HOPF MODULES AND REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

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Abstract. This project uses a modification of the approach developed by Andrey V. Zelevinsky in his book *Representations of Finite Classical Groups* to study the representation theory of finite wreath products, finite symplectic groups, and odd-dimensional finite orthogonal groups. The fundamental notion will be a particular type of Hopf module defined over and sharing many properties with a positive self-adjoint Hopf algebra (PSH-algebra), with the Hopf axiom replaced by a similar structure property in which the Hopf power map enters in an essential way. By passing to Grothendieck groups, it will be shown that the finite dimensional complex representations of finite wreath products $S_n \rtimes G^n$ can be used to construct such Hopf modules. These Hopf modules enjoy a strong compatibility relationship between the comultiplication and multiplication, quite analogous to the Hopf axiom for Hopf algebras, parameterized only by the order of the group G. Interestingly, this structure provides an alternate proof that the sum of the degrees of the irreducible representations of a finite group equals the order of the group, as is seen in a particular tensor product decomposition of these modules. A direct sum decomposition is proved for such modules in generality. These compatibility relations give rise to additional structure on the Grothendieck groups of categories of such modules. The observation that the Weyl groups of the symplectic and odd orthogonal groups over a finite field are themselves finite wreath products, and thus have representation theory described by such a Hopf module, motivates the construction of such a Hopf module from the representations of those finite groups of Lie type. The final section of this paper provides this construction, using the Bruhat decomposition in an essential way to reduce the question back to the case of the underlying Weyl groups.

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1. Introduction

Recall that an algebra is a vector space A with a multiplication $A \times A \to A$ satisfying certain familiar properties, including associativity and distributivity. In particular, this distributivity allows the multiplication to be described by a linear map $m\colon A \otimes A \to A$, again satisfying certain properties. For example, the associative property may be equivalently stated as the commutativity of the diagram:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{1\otimes m} & A \otimes A \\ \big\downarrow{\scriptstyle m\otimes 1} & & \big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A. \end{array}$$

Algebras can equally well be defined over arbitrary commutative rings, and it is algebras over the integers that will be of primary importance here.

This observation provides a method for creating new types of algebraic objects from old and a convenient language for working with these objects. One such object, which will be of fundamental importance in this paper, is the *coalgebra*, a dual notion of sorts to the notion of an algebra. The philosophy of dualizing is to reverse all arrows in the defining diagrams. Thus, a coalgebra C is again a vector space, now with a *comultiplication* $m^*\colon C \to C \otimes C$. All the algebra axioms are in play, now with arrows reversed, and for instance we require *coassociativity*:

$$\begin{array}{ccc} C & \xrightarrow{\quad m^* \quad} & C \otimes C \\ \big\downarrow{\scriptstyle m^*} & & \big\downarrow{\scriptstyle m^*\otimes 1} \\ C \otimes C & \xrightarrow{1\otimes m^*} & C \otimes C \otimes C. \end{array}$$

Comodules, for example, arise from applying this notion of dualization to the notion of ordinary modules.

A *Hopf algebra* H is an algebra that is simultaneously a coalgebra in which the algebra and coalgebra structures interact nicely. In particular, it is required that the comultiplication is a morphism of algebras with respect to the algebra structure on H and the induced algebra structure on $H \otimes H$ - this is the Hopf axiom. As it turns out, this structure arises naturally in the representation theory of the symmetric and finite general linear groups, in addition to many other areas of mathematics. Recall that if G is a group, a representation of G is a homomorphism

$$G \to GL(V)$$

for some vector space V.

Geissinger showed that the representations of the symmetric groups can be well understood by studying all of the groups simultaneously. The idea is to encode some of the key properties of the representations in the form of a Hopf algebra, and then to take advantage of the resulting rich algebraic structures to analyze this Hopf algebra and in turn realize results about the underlying representations. The elements of the Hopf algebra itself consist of formal sums or differences of representations. In view of the natural embedding $S_k \times S_l \hookrightarrow S_{k+l}$ and the natural isomorphism $\mathrm{Rep}(S_k) \otimes \mathrm{Rep}(S_l) \cong \mathrm{Rep}(S_k \times S_l)$ of representation rings, the multiplication map corresponds to induction of representations from the group $S_k \times S_l$ to S_{k+l}, and the comultiplication map similarly is constructed using restriction from

S_{k+l} to $S_k \times S_l$. The defining axioms of a Hopf algebra then reflect certain properties of representations and their interactions through induction and restriction. For instance, the associativity results from the associativity of induction of representations, and the Hopf axiom mentioned above is essentially Mackey's Theorem on the composition of induction and restriction. Crucially, there is even additional structure beyond the Hopf algebra structure - there is a natural grading in which representations of any particular group S_k are homogenous elements, and the multiplication and comultiplication are then graded maps, so the Hopf algebra is graded. Equally significantly, the standard "inner product" of representations descends to a genuine inner product on the Hopf algebra, and Frobenius reciprocity, a standard result describing the interaction of this "inner product" with restriction and induction, appears at the Hopf algebra level as the statement that the multiplication and comultiplication maps are adjoint in the standard sense of linear algebra. This fact turns out to be central, and these properties together give rise to tremendous rigidity in the structure of the Hopf algebra, shedding much light on the representations themselves.

Extending Geissinger's work, Zelevinsky showed that a very similar Hopf algebra construction can be achieved for the general linear groups over a fixed finite field as well as for finite wreath product groups, although in the case of the general linear groups the construction and resulting algebra itself are somewhat more complicated in a way that will be discussed later in detail. A wreath product of a group G is a semidirect product $S_k \rtimes G^k$ with respect to the action of S_k on G^k permuting the elements. The particular wreath products $S_k \rtimes (\mathbb{Z}/2\mathbb{Z})^k$, called the hyperoctohedral groups, are the Weyl groups of the symplectic and odd orthogonal groups, and thus are of particular importance.

