \geq \frac{r}{1-(1-\frac{r}{R_{min}})e^{rt}} - r \geq (R_{min} - r)e^{rt} \\ \text{If } r = 0, \text{ then} & R \geq \frac{R_{min}}{1-R_{min}t} \geq -\frac{1}{t} \\ \text{If } r > 0 \text{and } R_{min} < 0, \text{ then} & R \geq \frac{r}{1-(1-\frac{r}{R_{min}})e^{rt}} \geq R_{min}e^{-rt} \end{cases}$$

Note that these lower bounds are all uniform, so we have proved half of the curvature bound required for long-time existence.

The maximum principle can also be used to prove upper bounds on curvature (non-uniform in the $\chi > 0$ case), but some additional work is required. To begin, we say that a function $f : M \to \mathbb{R}$ is a *potential of the curvature* if f satisfies the PDE

$$\Delta f = R - r$$

Basic PDE theory shows that on the compact manifolds we are considering, such a solution f always exists since $\int_M (R-r)d\mu = 0$, and the solution is unique up to the addition of a function of time $c(t)$. It is easy to show (see [2] section 5.3) that we can choose $c(t)$ so that f satisfies the evolution equation

$$\frac{\partial}{\partial t}f = \Delta f + rf$$

and we will assume that this holds for all of our future potential functions.

We now define the quantity

$$H \doteqdot R - r + |\nabla f|^2.$$

This function, as well as the potential f, draw their definitions from Ricci solitons and behave particularly nicely on those special solutions; see [2] section 5.3 for a discussion.

H is a very useful quantity because of the nice evolution equation it satisfies:

Claim. H satisfies the equation

$$\frac{\partial}{\partial t}H = \Delta H - 2|\nabla\nabla f - \frac{1}{2}\Delta f \cdot g|^2 + rH$$

Proof. Compute using the evolution equation for R, the definition of f, and the commutator relations for ∇, Δ. □

From the weak maximum principle we now see that

$$R - r \leq H \leq H(0)e^{rt}.$$

This upper bound exists for all time, so we have finally demonstrated the boundedness of the curvature for solutions of the NRF on surfaces, and hence the long-time existence for such solutions. We summarize these bounds now:

Theorem 12. *We have the following curvature bounds for the solutions of the NRF on closed surfaces:*

$$\begin{cases} \textit{for } r < 0 & r - Ce^{rt} \leq R \leq r + Ce^{rt} \\ \textit{for } r = 0 & -\frac{C}{1+Ct} \leq R \leq C \\ \textit{for } r > 0 & -Ce^{-rt} \leq R \leq r + Ce^{rt} \end{cases}$$

where $C > 0$ is a constant depending only on the initial metric on the surface. The solution to the NRF on a closed surface exists for all time.

8. Convergence for $\chi(M) \leq 0$

Our goal is to prove the following result:

Theorem 13. *If $(M, g(t))$ is a solution to the normalized Ricci flow on a closed surface M with $\chi(M) \leq 0$, then g converges uniformly in any C^k norm to a smooth constant-curvature metric as $t \to \infty$.*

As in our analysis for long-time existence, it will suffice to prove that all metric derivatives $|\nabla^k R|^2$ are decaying uniformly. The $\chi < 0$ and $\chi = 0$ cases both rely on maximum principle arguments to prove this decay, though the curvature quantities to which the principle is applied vary in the two cases, yielding different decay rates. The trick in the proofs is simply defining the correct curvature quantities, so I will omit the large amounts of calculation involved and refer the reader to [2] Sections 5.5 and 5.6 where they are done in full.

Proposition 14. *If $\chi < 0$, then for every integer $m \geq 1$, there is a constant C_m such that for all $(x, t) \in M \times [0, \infty)$ we have the bound*

$$|\nabla^m R(x,t)|^2 \leq C_m e^{rt/2}$$

Proof. The $m = 1$ case follows from the exponential curvature convergence $|R - r| \leq Ce^{rt}$ and the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla R|^2 &= \Delta|\nabla R|^2 - 2|\nabla\nabla R|^2 + (4R - 3r)|\nabla R|^2 \\ &\leq \Delta|\nabla R|^2 + (4Ce + r)|\nabla R|^2 \\ &\leq \Delta|\nabla R|^2 + \frac{r}{2}|\nabla R|^2 \text{ for large } t. \end{aligned}$$

The $m > 1$ case uses a similar evolution equation satisfied by the quantity $\Phi = |\nabla^m R|^2 - (m+1)r|\nabla^{m-1} R|^2$; applying the inductive hypothesis eventually yields an inequality

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + \frac{kr}{2}\Phi + C' e^{rt}$$

to which the maximum principle can be applied. This finishes the $\chi < 0$ case. □

The $\chi = 0$ case requires a bit more work. Recall that we previously defined a function $f(x,t)$ called the "potential" of the curvature: it satisfied

$$\begin{aligned} \Delta f &= R - r = R \\ \frac{\partial}{\partial t} f &= \Delta f + rf = \Delta f \end{aligned}$$

We begin by showing that ∇f decays uniformly:

Lemma 15. *If $\chi = 0$ then there is a constant $C < \infty$ depending only $g(0)$ such that for all $(x,t) \in M \times [0,\infty)$ we have*

$$|\nabla f(x,t)|^2 \leq \frac{C}{1+t}$$

Proof. Use the evolution equation

$$\frac{\partial}{\partial t}|\nabla f|^2 = \Delta|\nabla f|^2 - 2|\nabla\nabla f|^2$$

and apply the maximum principle to bound $|\nabla f|^2$. A calculation then yields the differential inequality

$$\frac{\partial}{\partial t}(t|\nabla f|^2 + f^2) \leq \Delta(t|\nabla f|^2 + f^2)$$

which in turn implies that $|\nabla f|^2 \leq \frac{C_1}{t}$. □

Recall that we previously derived a curvature bound for the $r = 0$ case of the form

$$-\frac{D}{1+Dt} \leq R \leq D.$$

The next step in our convergence proof is extending the curvature decay to the upper bound.

Proposition 16. *Given the above assumptions, $\exists C < \infty$ such that for all $(x,t) \in M \times [0,\infty)$ we have the bound*

$$|R| + |\nabla f|^2 \leq \frac{C}{1+t}.$$

Proof. Demonstrate that the maximum principle applies to the quantity $t(R + 2|\nabla f|^2)$ using the previous lemma; we already know the lower bound for R. □

Finally, we extend the decay to all derivatives of the curvature, where it becomes more rapid:

Proposition 17. *Given our assumptions, for each $m \geq 1$ there is a constant $C < \infty$ such that for all $(x,t) \in M \times [0,\infty)$ we have the bound*

$$|\nabla^k R|^2 \leq \frac{C}{(1+t)^{k+2}}$$

Proof. Derive a heat-type evolution equation for the curvature quantity $\Phi = t^{k+3}|\nabla^k R|^2 + Nt^{k+2}|\nabla^{k-1}R|^2$ (N a constant to be chosen); use induction and the preceding proposition to show that we get a differential inequality of the form

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + D.$$

Apply the maximum principle and the inductive hypothesis again. □

This finishes our proof of convergence of the normalized Ricci flow on surfaces for the two cases $\chi(M) < 0$ (where convergence is exponential) and $\chi(M) = 0$ (where convergence goes more slowly, like inverse polynomials). It is time to begin the far more difficult analysis for the $\chi > 0$ case, where the only absolute bound we have on R is an increasing exponential. Perhaps surprisingly, our end result will be that the metric still converges, this time exponentially fast as with $\chi < 0$.

9. Convergence for $\chi > 0$: an introduction

This section is drawn almost entirely from [4], a wonderful short paper by Ben Andrews and Paul Bryan. We will be considering the normalized Ricci flow on a sphere metric $(M, g) = (S^2, g)$, with area normalized to 4π. By the Gauss-Bonnet theorem, this implies that the average Gauss curvature $K = \frac{1}{2}R$ will be 1; we want to show that flow converges nicely, so we expect that $K \to 1$.

The main idea of the proof is to consider the *isoperimetric profile* of a solution; this is a function that associates an area to the smallest perimeter of any domain in the solution that has that area (e.g. the isoperimetric profile of the plane would send $\xi \to 2\sqrt{\pi\xi}$, the relation between the area and perimeter of a circle). The authors demonstrate that for axially-symmetric, positively-curved solutions, the isoperimetric profiles have a comparison property: any other solution with an initial profile bounded below by the "special" profile will remain bounded below by that profile for all time. By noting that the asymptotic behavior of the profiles involves the curvature, they show that the maximum curvature on the other solution is bounded above by the maximum curvature on the "special" profile. This technique is then applied to the Rosenau solution, a special positive-curvature axially-symmetric metric that approaches a thin cylinder as $t \to -\infty$, allowing its isoperimetric profile to bound any other fixed profile from below by selecting an appropriate starting time.

10. The Isoperimetric Profile

The *isoperimetric profile* $h : (0,1) \to (0,\infty)$ for a compact Riemannian surface M is defined as

$$h(\xi) = inf\{|\partial\Omega| : \Omega \subset M, \frac{|\Omega|}{|M|} = \xi\};$$

recall that in our particular case $|M| = 4\pi$ for all time, so ξ is $\frac{area(\Omega)}{4\pi}$. Note that $h(\xi) = h(1-\xi)$ since a set shares a boundary with its complement.

It is natural to ask questions concerning the nature of sets Ω minimizing perimeter with respect to a given area (I'll call these *isoperimetric domains*): most important for our use are the facts that (a) a perimeter-minimizing set Ω exists for any area ξ and (b) for such a minimizer $\partial\Omega$ always consists of smooth curves. The former fact is an easy consequence of some compactness results, but the latter is much more tricky. One method to approach the proof of this fact would be to utilize the full machinery of Geometric Measure Theory, which allows you to demonstrate that isoperimetric domains have smooth boundaries in compact manifolds of dimension $n \leq 7$. Unfortunately, proving this fact requires some very powerful and difficult techniques (see [5] for a discussion). The surface case, however, is simple enough to be tackled using a more elementary analysis, which can be found in its entirety in [6]. This smoothness will be essential in our analysis since we will be relying on variations of the boundary of our isoperimetric domains.