That the Hopf algebra structure for the symmetric groups has a close analog in the case of the finite general linear groups should not be viewed as an accident. Indeed, the symmetric groups are the Weyl groups of the general linear groups, and, with Lie theory in mind, the former should therefore be viewed as the "skeletons" of the latter, with many properties carrying over, in this case in particular the Hopf algebra structure, with thanks largely due to the Bruhat decomposition theorem (discussed in the final section). Therefore, given Zelevinsky's Hopf algebra for the Weyl groups of the finite symplectic and odd orthogonal groups, it therefore seems reasonable to try to extend the method to those finite groups of Lie type. Sadly, later when the construction for the general linear groups is presented, it will be clear that the construction as such does not carry through to the case of the symplectic and odd orthogonal groups.

The final section of this paper solves this problem by constructing an alternative algebraic structure in those cases, where now the role of the Hopf algebra is replaced by the role of a Hopf module - a module, simultaneously a comodule, defined over a Hopf algebra. In view of the close connection between the groups of Lie type and their Weyl groups, the approach of this paper diverges from Zelevinsky's treatment of wreath products via Hopf algebras in favor of a Hopf module approach, which is much more naturally generalized to the associated finite groups of Lie type. Nevertheless, these Hopf modules will encode the same sorts of information about the underlying representations as do Zelevinsky's Hopf algebras, and, in fact, his results on the structure of those Hopf algebras prove very useful in analyzing these Hopf modules. The particular type of Hopf module structure which emerges from this approach is a new and very interesting algebraic object, and some effort is spent discussing these novel objects in their own right.

Roughly speaking, the use of Hopf modules allow representations of groups belonging to different families to be multiplied, providing an action of one type of representation on another. In the case of wreath products, for example, similar to the embedding of products of symmetric groups, the product

$$S_k \times (S_l \rtimes G^l)$$

naturally embeds in the wreath product $S_{k+l} \rtimes G^{k+l}$. As such, and again analogously to the case of the symmetric groups, representations of S_k and of $S_l \rtimes G^l$ may be "multiplied," using induction, to yield a representation of $S^{k+l} \rtimes G^{k+l}$. Comultiplication can then also be defined similarly, again using restriction. In this way, we will see that a Hopf module can be constructed from the representations of wreath products of a fixed finite group, and this Hopf module will share many of the important properties of Zelevinsky's Hopf algebras. A central result of this paper describes a compatibility between the multiplication and comultiplication, much like the Hopf axiom but involving a "twist" by the Hopf $|G|^{th}$-power map. Via a somewhat more complicated construction, a similar action of the representations of the general linear groups on representations of the symplectic or odd orthogonal groups will be used in defining the Hopf modules in the case of those finite groups of Lie type. It will be seen that these modules then share significant structural properties with the hyperoctohedral Hopf module, both depending in an essential way on the Hopf square map.

Finally, I would like to express my gratitude to my adviser, Professor Daniel Bump, for suggesting this project, for sharing his thoughts and insights, and for his consistent support over the last year.

2. Fundamentals

This first section summarizes most of the results from Zelevinsky's book which will be important in subsequent sections. First we need some notation. Let S_n denote the symmetric group on n letters. Using Zelevinsky's notation, let $R_0 = \mathbb{Z}$, for $n > 0$ let R_n be the Grothendieck group of the category of finite dimensional complex representations of S_n, and let R be the graded abelian group

$$R = \bigoplus_{n \geq 0} R_n.$$

R has a natural basis consisting of 1 along with representatives of the irreducible representations of all the S_n, where the addition is given by the direct sum.

If $k + l = n$ then the group $S_k \times S_l$ can be identified with the subgroup of S_n stabilizing $\{1, 2, \dots, k\}$. Recall that if G and H are finite groups then the irreducible representations of $G \times H$ are precisely those of the form $\sigma \otimes \tau$, where σ is an irreducible representation of G and τ is an irreducible representation of H. We can therefore define multiplication and comultiplication on R as follows.

Let $m\colon R \otimes R \to R$ be the multiplication defined by setting, for an irreducible representation $\alpha \otimes \beta$ of $S_k \times S_l$,

$$m(\alpha \otimes \beta) = \mathrm{Ind}_{S_k \times S_l}^{S_n}(\alpha \otimes \beta)$$

and extending by linearity. This expression clearly respects the direct sum, so the above formula extends identically to representations which are not irreducible.

Proposition 1. *m is associative, i.e. the following diagram commutes:*

$$\begin{array}{ccc} R\otimes R\otimes R & \xrightarrow{1\otimes m} & R\otimes R \\ \big\downarrow{\scriptstyle m\otimes 1} & & \big\downarrow{\scriptstyle m} \\ R\otimes R & \xrightarrow{\quad m \quad} & R \end{array}$$

Proof. Let α, β, and γ be representations of S_a, S_b, and S_c, respectively. From the general identity

$$\mathrm{Ind}_A^B(\sigma)\otimes \mathrm{Ind}_C^D(\tau) \cong \mathrm{Ind}_{A\times C}^{B\times D}(\sigma\otimes\tau),$$

and the transitivity of induction, it follows that

$$\begin{aligned} m(m(\alpha\otimes\beta)\otimes\gamma) &= \mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}}(\mathrm{Ind}_{S_a\times S_b}^{S_{a+b}}(\alpha\otimes\beta))\otimes\gamma) \\ &= \mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}}(\mathrm{Ind}_{S_a\times S_b}^{S_{a+b}}(\alpha\otimes\beta))\otimes \mathrm{Ind}_{S_c}^{S_c}(\gamma)) \\ &= \mathrm{Ind}_{S_{a+b}\times S_c}^{S_{a+b+c}}(\mathrm{Ind}_{S_a\times S_b\times S_c}^{S_{a+b}\times S_c}(\alpha\otimes\beta\otimes\gamma)) \\ &= \mathrm{Ind}_{S_a\times S_b\times S_c}^{S_{a+b+c}}(\alpha\otimes\beta\otimes\gamma) \end{aligned}$$

and similarly

$$m(\alpha\otimes m(\beta\otimes\gamma)) = \mathrm{Ind}_{S_a\times S_b\times S_c}^{S_{a+b+c}}(\alpha\otimes\beta\otimes\gamma).$$

This proves the associative law in this case. □

Comultiplication is similarly defined using restriction. In particular, if π is a representation of S_n, then $m^*\colon R\to R\otimes R$ is defined by setting

$$m^*(\pi) = \sum_{k+l=n} \mathrm{Res}_{S_n}^{S_k\times S_l}(\pi)$$

and extending by linearity.

Proposition 2. *m^* is coassociative, i.e. the following diagram commutes:*

$$\begin{array}{ccc} R & \xrightarrow{\quad m^* \quad} & R\otimes R \\ \big\downarrow{\scriptstyle m^*} & & \big\downarrow{\scriptstyle m^*\otimes 1} \\ R\otimes R & \xrightarrow{1\otimes m^*} & R\otimes R\otimes R \end{array}$$

Proof. By linearity of m^*, it suffices to verify this for a representation π of S_n. We see

$$\begin{aligned} (1\otimes m^*)(m^*(\pi)) &= \sum_{k+l+m=n} \mathrm{Res}_{S_n}^{S_k\times S_l\times S_m}(\pi) \\ &= (m^*\otimes 1)(m^*(\pi)), \end{aligned}$$

as needed. □

We will identify S_n with the group of $n\times n$ permutation matrices, $S_k\times S_l$ as the subgroup of permutation matrices with all nonzero entries occurring in the upper-left $k\times k$ and lower-right $l\times l$ blocks, and similarly for $S_p\times S_q$.

Proposition 3. *Let k, l, p, q be nonnegative integers with $k+l=p+q=n$. The double-coset space $S_p\times S_q\backslash S_n/S_k\times S_l$ has a complete set of representatives of the* *form*

$$\begin{bmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & I_b & 0 & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix}$$

where $a+b=k,\ c+d=l,\ a+c=p,\ b+d=q.$

Proof. $S_k \times S_l$ acts on S_n by right multiplication permuting the first k and last l columns, and $S_p \times S_q$ acts similarly by left multiplication permuting the first p and last q rows. Therefore together these actions stabilize the upper-left $p \times k$, lower-left $q \times k$, upper-right $p \times l$, and lower-right $q \times l$ blocks. In particular, by permuting the first k and last l columns of a permutation matrix, the nonzero entries in the upper-left $p \times k$ and lower-left $q \times k$ blocks can be arranged to occur in columns successively from the right of the left side and from the left of the right side, respectively, and similarly for the corresponding right blocks as in the matrix above. By permuting rows, it follows that every double-coset has a representative as above. The conditions on a, b, c, d are necessary since a permutation matrix must have full rank. Furthermore, since the left- and right-actions preserve the rank in each block, it follows that the integers a, b, c, d are uniquely determined for each double-coset, completing the proof. □

We will need the following fact.

Proposition 4. (Mackey's Theorem) *Let G be a finite group, and let H_1 and H_2 be subgroups. Let (π, V) be an irreducible representation of H_1. Let $\gamma_1, \dots, \gamma_h$ be a complete set of representatives of the double coset space $H_2\backslash G/H_1$, let $H_\gamma = H_2 \cap \gamma H_1 \gamma^{-1}$, and let $\pi^\gamma \colon H_\gamma \to GL(V)$ be the representation $\pi^\gamma(g) = \pi(\gamma^{-1} g \gamma)$. Then we have the following equivalence of representations:*

$$\mathrm{Res}_G^{H_2}(\mathrm{Ind}_{H_1}^G(\pi)) = \bigoplus_{1 \le i \le h} \mathrm{Ind}_{H_{\gamma_i}}^{H_2}(\pi^{\gamma_i}).$$

Proof. See Bump's *Lie Groups*, Chapter 34, Theorem 34.2. □

Proposition 5. *The multiplication m and comultiplication m^* satisfy the Hopf axiom. In other words, if $R \otimes R$ is given the algebra structure of component-wise multiplication then m^* is an algebra homomorphism, i.e. the following diagram commutes:*

$$\begin{array}{ccccc} R \otimes R & \xrightarrow{m^* \otimes m^*} & R \otimes R \otimes R \otimes R & \xrightarrow{1 \otimes \tau \otimes 1} & R \otimes R \otimes R \otimes R \\ \downarrow m & & & & \downarrow m \otimes m \\ R & & \xrightarrow{\quad m^* \quad} & & R \otimes R \end{array}$$

where $\tau \colon R \otimes R \to R \otimes R$ is the transposition $\tau(x \otimes y) = y \otimes x$.

Proof. Let $\bar{m} \colon R \otimes R \to R \otimes R$ denote the multiplication on $R \otimes R$ given by $(m \otimes m) \circ (1 \otimes \tau \otimes 1)$. Let M be the double coset representative from the previous proposition. For integers $s, t \in \mathbb{Z}$ let $[s, t]$ be the set $\{r \in \mathbb{Z} \colon s \le r \le t\}$. M is the following permutation of $[1, n]$:

$$i \mapsto i, i \in [1, a] \cup [a + b + c + 1, n]$$

$$i \mapsto i + c, i \in [a+1, a+b]$$
$$i \mapsto i - b, i \in [a+b+1, a+b+c]$$

Then $M(S_k \times S_l)M^{-1}$ is the group of permutations of $[1,n]$ stabilizing $[1,a] \cup [p+1, p+b]$ and $[a+1,p] \cup [p+b+1,n]$. Therefore, $M(S_k \times S_l)M^{-1} \cap S_p \times S_q$ is $S_a \times S_c \times S_b \times S_d$. Let π be the representation of this group given by $\pi(g) = (\alpha \otimes \beta)(M^{-1}gM)$. Identifying the group $S_a \times S_b \times S_c \times S_d$ with the group $S_a \times S_c \times S_b \times S_d$ via the automorphism of S_n given by conjugation by M, the representation π is equivalent to the representation of the group $S_a \times S_b \times S_c \times S_d$ given by $\mathrm{Res}_{S_k}^{S_a \times S_b}(\alpha) \otimes \mathrm{Res}_{S_l}^{S_c \times S_d}(\beta)$. Under this same isomorphism, the we may consider $S_p \times S_q$ as the subgroup of S_n stabilizing $[1,a] \cup [k+1, k+c]$. Therefore, by Mackey's Theorem on the composition of restriction and induction, we have

$$\mathrm{Res}_{S_n}^{S_p \times S_q}(\mathrm{Ind}_{S_k \times S_l}^{S_n}(\alpha \otimes \beta)) \cong \bigoplus \mathrm{Ind}_{S_a \times S_b \times S_c \times S_d}^{S_p \times S_q}(\mathrm{Res}_{S_k}^{S_a \times S_b}(\alpha) \otimes \mathrm{Res}_{S_l}^{S_c \times S_d}(\beta))$$

where the direct sum on the right-hand side runs over non-negative integers a, b, c, d subject to the conditions

$$a + b = k, \ c + d = l$$
$$a + c = p, \ b + d = q.$$

Furthermore, using the equivalence of $\mathrm{Ind}_A^B(\sigma) \otimes \mathrm{Ind}_C^D(\tau)$ and $\mathrm{Ind}_{A \times B}^{C \times D}(\sigma \otimes \tau)$ and noting that the above embedding of $S_p \times S_q$ in S_n corresponds to the transposition of middle factors in the definition of the multiplication $\bar{m}$, the summand on the right side is given by

$$\mathrm{Ind}_{S_a \times S_b \times S_c \times S_d}^{S_p \times S_q}(\mathrm{Res}_{S_k}^{S_a \times S_b}(\alpha) \otimes \mathrm{Res}_{S_l}^{S_c \times S_d}(\beta)) = \bar{m}(\mathrm{Res}_{S_k}^{S_a \times S_b}(\alpha) \otimes \mathrm{Res}_{S_l}^{S_c \times S_d}(\beta)).$$

The above calculations show

$$\begin{aligned} m^*(m(\alpha \otimes \beta)) &= \sum_{p+q=n} \mathrm{Res}_{S_n}^{S_p \times S_q}(\mathrm{Ind}_{S_k \times S_l}^{S_n}(\alpha \otimes \beta) \\ &= \sum_{a+b=k, \ c+d=l} \bar{m}(\mathrm{Res}_{S_k}^{S_a \times S_b}(\alpha) \otimes \mathrm{Res}_{S_l}^{S_c \times S_d}(\beta)) \\ &= \bar{m}\left(\left(\sum_{a+b=k} \mathrm{Res}_{S_k}^{S_a \times S_b}(\alpha)\right) \otimes \left(\sum_{c+d=l} \mathrm{Res}_{S_l}^{S_c \times S_d}(\beta)\right)\right) \\ &= \bar{m}(m^*(\alpha) \otimes m^*(\beta)). \end{aligned}$$

Thus the Hopf axiom holds for the simple tensor $\alpha \otimes \beta$ and hence in general by $\mathbb{Z}$-linearity of m, m^*, and $\bar{m}$. □

Using Zelevinsky's notation, let $\Omega(R_0) = \{1\}$ and for $n > 0$ let $\Omega(R_n)$ be the complete set of (isomorphism classes of) irreducible representations of S_n, and let

$$\Omega(R) = \bigcup_{n \geq 0} \Omega(R_n).$$

Then $\Omega(R_n)$ and $\Omega(R)$ are $\mathbb{Z}$-bases of R_n and R, respectively, and these are called the *irreducible* elements in R. This induces an inner product $\langle \, , \, \rangle$ on R by declaring $\Omega(R)$ to be an orthonormal basis. Naturally, $\Omega(R_n) \times \Omega(R_m)$ is a $\mathbb{Z}$-basis of $R_n \otimes R_m$ and $\Omega(R) \times \Omega(R)$ is a $\mathbb{Z}$-basis of $R \otimes R$, and these elements are also called irreducible. We then have an inner product on $R \otimes R$ in which $\Omega(R) \times \Omega(R)$ is an orthonormal basis. Note then that the subspaces R_n of R are mutually orthogonal, as are the subspaces $R_n \otimes R_m$ of $R \otimes R$.

$\Omega(R)$ also induces a partial ordering on R. In particular, for elements $x,\ y \in R$ define $x \geq 0$ if and only if $\langle x, \omega \rangle \geq 0$ for every $\omega \in \Omega(R)$, and then $x \geq y$ if and only if $x - y \geq 0$. Similarly, define $x > 0$ if and only if $x \geq 0$ and there exists $\omega \in \Omega(R)$ such that $\langle x, \omega \rangle > 0$, and $x > y$ if and only if $x - y > 0$. Define an analogous partial ordering on $R \otimes R$. An element x is called *positive* if $x > 0$, and an operator is called *positive* if it sends positive elements to positive elements. Observe that the positive elements are precisely those that correspond to actual representations (or, more precisely, collections of representations).

Before stating more results, we recall the following fact.