One observation about the connectivity of isoperimetric domains which will be necessary later is the following:

Theorem 18. *Suppose (M, g) is a compact Riemannian manifold with isoperimetric profile h. Suppose further that for some strictly concave function ϕ on $(0, 1)$, also satisfying $\phi(\xi) = \phi(1 - \xi)$, we have $h \geq \phi$ and $h(\xi_0) = \phi(\xi_0)$ for some $\xi_0 \in (0, 1)$. Then it follows that any isoperimetric domain Ω with fractional area ξ_0 is connected and has connected complement; in particular this implies that if $M = S^2$ then Ω is simply connected.*

Proof. Suppose Ω disconnects as $\Omega_1 \cup \Omega_2$, so $|\Omega| = |\Omega_1| + |\Omega_2|$ and $|\partial\Omega| = |\partial\Omega_1| + |\partial\Omega_2|$ with all values non-zero. Then

$$\begin{aligned}
\phi(\frac{|\Omega_1|}{|M|}) + \phi(\frac{|\Omega_2|}{|M|}) &\leq h(\frac{|\Omega_1|}{|M|}) + h(\frac{|\Omega_2|}{|M|}) \\
&\leq |\partial\Omega_1| + |\partial\Omega_2| \\
&= |\partial\Omega| \\
&= h(\frac{|\Omega|}{|M|}) \text{ } (\Omega \text{optimal}) \\
&= \phi(\frac{|\Omega|}{|M|}) \\
&= \phi(\frac{|\Omega_1| + |\Omega_2|}{|M|}) \\
&< \phi(\frac{|\Omega_1|}{|M|}) + \phi(\frac{|\Omega_2|}{|M|}) \text{ by concavity.}
\end{aligned}$$

This contradiction implies that Ω is in fact connected. Using the symmetry of h and ϕ under reflection through $\xi = \frac{1}{2}$ then yields the connectedness of the complement of Ω as well. $\square$

Our reason for examining the isoperimetric profile in the context of convergence of the Ricci flow is due to the connection between the asymptotics of the profile and the maximum curvature of (M, g):

Theorem 19. *Let M be a compact Riemannian surface. If h is the isoperimetric profile of M, then*

$$h(\xi) = \sqrt{4\pi|M|\xi} - \frac{|M|^{3/2} \sup_M K}{4\sqrt{\pi}} \xi^{3/2} + O(\xi^2) \text{ as } \xi \to 0.$$

Proof. We know that $h(\xi) \leq |\partial B_r(p)|$ for geodesic balls $B_r(p)$ of area $\xi|M|$. By [7] Lemma 5.5 we can expand the metric in normal coordinates around any point p as

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iwvj} x^w x^v - \frac{1}{6} R_{iwvj,k} x^w x^v x^k + O(r^4)$$

where the curvature is evaluated at p; this expansion is derived by analyzing Jacobi fields on geodesic rays from p. Taking the determinant of this expression yields the asymptotics

$$\begin{aligned} det(g_{ij}) &= 1 - \frac{1}{3}R_{ij}x^i x^j + \text{(degree-3 terms)} + O(r^4) \\ \sqrt{det(g_{ij})} &= 1 - \frac{1}{6}R_{ij}x^i x^j + \text{(degree-3 terms)} + O(r^4) \\ &= 1 - \frac{1}{6}Kr^2 + ... \end{aligned}$$

Integrating first over the geodesic sphere $|x| = r'$ and then from $r' = 0$ to $r' = r$ yields the asymptotic formulae

$$\begin{aligned} |\partial B_r(p)| &= 2\pi r(1 - \frac{K(p)}{6}r^2 + O(r^4)) \\ |B_r(p)| &= \pi r^2(1 - \frac{K(p)}{12}r^2 + O(r^4)). \end{aligned}$$

Invert the second power series to find the radius of a small geodesic ball as a function of volume:

$$r(V) = \sqrt{\frac{V}{\pi}} + \frac{K(p)}{24}(\frac{V}{\pi})^{3/2} + O(V^{5/2})$$

Plugging into the asymptotics for the boundary length yields the boundary of a small geodesic ball as a function of volume:

$$|\partial B_r(p)|(V) = \sqrt{4\pi}V^{1/2} - \frac{K(p)}{4\sqrt{\pi}}V^{3/2} + O(V^{5/2}),$$

whence setting $V = \xi|M|$ and taking the infimum over all points p yields the desired upper bound

$$h(\xi) \leq \sqrt{4\pi|M|\xi} - \frac{|M|^{3/2}\sup_M K}{4\sqrt{\pi}}\xi^{3/2} + O(\xi^2).$$

We now prove the lower bound for h. For small enough ξ, an isoperimetric domain $|\Omega|$ of area $\xi|M|$ must lie inside a small geodesic ball around a point p, where g varies from a metric of constant curvature $K(p)$ by a multiplicative factor $(1+O(|\partial\Omega|^3))$. Letting bars denote quantities in the constant-curvature metric, we have

$$\begin{aligned} |\partial\bar{\Omega}| &= |\partial\Omega|(1 + O(|\partial\Omega|^3)) \\ |\bar{\Omega}| &= |\Omega|(1 + O(|\partial\Omega|^3)) \end{aligned}$$