Proposition 6. (Frobenius Reciprocity) *If G is a finite group, $H < G$ is a subgroup, π is a representation of G, and σ is a representation of H, then*

$$\langle \pi, \mathrm{Ind}_H^G(\sigma) \rangle_G = \langle \mathrm{Res}_G^H(\pi), \sigma \rangle_H$$

Proof. See Lang's *Algebra*, Chapter XVIII, Theorem 6.1. □

Proposition 7. *m and m^* are positive operators.*

Proof. This is simply a paraphrasing of the fact that restriction and induction send representations to representations. □

Proposition 8. *m and m^* are adjoint with respect to the inner products on R and $R \otimes R$.*

Proof. This is a restatement of Frobenius reciprocity. □

Since R is a graded ring it follows that

$$I = \bigoplus_{n>0} R_n$$

is an ideal. The *primitive* elements of R are defined to be the elements in the orthogonal complement of $I^2 = m(I \otimes I)$ in I. Since m and m^* are adjoint it follows that $z \in R$ is primitive if and only if $m^*(z) = 1 \otimes z + z \otimes 1$. Clearly, the primitive elements of R form a subgroup (but NOT a subring). Let P denote this subgroup of primitive elements. Zelevinsky characterized the primitive elements in R as follows.

Proposition 9. *For each $n \geq 0$ there exists a unique primitive element $z_n \in R$ such that $\langle z, x_n \rangle = 1$, and $P \cap R_n = \mathbb{Z} z_n$. Furthermore, $\mathbb{Q} \otimes R$ is isomorphic as a $\mathbb{Q}$-algebra to a polynomial ring over $\mathbb{Q}$ in the variables z_n.*

Proof. See Zelevinsky, Chapter 1, Proposition 3.15.a. □

Similarly to the x's and y's, given a partition $\lambda = (l_1, \ldots, l_r)$, define $z_\lambda = z_{l_1} \cdots z_{l_r}$, where as usual an empty product is interpreted to be 1. Zelevinsky also proved the following proposition.

Proposition 10. *If $S \subset R$ is a set of mutually orthogonal primitive elements and if $p_1, \ldots, p_r, q_1, \ldots, q_s \in S$ then $\langle p_1 \cdots p_r, q_1 \cdots q_s \rangle = 0$ unless the p's and q's are the same up to reordering. In particular, $\langle z_\lambda, z_\mu \rangle = 0$ unless $\lambda = \mu$.*

Proof. See Zelevinsky, Chapter 1, Proposition 2.3. □

Zelevinsky proved the following fact.

Proposition 11. *Given $x \in R$, the linear map $R \to R$ given by multiplication by x has an adjoint map $x^*\colon R \to R$, i.e. a linear map satisfying*

$$\langle xy, z\rangle = \langle y, x^*(z)\rangle$$

for all $y, z \in R$. The adjoint map of a positive element is a positive map. Furthermore, this association satisfies the properties

$$(1) \quad x \in R_k \Rightarrow x^*(R_n) \subset R_{n-k}$$

$$(2) \quad y^* \circ x^* = (xy)^* = (yx)^* = x^* \circ y^*$$

$$(3) \quad m^*(x) = \sum_i x'_i \otimes x''_i \Rightarrow x^*(yz) = \sum_i x'^*_i(y) x''^*_i(z).$$

In addition, if $p \in R$ is primitive then p^* is a derivation of R.

Proof. See Zelevinsky, Chapter 1, Proposition 1.9.a. □

As a final note, Zelevinsky proved the following decomposition theorem for PSH-algebras:

Proposition 12. *Suppose H is a PSH-algebra. If H has one primitive irreducible element then H is determined as a PSH-algebra up to a scaling of grading, and H has exactly two PSH-algebra automorphisms. In particular, if the unique primitive irreducible element of H has degree one then $H \cong R$ as PSH-algebras.*

More generally, if $\mathcal{C} \subset H$ is the complete set of primitive irreducible elements of H and if for $c \in \mathcal{C}$ $H(c)$ is the PSH-subalgebra of H whose irreducible elements are precisely those occurring in powers of c then we have the following isomorphism of PSH-algebras:

$$H \cong \bigotimes_{c \in \mathcal{C}} H(c).$$

The righthand-side of the above decomposition has component-wise multiplication β, comultiplication β^* given by

$$\beta^*(r_1 \otimes \cdots \otimes r_k) = r_1^{(1)} \otimes \cdots \otimes r_k^{(1)} \otimes r_1^{(2)} \otimes \cdots \otimes r_k^{(2)},$$

where the superscript notation refers to the corresponding tensor factor in the comultiplication of the element in H (subscripts omitted), obvious unit and counit (note that $H_0 = \mathbb{Z}$), and with irreducible elements given by

$$\{\omega_1 \otimes \cdots \otimes \omega_k \colon \omega_i \in \Omega(H)\}.$$

Proof. See Zelevinsky, Chapter 1, Sections 2 and 3. □

3. Motivation and Wreath Products

Recall that given a group G, a *wreath product* of G is a semidirect product $W = G^n \rtimes S_n$ induced by the action of S_n on G^n given by $\sigma(g_1, \dots, g_n) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)})$. In particular, if there is an action of G on a set X then there is an action of W on X^n given by

$$(g_1, \dots, g_n, \sigma)(x_1, \dots, x_n) = (g_1(x_{\sigma^{-1}(1)}), \dots, g_n(x_{\sigma^{-1}(n)})).$$

Observe that W may be identified as the subgroup of the matrix group $GL_n(\mathbb{Z}[G])$ consisting of those matrices containing exactly one nonzero element, equal to an element of G, in each row and column.

For the remainder of this section, fix a finite group G and let W_n denote the wreath product $S_n \rtimes G^n$, realized as the group of matrices described above. Also, fix a normal subgroup $H \triangleleft G$ and set $V_n = S_n \rtimes H^n$ as the subgroup of W_n consisting of those matrices whose elements are all 0 or elements of H. Identify S_n as the subgroup of W_n or V_n consisting of the permutation matrices, and identify G^n (H^n) as the diagonal subgroup of W_n (V_n, respectively). Let $k = |G|$.