Applying the constant-curvature isoperimetric inequality $L^2 \geq 4\pi A - KA^2$ to the barred quantities, we see that

$$\begin{aligned} |\partial\Omega|^2 &\geq 4\pi|\Omega| - K(p)|\Omega|^2 + O(|\partial\Omega|^5) + |\Omega|O(|\partial\Omega|^3) \\ &= 4\pi|\Omega| - K(p)|\Omega|^2 + O(|\Omega|^{5/2}) \text{ (using the upper bound for } h) \\ &= 4\pi|M|\xi(1 - \frac{K(p)}{4\pi}|M|\xi + O(\xi^{3/2})) \end{aligned}$$

Taking the square root then gives

$$\begin{aligned} h(\xi) &= |\partial\Omega| \geq \sqrt{4\pi|M|}\xi^{1/2}(1-\frac{K(p)|M|}{8\pi}\xi+O(\xi^{3/2})). \\ &\geq \sqrt{4\pi|M|}\xi^{1/2}(1-\frac{\sup_M K\cdot|M|}{8\pi}\xi+O(\xi^{3/2})) \end{aligned}$$

□

This theorem tells us that we can extract information about the maximum curvature on a surface if we know the asymptotics of h; we will put this fact to good use soon.

11. An isoperimetric comparison theorem

We now wish to demonstrate that we can bound one isoperimetric function by another and then maintain that bound as the two metrics evolve under the normalized Ricci flow. We will prove this by analyzing solutions to a differential equality which will be satisfied by the isoperimetric function for our "special" comparison metric:

Theorem 20. *Let $\phi : (0,1)\times[0,\infty)\to\mathbb{R}$ be smooth with $\phi(\cdot,t)$ strictly concave for all t, $\phi(\xi)=\phi(1-\xi)$, $\limsup_{\xi\to 0}\frac{\phi(\xi,t)}{4\pi\sqrt{\xi}}<1$ for all t, and*

$$\frac{\partial}{\partial t}\phi < \frac{\phi^2\phi''-\phi(\phi')^2}{(4\pi)^2}+\phi+\phi'(1-2\xi).$$

Then if $(M,g(t))$ is any solution to the NRF on S^2 with $h_{g(0)}>\phi(\cdot,0)$, then $h_{g(t)}>\phi(\cdot,t)$ for all t.

Proof. First note that $h_{g(t)}(\xi)$ is a continuous function of t and ξ; furthermore, by our results on the asymptotics of h we know that

$$\frac{h_{g(t)}(\xi)}{\phi(\xi,t)}>1$$

for $\xi\in(0,\epsilon(t))$ with $\epsilon>0$ a continuous function of time. Thus if the theorem doesn't hold, there is some first time $t_0>0$ and fractional area $\xi_0\in(0,1)$ such that $h_{g(t)}>\phi(\cdot,t)$ for all $t<t_0$ and $h_{g(t_0)}(\xi_0)=\phi(\xi_0,t_0)$.

If Ω_0 is the isoperimetric region corresponding to (ξ_0,t_0), define γ_0 to be the smooth curve $\partial\Omega_0$. Then by assumption

$$|\gamma_0|_{g(t)}=h_{g(t)}(\xi_0)\geq\phi(\frac{|\Omega_0|_{g(t)}}{4\pi},t)$$

for all t, with equality when $t=t_0$. Differentiating the left and right sides of this inequality yields

$$\frac{\partial}{\partial t}|\gamma_0|_{g(t)}|_{t=t_0}\leq\frac{\partial}{\partial t}\phi+\frac{1}{4\pi}\phi'(\xi_0,t_0)\frac{\partial}{\partial t}|\Omega_0|_{g(t)}|_{t=t_0}. \tag{11.1}$$

The derivative of the length of γ_0 can be computed as

$$\begin{aligned}\frac{\partial}{\partial t}|\gamma_0|_{g(t)}|_{t=t_0} &= \frac{\partial}{\partial t}\int_{\gamma_0}\sqrt{g_t(\gamma'(u),\gamma'(u))}du = \int_{\gamma_0}\frac{-2(K-1)g_{t_0}(\gamma'(u),\gamma'(u))}{2\sqrt{g_{t_0}(\gamma'(u),\gamma'(u))}}du \\ &= -\int_{\gamma_0}(K-1)ds \\ &= -\int_{\gamma_0}Kds+\phi\end{aligned}$$

Meanwhile, applying the NRF area variation formula $\frac{\partial}{\partial t}d\mu = -(R-r)d\mu = -2(K-1)d\mu$ we have previously proved, we see that

$$\begin{aligned}\frac{\partial}{\partial t}|\Omega_0|_{g(t)} &= \frac{\partial}{\partial t}\int_{\Omega_0}d\mu = -2\int_{\Omega_0}(K-1)d\mu \\ &= 2|\Omega_0| - 2(2\pi - \int_{\gamma_0}k\,ds) \\ &= 8\pi\xi_0 - 4\pi + 2\int_{\gamma_0}k\,ds\end{aligned}$$

where in the second line we applied Gauss-Bonnet with k the geodesic curvature of γ_0, using the simple-connectedness of Ω_0 implied by one of our previous theorems. Inequality 11.1 therefore becomes

$$-\int_{\gamma_0}Kds+\phi \le \frac{\partial}{\partial t}\phi + \frac{\phi'}{4\pi}(8\pi\xi_0 - 4\pi + 2\int_{\gamma_0}k\,ds) \tag{11.2}$$

We will now simplify the curvature integrals in this equation by using the fact that Ω_0 is an isoperimetric domain. We first reprove the well-known result that a minimal boundary has constant geodesic curvature: consider a smooth family Ω_s of domains formed by flowing points x on $\partial\Omega_0$ by signed distance $\eta(x)\cdot s$ along the outward-pointing unit normal. Let γ_s be the flow image of the boundary. Since $|\partial\Omega_s| \ge h(\frac{|\Omega_s|}{4\pi}) \ge \phi(\frac{|\Omega_s|}{4\pi}, t_0)$ for all s with equality when $s=0$, we know that the difference between the left and right sides has derivative with respect to s zero and second derivative non-negative when $s=0$. We know the basic geometric formulae

$$\begin{aligned}\frac{d}{ds}|\Omega_s||_{s=0} &= \int_{\gamma_0}\eta\,ds \\ \frac{d}{ds}|\gamma_s||_{s=0} &= \int_{\gamma_0}k\eta\,ds\end{aligned}$$

so the vanishing of the first derivative yields the equation

$$0 = \int_{\gamma_0}k\eta\,ds - \frac{\phi'}{4\pi}\int_{\gamma_0}\eta\,ds.$$

The arbitrary nature of η means that we must in fact have $k = \frac{\phi'}{4\pi}$ on all of $\partial\Omega$.

Now we specialize our variation to have $\eta \equiv 1$. Since $\frac{d}{ds}|\Omega_s| = \int_{\gamma_s}\eta\,ds = |\gamma_s|$, we see that

$$\begin{aligned}\frac{d^2}{ds^2}|\Omega_s||_{s=0} &= \int_{\gamma_0}k\,ds = \frac{\phi'}{4\pi}|\gamma_0| \\ &= \frac{\phi\phi'}{4\pi}\end{aligned}$$

Additionally, since $\frac{d}{ds}|\gamma_s| = \int_{\gamma_s} k\, ds = 2\pi - \int_{\Omega_s} K d\mu$ for all s we derive

$$\frac{d^2}{ds^2}|\gamma_s||_{s=0} = -\int_{\gamma_0} K\, ds$$

The non-negativity of this term minus $\frac{d^2}{ds^2}\phi(\frac{|\Omega_s|}{4\pi}, t_0)|_{s=0} = \frac{d}{ds}[\frac{1}{4\pi}|\gamma_s|\phi'(\frac{|\Omega_s|}{4\pi}, t_0)]|_{s=0} = \frac{\phi(\phi')^2}{(4\pi)^2} + \frac{\phi^2\phi''}{(4\pi)^2}$ implies that

$$0 \leq -\int_{\gamma_0} K\, ds - \frac{\phi(\phi')^2}{(4\pi)^2} - \frac{\phi^2\phi''}{(4\pi)^2}.$$

Combining this inequality with 11.2 gives us

$$\frac{\phi(\phi')^2}{(4\pi)^2} + \frac{\phi^2\phi''}{(4\pi)^2} + \phi \leq \frac{\partial}{\partial t}\phi + \frac{\phi'}{4\pi}(8\pi\xi_0 - 4\pi + 2\frac{\phi\phi'}{4\pi}ds)$$

which after a bit of rearrangement yields a contradiction in the form of the inequality

$$\frac{\partial}{\partial t}\phi \geq \frac{\phi^2\phi'' - \phi(\phi')^2}{(4\pi)^2} + \phi + \phi'(1 - 2\xi_0)$$

□

12. Construction of the general comparison profile

It is now time to construct isoperimetric profiles suitable for comparison purposes, i.e. profiles that will satisfy the conditions on ϕ in the previous theorem. These profiles will be constructed from axially-symmetric solutions to the NRF on S^2: define a metric $\bar{g}$ by

$$\begin{aligned} \bar{g} &= e^{-2u(\phi,t)} g_{S^2} \\ &= e^{-2u(\phi,t)}(d\phi^2 + \sin^2\phi d\theta^2) \end{aligned}$$

where θ, ϕ are the spherical coordinates of azimuth and polar inclination, respectively, and u is some smooth even 2π-periodic function whose initial values $u(\cdot, 0)$ are given. We also require that the area of this initial metric is 4π. A brief calculation of the Gaussian curvature for this metric shows that the NRF on $(S^2, \bar{g})$ is equivalent to a PDE for u of the form

$$\frac{\partial}{\partial t}u = e^{-2u}(\frac{\partial^2}{\partial\phi^2}u + \cot\phi\frac{\partial}{\partial\phi}u - 1) + 1.$$

Short-time existence results applied to this equation, along with the long-time existence and uniqueness results for the NRF on S^2, show that $\bar{g}$ remains radially symmetric and that a unique solution u exists for all time.