Definition 1. *Let $\psi_n = Res^{S_n}_{W_n}(Ind^{W_n}_{S_n}(1))$ and let θ_n be its character, where 1 denotes the trivial representation of S_n.*

Proposition 13. *Let χ be a character of S_n. Then we have the following equality of characters:*

$$Res^{S_n}_{W_n}(Ind^{W_n}_{S_n}(\chi)) = \theta_n \chi.$$

Proof. Observe that G^n is a complete set of coset representatives for W_n/S_n. Let $\bar{\chi} = Res^{S_n}_{W_n}(Ind^{W_n}_{S_n}(\chi))$. Let $\chi' \colon W_n \to \mathbb{C}$ be the function which agrees with χ on S_n and is zero elsewhere, and let 1_{S_n} be the indicator function of S_n. Note that conjugation by $d \in G^n$ can only scale the entries of $\sigma \in S_n$, so it follows that $d\sigma d^{-1} \in S_n$ if and only if $d\sigma d^{-1} = \sigma$. In particular, if follows that

$$\begin{aligned} \bar{\chi}(\sigma) &= \sum_{d \in G^n} \chi'(d\sigma d^{-1}) \\ &= \sum_{d \in G^n} \chi(\sigma) 1_{S_n}(d\sigma d^{-1}) \\ &= \chi(\sigma) \sum_{d \in G^n} 1_{S_n}(d\sigma d^{-1}) \\ &= (\chi\theta_n)(\sigma), \end{aligned}$$

as needed. □

Proposition 14. *Let π be a representation of S_n. Then*

$$Res^{S_n}_{W_n}(Ind^{W_n}_{S_n}(\pi)) \cong \psi_n \otimes \pi.$$

Proof. This is an immediate consequence of the previous proposition, the fact that the character of a tensor product of representations is equal to the product of the characters of the representations involved, and that representations are uniquely determined by their characters. □

Definition 2. *Let $cycles(\sigma)$ denote the number of distinct cycles appearing in the cycle decomposition of $\sigma \in S_n$.*

Proposition 15. *For $\sigma \in S_n$, we have $\theta_n(\sigma) = k^{cycles(\sigma)}$.*

Proof. Since both the number of cycles in the cycle decomposition and the value of a character are constant on conjugacy classes, and since any permutation matrix is conjugate to a block diagonal permutation matrix in which each block corresponds to a cycle, it suffices to check this identity for a such a block diagonal permutation matrix $\sigma \in S_n$.

So, let $\sigma \in S_n$ be such a block diagonal permutation matrix. Then $cycles(\sigma)$ is equal to the number of blocks along the diagonal of σ. If the i^{th} diagonal entry of d is d_i and if $\sigma = (\sigma_{ij})$ then $d\sigma d^{-1} = (d_i \sigma_{ij} d_j^{-1})$. Observe then that if $d \in G^n$ then $d\sigma d^{-1}$ is a block diagonal matrix of the same form. If conjugation by d preserves one of the blocks in σ then we obtain collection of equations $d_i = d_j = d_{\sigma(i)}$ for all rows i corresponding to this block, and since these row numbers form a complete cycle of σ this implies $d_i = d_{i'}$ for all rows i, i' in this block. Recall also $d\sigma d^{-1} \in S_n$ if and only if $d\sigma d^{-1} = \sigma$. Therefore those $d \in G^n$ such that $d\sigma d^{-1} \in S_n$ are determined by a choice of a group element for each of the k blocks, so there are $k^{cycles(\sigma)}$ such $d \in G^n$. The formula for induced characters used in the previous proof therefore gives $\theta_n(\sigma) = k^{cycles(\sigma)}$. □

Proposition 16. *$\theta_a\theta_b$ and θ_{a+b} agree on $S_a \times S_b$.*

Proof. Let $\sigma \in S_a$ and $\tau \in S_b$. Then $cycles(\sigma\tau) = cycles(\sigma) + cycles(\tau)$. The claim then follows from the previous proposition. □

θ_n can also be understood using Mackey theory. We first need to understand the double-coset space $S_n \backslash W_n / S_n$.

Proposition 17. *Let $G = \{g_1, \dots, g_k\}$. The double coset-space $S_n \backslash W_n / S_n$ has a complete set of representatives of parameterized by k-tuples of nonnegative integers $(a_1, \dots, a_k)$ satisfying $a_1 + \cdots + a_k = n$, where $(a_1, \dots, a_k)$ corresponds to the $n \times n$ block diagonal matrix M with j^{th} block equal to $g_j I_{a_j}$ (I_{a_j} is the $a_j \times a_j$ identity matrix).*

Proof. Left-multiplication by elements of S_n permutes rows and right-multiplication permutes columns. Therefore, since every element of W_n has exactly one nonzero entry in each row and column, any element of $S_n \backslash W_n / S_n$ clearly has a diagonal representative. Furthermore, transposing the i^{th} and j^{th} rows and columns transposes the i^{th} and j^{th} diagonal elements. Hence, each double-coset has a representative satisfying the condition that if g_j and g_l both occur on the diagonal and $1 \le j < l \le k$ then the former occurs in a higher row. The proposition follows. □

Definition 3. *Let x_j be the trivial representation of S_j, let y_j be the sign character of S_j, considered as elements of the Hopf algebra R. Given a partition $\lambda = (l_1, \dots, l_r)$ of n, let $x_\lambda = x_{l_1} \cdots x_{l_r}$ and similarly let $y_\lambda = y_{l_1} \cdots y_{l_r}$. Let s_λ be the character of the irreducible representation of S_n characterized as the unique irreducible constituent appearing in the decomposition of both x_λ and y_λ.*

In terms of this notation, $\psi_n = \mathrm{Res}^{S_n}_{W_n}(\mathrm{Ind}^{W_n}_{S_n}(x_n))$.

Proposition 18. *Let π be a representation of S_n. Then*

$$\mathrm{Res}^{S_n}_{W_n}(\mathrm{Ind}^{W_n}_{S_n}(\pi)) = \sum_{a_1+\cdots+a_k=n} \mathrm{Ind}^{S_n}_{S_{a_1}\times\cdots\times S_{a_k}}(\mathrm{Res}^{S_{a_1}\times\cdots\times S_{a_k}}_{S_n}(\pi)).$$

In particular,

$$\psi_n = \sum_{a_1+\cdots+a_k=n} x_{(a_1,\dots,a_k)}.$$

Proof. Notice that if $\tau \in S_n$ and if d is a representative of the double-coset space $S_n \backslash W_{n,k} / S_n$ as in the previous proposition then $d\tau d^{-1} \in S_n$ if and only if τ is block diagonal with block sizes $a_1, \dots, a_k$. Therefore $dS_n d^{-1} \cap S_n = S_{a_1} \times \cdots \times S_{a_k}$ and conjugation by d in fact fixes each element of $S_{a_1} \times \cdots \times S_{a_k}$. The result is then immediate from Mackey's theorem. □

We have the following fact.

Proposition 19. (Jacobi-Trudi Identity) *Let $\lambda = (l_1, \dots, l_r)$ and $\mu = (m_1, \dots, m_s)$ be conjugate partitions of k, i.e. the transpose of the Young diagram of λ is the Young diagram of μ. For $n < 0$ define $x_n = y_n = 0$. Then we have the following equality of representations of S_k:*

$$s_\lambda = \det(x_{l_i - i + j})_{1 \le i,j \le r} = \det(y_{m_i - i + j})_{1 \le i,j \le s}.$$

Proof. See Bump's *Lie Groups*, Chapter 37, Theorem 37.1. □

Proposition 20. *Suppose $|G| = 2$. Then $\psi_n = \sum_{0 \le l \le n/2} (n + 1 - 2l) s_{(n-l,l)}$.*

Proof. The Jacobi-Trudi identity gives

$$s_{(n-l,l)} = det\left(\begin{bmatrix} x_{n-l} & x_{n-l+1} \\ x_{l-1} & x_l \end{bmatrix}\right) = x_{n-l}x_l - x_{n-l+1}x_{l-1},$$

where by definition $x_l = 0$ for $l < 0$. Therefore, using the last proposition we calculate

$$\begin{aligned} & \sum_{0 \le l \le n/2} (n + 1 - 2l) s_{(n-l,l)} \\ = & \sum_{0 \le l \le n/2} (n + 1 - 2l) x_{n-l} x_l - x_{n-l+1} x_{l-1} \\ = & \sum_{0 \le l \le n/2} (n + 1 - 2l) x_{n-l} x_l - \sum_{-1 \le l < n/2} (n - 1 - 2l) x_{n-l} x_l \\ = & \; 2 \sum_{0 \le l < n/2} x_{n-l} x_l + [x_{n/2} x_{n/2}] \\ = & \sum_{a+b=n} x_a x_b \\ = & \; \psi_n, \end{aligned}$$

where the brackets in the fourth line indicate that the term is included exactly when $n/2$ is an integer. □

Definition 4. *Let $M_0(G) = \mathbb{Z}$, for $n > 0$ let $M_n(G)$ be the Grothendieck group of the category of finite dimensional complex representations of W_n, and let*

$$M(G) = \bigoplus_{n \ge 0} M_n(G).$$

$M(G)$ can be seen as the graded free abelian group on a set consisting of 1 along with (representatives of) all the irreducible representations of all the W_n over all n, with addition given by the direct sum. Recall that H is a normal subgroup of G and hence $V_n = S_n \rtimes H^n$ is a normal subgroup of W_n. In particular, if $a + b = n$ then we may view $V_a \times W_b$ as a subgroup of W_b by identifying the element $(\sigma, \tau) \in V_a \times W_b$ with the block diagonal matrix with upper-left block σ and lower-right block τ. Then, analogously with R, we have an action

$$\alpha_{H,G} \colon M(H) \otimes M(G) \to M(G)$$

of $M(H)$ on $M(G)$ defined, for representations π of V_a and ϕ of W_b by

$$\alpha_{H,G}(\pi, \phi) = \mathrm{Ind}_{V_a \times W_b}^{W_n} (\pi \otimes \phi)$$

and extending by linearity. When the meaning is clear from the context, $\alpha_{H,G}$ will sometimes be written using juxtaposition, i.e. as $\alpha_{H,G}(\pi, \phi) = \pi\phi$. Similarly, define the coaction $\alpha^*_{H,G}\colon M(G) \to M(H) \otimes M(G)$ by setting, for a representation ρ of W_n,

$$\alpha^*_{H,G}(\rho) = \sum_{a+b=n} \mathrm{Res}^{V_a \times W_b}_{W_n}(\rho)$$

and extending by linearity. When $H = G$, set $\alpha_G = \alpha_{G,G}$ and $\alpha^*_G = \alpha^*_{G,G}$.