For such a metric $\bar{g}$, we can define functions $L_u(\phi_0, t)$, $A_u(\phi_0, t)$ giving the boundary length and area (respectively) of a polar cap of angle ϕ_0 at time t, i.e. of the set $\{x \in S^2 : \phi(x) \leq \phi_0\}$:

$$\begin{aligned} L_u(\phi_0, t) &= 2\pi e^{-u(\phi,t)}\sin\phi \\ A_u(\phi_0, t) &= \int_0^{\phi_0} 2\pi e^{-2u(\phi,t)}\sin\phi d\phi \end{aligned}$$

Now define a smooth function $\psi_u : [0, 4\pi]\times[0,\infty) \to \mathbb{R}$ by $\psi_u(\xi) = L_u \circ A_u^{-1}(4\pi\xi)$: thus $\psi_u(\cdot, t)$ assigns a fractional area ξ to the length of the boundary of the polar cap with that fractional area at time t. $\psi$is therefore the isoperimetric profile for $(S^2, \bar{g})$ restricted only to polar cap regions; it turns out that certain curvature conditions imply that $\psi_u$is the actual isoperimetric profile for the manifold, but we don't need that information here.

This function ψ_u is almost the comparison function that we are looking for:

Theorem 21. *If $\bar{g} = e^{-2u(\phi,t)}g_{S^2}$ satisfies the NRF on $(S^2, \bar{g})$ with area 4π, then the function ψ_u satisfies the differential equality*

$$\frac{\partial}{\partial t}\psi_u = \frac{\psi_u^2\psi_u'' - \psi_u(\psi_u')^2}{(4\pi)^2} + \psi_u + \psi_u'(1-2\xi).$$

We also have asymptotics of the form

$$\psi_u(\xi,t) = 4\pi\sqrt{\xi}(1 - \frac{1}{2}(1-2u''(0))\xi + O(\xi^2)) \text{ as } \xi \to \infty.$$

Proof. Note that the identity $|\partial\Omega(s)| = \psi_u(\frac{|\Omega(s)|}{4\pi}, t)$ holds for all values of s, t when $\Omega(s)$ is the polar cap with $\phi = s$. The time derivatives of the two sides are therefore equal; furthermore, if we perform a unit-speed normal variation on a polar cap it remains a polar cap by the symmetry of $\bar{g}$, so this process also keeps the two sides of the equation equal. We can therefore perform the almost exact same proof as the previous theorem except with equalities in place of the inequalities; the only change is that we now know by symmetry that k is constant on the boundary of the caps. Thus we get the PDE satisfied by ψ_u as shown above.

To show the asymptotics for $\psi_u(\xi,t)$, we first calculate the asymptotics for $A_u(\phi,t)$ and $L_u(\phi,t)$ as $\phi \to 0$: recalling that u is even, we find that as $\phi \to 0$ we have

$$\begin{aligned} A_u(\phi,t) &= \pi e^{-u(0)}[\phi^2 + \frac{1}{2}(-u''(0) - \frac{1}{6})\phi^4 + O(\phi^5)] \\ L_u(\phi,t) &= 2\pi e^{-u(0)}[\phi + \frac{1}{2}(-u''(0) - \frac{1}{3})\phi^3 + O(\phi^4)] \end{aligned}$$

Invert A_u near $\phi = 0$ to find that

$$\phi_u(A,t) = \sqrt{\frac{A}{\pi e^{-2u(0)}}} + \frac{1}{4}(u''(0) + \frac{1}{6})(\frac{A}{\pi e^{-2u(0)}})^{3/2} + O(A^2)$$

Substitute this expression into L_u and set $A = 4\pi\xi$ to get the desired asymptotics for ψ_u. ∎

As I stated above, this theorem shows that ψ_u is almost the comparison function we are looking for: if we wished to apply our comparison theorem to ψ_u, recall that we need three important facts: the first is the strict inequality

$$\frac{\partial}{\partial t}\psi_u < \frac{\psi_u^2\psi_u'' - \psi_u(\psi_u')^2}{(4\pi)^2} + \psi_u + \psi_u'(1-2\xi).$$

which we have shown does not hold. We also would need to have the two conditions

$$\begin{aligned} \limsup_{\xi\to 0}\psi_u(\xi,t) &< 1 \text{ for all } t \\ \psi_u'' &< 0 \quad \text{for all } t \end{aligned}$$

the first of which we also just showed was false.

The strict concavity condition on ψ_u can be satisfied by choosing u so that $\bar{g}$ has positive curvature: to see this, recall that we have equalities everywhere if we replace ϕ by ψ_u in the proof of the comparison theorem, so in particular

$$-\int_{\Omega(s_0)} \bar{K} ds - \frac{\psi_u^2 \psi_u'' - \psi_u(\psi_u')^2}{(4\pi)^2} = 0$$

which quickly implies that $\psi_u'' < 0$ if $\bar{K} > 0$.

The other two conditions for applying the comparison theorem can be easily met by scaling ψ_u down: for any $\epsilon \in (0,1)$ define $\psi_{u,\epsilon} = (1-\epsilon)\psi_u$. Then it is immediate from the previous theorem that $\limsup_{\xi \to 0} \psi_{u,\epsilon}(\xi, t) = 1 - \epsilon < 1$ for all t, and we also see that

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\psi_{u,\epsilon} - \tfrac{\psi_{u,\epsilon}^2 \psi_{u,\epsilon}'' - \psi_{u,\epsilon}(\psi_{u,\epsilon}')^2}{(4\pi)^2} - \psi_{u,\epsilon} - \psi_{u,\epsilon}'(1-2\xi) \\
&= (1-\epsilon)\left[\tfrac{\psi_u^2\psi_u'' - \psi_u(\psi_u')^2}{(4\pi)^2} + \psi_u + \psi_u'(1-2\xi)\right] \\
&-(1-\epsilon)^3 \tfrac{\psi_u^2\psi_u'' - \psi_u(\psi_u')^2}{(4\pi)^2} - (1-\epsilon)\psi_u - (1-\epsilon)\psi_u'(1-2\xi) \\
&= \tfrac{1}{(4\pi)^2}(2\epsilon - 3\epsilon^2 + \epsilon^3)\left[\psi_u^2\psi_u'' - \psi_u(\psi_u')^2\right] \\
&= \tfrac{\epsilon(2-\epsilon)(1-\epsilon)}{(4\pi)^2}\left[\psi_u^2\psi_u'' - \psi_u(\psi_u')^2\right] \\
&< 0
\end{aligned}$$

where in the last line we use $\psi_u'' < 0$.

Given some NRF solution metric $g(t)$ with isoperimetric profile h that satisfies $h(\cdot, 0) \geq \psi_u(\cdot, 0)$, the above results allow us to apply the comparison theorem to conclude that for any $\epsilon \in (0,1)$ it is true that $h(\cdot, t) > \psi_{u,\epsilon}(\cdot, t)$ for all $t \geq 0$. Letting $\epsilon \to 0$, we deduce the following result:

Theorem 22. *Let $\bar{g}(t) = e^{-2u(\phi,t)} g_{S^2}$ be an axially-symmetric solution of the NRF on S^2 with positive curvature. Suppose $g(t)$ is another solution of the NRF on S^2 with isoperimetric profile h satisfying $h(\cdot, 0) \geq \psi_u(\cdot, 0)$. Then $h(\cdot, t) \geq \psi_u(\cdot, t)$ for all $t \geq 0$.*

13. The Rosenau Solution

The positive-curvature, axially-symmetric metric that we will use in our application of the preceding comparison theorem is an explicit solution called the *Rosenau solution*. It is an eternal solution to the NRF on S^2, i.e. the solution is defined for all times, even large negative ones. This will be a key property of the solution, as we will demonstrate that the Rosenau metric behaves like a long, thin cigar-shape that decreases in radius as $t \to -\infty$; thus the isoperimetric profile for the solution becomes uniformly small as $t \to -\infty$.

The standard explicit formula for the Rosenau solution is actually given as a smooth solution to the NRF on the cylinder $\mathbb{R} \times S^1$:

$$g(t) = \frac{\sinh(e^{-2t})}{e^{-2t}(\cosh(x) + \cosh(e^{-2t}))}(dx^2 + d\theta^2)$$

This metric does in fact extend to a smooth metric on the 2-sphere, however. One can check from this equation that $g(t)$ is a solution of the NRF with area 4π, though we will not do so here. Another calculation of the areas and boundary lengths of the sets $\{(x, \theta) : x \leq x_0\}$ yields

$$\begin{aligned} A(x,t) &= 2\pi + 2\pi e^{-2t} \log\left(\frac{1+\cosh(e^{-2t}+x)}{1+\cosh(e^{-2t}-x)}\right) \\ L(x,t) &= 2\pi e^t \sqrt{\frac{\sinh(e^{-2t})}{\cosh(x)+\cosh(e^{-2t})}} \end{aligned}$$

Solving for x in terms of $\xi = A/4\pi$ and substituting into L yields our isoperimetric comparison function:

$$\psi(\xi,t) = 4\pi\sqrt{\frac{\sinh(\xi e^{-2t})\sinh((1-\xi)e^{-2t})}{\sinh(e^{-2t})e^{-2t}}}$$

If we expand this function as an asymptotic series near $\xi = 0$, we see that

$$\psi(\xi,t) = 4\pi\sqrt{\xi}\left(1 - \frac{1}{2}e^{-2t}\coth(e^{-2t})\xi + O(\xi^2)\right)$$

Examining the area and boundary length expressions for these polar caps, it is clear that the Rosenau solution behaves like a long, thin cylinder as $t \to -\infty$; the maximum boundary length of any of the polar caps is $2\pi e^t\sqrt{\tanh(e^{-2t})} \sim 2\pi e^t$ as $t \to -\infty$. Thus it is to be expected that this solution gives an excellent isoperimetric comparison function:

Theorem 23. *For any metric g on S^2, there is a time t_0 such that $h_g \geq \psi(\cdot, t_0)$, where ψ is the isoperimetric comparison function for the Rosenau solution as defined above.*

Proof. A brief calculation shows that $\psi(\xi,t)$ is a strictly increasing function of t; furthermore, we see that $\lim_{t\to-\infty}\psi(\xi,t)=0$ and $\lim_{t\to\infty}\psi(\xi,t) = 4\pi\sqrt{\xi(1-\xi)}$. We can therefore define a continuous function $t(\xi)$ from $(0,1) \to \mathbb{R}\cup\{\infty\}$ by letting $t(\xi)$ be the unique time for which $\psi(\xi,t(\xi)) = h_g(\xi)$ if $h_g(\xi) < 4\pi\sqrt{\xi(1-\xi)}$ and $t(\xi) = \infty$ otherwise. Our asymptotic calculations of ψ and h_g as $\xi \to 0$ show that $t_* \doteqdot \lim_{\xi\to 0} t(\xi)$ exists; this is because $e^{-2t}\coth(e^{-2t})$ decreases from ∞ to 1 as t increases from $-\infty$, so our asymptotics show that t_* exists and is the unique solution of $e^{-2t_*}\coth(e^{-2t_*}) = \sup_M K$. t therefore extends to a continuous function from $[0,1] \to (-\infty,\infty]$, hence it is bounded below. Set t_0 to be less than this bound. $\square$

Combining this theorem with our isoperimetric comparison results yields our primary result in the $\chi > 0$ case:

Theorem 24. *Let $g(t)$ be any solution of the NRF on S^2. Then there is some t_0 such that*

$$h_{g(t)}(\xi) \geq \psi(\xi, t+t_0)$$

for all $t > 0$. By our results on the asymptotics of these two functions, this implies that

$$K(x,t) \leq coth(e^{-2(t+t_0)})e^{-2(t+t_0)} \leq 1 + \frac{1}{2}e^{-4(t+t_0)}.$$

14. Exponential Convergence of the metric for $\chi > 0$

To summarize, at this point we have analyzed the NRF on S^2 metrics, deriving bounds of the form

$$-Ce^{-2t} \leq K(x,t) \leq 1 + Ce^{-4t}$$

which are valid for all $t \geq 0$ (the lower bound comes from our previous maximum-principle work). Next, note that the same maximum principle argument that we made in the general long-time existence section allows us to turn the curvature bound into bounds on higher derivatives: specifically

$$|\nabla^k K|^2 \leq C_k(1 + t^{-k})$$

where C_k depends only on k and the initial bounds for K.

We can now apply Gauss-Bonnet to obtain exponential L^1 convergence of K to the expected limiting value of 1:

$$\begin{aligned} 0 &= \int_M (K-1)d\mu \text{ (Gauss-Bonnet with |M|=}2\pi\chi\text{(M)=}4\pi) \\ &= \int_{K\leq 1}(K-1)d\mu + \int_{K>1}(K-1)d\mu \\ &\leq -\int_{K\leq 1}|K-1| + Ce^{-4t} \end{aligned}$$

Thus $\int_M |K-1|d\mu \leq Ce^{-4t} + 4\pi Ce^{-4t} = \hat{C}e^{-4t}$.

We now have rapid L^1 convergence of K and upper bounds on all derivatives of K. To get exponential convergence in all C^k norms, Andrews and Bryan employ an interpolation inequality known as a *Gagliardo-Nirenberg inequality.* The statement of the inequality in its full generality, and its proof, can be found in [8], Theorem 12, but for our purposes we need only a slightly simplified version: on any fixed compact manifold M, it is true that for any integers $m > k \geq 1$ there exists a constant $C_{k,m} < \infty$ such that for any smooth function f

$$||\nabla^k f||_\infty \leq C_{k,m}||f||_{L^1}^{\frac{m-k}{m+2}}||\nabla^m f||_\infty^{\frac{k+2}{m+2}}$$

. This result is a kind of generalization of the Sobolev inequality to higher-order derivatives, so it is not surprising that the constant $C_{k,m}$ depends only on k, m, and the *Sobolev constant* $S(M)$ of the manifold, given by

$$S(M) \doteqdot \inf_{f \in C^\infty(M)} \frac{\int |\nabla f|^n}{(\int f^{\frac{n}{n-1}})^{n-1}}.$$

The Sobolev constant is in turn controlled by the *isoperimetric constant*

$$I(M) \doteqdot \inf_{\Omega \subset M} \frac{|\partial\Omega|^2}{\min\{|\Omega|, |M \setminus \Omega|\}}$$

Returning to our solution $g(t)$ of the NRF on $M = S^2$, we note that our lower bound of $h_{g(t)}$ by the Rosenau profile ψ yields a positive lower bound for $I(M,t)$ for all $t \geq 0$, so we also bound the Sobolev constant from below. Thus we can choose constants $C_{k,m}$ in the Gagliardo-Nirenberg inequality that are applicable for all positive times, so for all $t \geq 0$ and each $k \geq 1$

$$\begin{aligned} ||\nabla^k K||_\infty &\leq C_{k,m}||K-1||_{L^1}^{\frac{m-k}{m+2}}||\nabla^m K||_\infty^{\frac{k+2}{m+2}} \\ &\leq \tilde{C}_{k,m}e^{\frac{-4(m-k)}{m+2}t}(1+t^{-\frac{m(k+2)}{2(m+2)}}) \\ &\leq \bar{C}_k e^{-2t} \end{aligned}$$

by choosing m large enough.

Once we know that the curvature is decaying exponentially in any C^k norm, exponential convergence of the metric follows as in the section on general long-time existence. Thus concludes our proof of the convergence of the NRF on surfaces of positive Euler characteristic. We have completed the final case of our demonstration of long-time existence and convergence of the NRF on surfaces, and thus have established the Uniformization Theorem for compact surfaces via the Ricci flow.

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