Proofs nearly identical to the proofs of the associativity of m and coassociativity of m^* for R show that $\alpha_{H,G}$ is associative and $\alpha^*_{H,G}$ is coassociative. In other words, we have the following two propositions:

Proposition 21. *$\alpha_{H,G}$ is associative; i.e. the following diagram commutes:*

$$\begin{array}{ccc} M(H) \otimes M(H) \otimes M(H) & \xrightarrow{1 \otimes \alpha_{H,G}} & M(H) \otimes M(G) \\ \downarrow{\scriptstyle \alpha_H \otimes 1} & & \downarrow{\scriptstyle \alpha_{H,G}} \\ M(H) \otimes M(G) & \xrightarrow{\alpha_{H,G}} & M(G) \end{array}$$

Proposition 22. *$\alpha^*_{H,G}$ is coassociative; i.e. the following diagram commutes:*

$$\begin{array}{ccc} M(G) & \xrightarrow{\alpha^*_{H,G}} & M(H) \otimes M(G) \\ \downarrow{\scriptstyle \alpha^*_{H,G}} & & \downarrow{\scriptstyle \alpha^*_H \otimes 1} \\ M(H) \otimes M(G) & \xrightarrow{1 \otimes \alpha^*_{H,G}} & M(H) \otimes M(H) \otimes M(G) \end{array}$$

Zelevinsky proved the following proposition:

Proposition 23. *When $H = G$ in the above setting, $M(G)$ is a PSH-algebra with primitive irreducible elements forming a basis of $M_1(G)$ and corresponding to the irreducible representations of the group G. Let $\Omega(G)$ denote the set of irreducible representations of G. The decomposition theorem for PSH-algebras gives the following isomorphism of PSH-algebras:*

$$M(G) \cong \bigotimes_{\omega \in \Omega(G)} R(\omega).$$

Proof. See Zelevinsky, Chapter II, Proposition 7.2. □

In the rest of this section we will show how the properties of $M(G)$ as a PSH-algebra lead naturally to a more general notion of a particular type of module over a PSH-algebra. In particular, there is a compatibility between $\alpha_{H,G}$ and $\alpha^*_{H,G}$ analogous to the Hopf axiom for m and m^* in R. This is established in the following two propositions.

Proposition 24. *Let $H \trianglelefteq G$, $|H| = h$, let $|G| = k$, let $[G\colon H] = t$, and let $\{g_1, \dots, g_t\}$ be a complete set of representatives for the elements of the quotient group G/H. Suppose $p + q = r + s = n$ and $k = ht$. Then the double-coset space $V_p \times W_q \backslash W_n / V_r \times W_s$ has a complete set of representatives parametrized by tuples $(a_1, \dots, a_t, b, c, d)$ of nonnegative integers satisfying the conditions*

$$a_1 + \cdots + a_t + b = r, c + d = s, a_1 + \cdots + a_t + c = p, b + d = q,$$

where the tuple $(a_1, \dots, a_t, b, c, d)$ *corresponds to the representative*

$$\begin{bmatrix} g_1 I_{a_1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & g_2 I_{a_2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_t I_{a_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & 0 & I_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_d \end{bmatrix}.$$

Proof. The elements of W_n are invertible matrices with exactly one nonzero entry in each row and column, each equal to some $g \in G$. Left-multiplication by $V_p \times W_q$ permutes and scales by elements $h \in H$ the first p rows and permutes and scales by elements $g \in G$ the last q rows. Right-multiplication by $V_r \times W_s$ acts similarly on the first r and last s columns. So, these actions together stabilize the upper-left $p \times r$, lower-left $q \times r$, upper-right $p \times s$, and lower-right $q \times s$ blocks. By scaling the rows and columns and arranging the blocks of rows and columns similarly to the previous analysis of the double-coset space $S_p \times S_q \backslash S_n / S_r \times S_s$, it is clear that the proposition follows. □

Definition 5. *Let A be a Hopf algebra (e.g. a PSH-algebra) with multiplication m and comultiplication m^*. For $t \geq 1$, let $\Psi^t \colon A \to A$ be the Hopf t^{th} power map, i.e. $\Psi^t = m^{(t)} \circ m^{*(t)}$ where $m^{*(t)} \colon A \to A^{\otimes t}$ is the linear map given by*

$$A \xrightarrow{m^*} A \otimes A \xrightarrow{m^* \otimes 1} A \otimes A \otimes A \ \cdots \ \xrightarrow{m^* \otimes 1 \otimes \cdots \otimes 1} A^{\otimes t}$$

and $m^{(t)}$ is defined similarly. Note that by the associativity of m and the coassociativity of m^* the ordering of the 1's and the m^* in each of the maps in the above composition is irrelevant, and similarly for $m^{(t)}$.

Definition 6. *Given a sequence of integers $(n_1, \dots, n_l)$ let $W_{(n_1,\dots,n_l)}$ denote the direct product $W_{n_1} \times \cdots \times W_{n_l}$ and similarly for $V_{(n_1,\dots,n_l)}$.*

In the case $A = M(H)$ with the PSH-algebra structure as mentioned previously, we have the following description of $\Psi^t(\pi)$ for a representation $\pi \in M_n(H)$ of V_n:

$$\Psi^t(\pi) \quad = \sum_{a_1 + \cdots + a_t = n} \mathrm{Ind}_{V_{a_1} \times \cdots \times V_{a_t}}^{V_n} (\mathrm{Res}_{V_n}^{V_{a_1} \times \cdots \times V_{a_t}} (\pi)).$$

We will use the first equality above to prove the following proposition.

Proposition 25. *Suppose again that $|G| = k$, $|H| = h$, and $[G \colon H] = t$. Let G and H have the property that every inner automorphism of G restricts to an inner automorphism of H. Then following diagram commutes:*

$$\begin{array}{ccc} M(H) \otimes M(G) & \xrightarrow{\alpha_H^* \otimes \alpha_{H,G}^*} & M(H) \otimes M(H) \otimes M(H) \otimes M(G) \\ & & \big\downarrow {\scriptstyle \Psi^t \otimes \tau \otimes 1} \\ \big\downarrow {\scriptstyle \alpha_{H,G}} & & M(H) \otimes M(H) \otimes M(H) \otimes M(G) \\ & & \big\downarrow {\scriptstyle \alpha_H \otimes \alpha_{H,G}} \\ M(G) & \xrightarrow{\qquad \alpha_{H,G}^* \qquad} & M(H) \otimes M(G). \end{array}$$

Note that in the case $t = 1$ this is exactly the Hopf axiom for $M(G)$.

Proof. By the linearity of the maps involved, it suffices to verify that the above maps are equal on $\pi\otimes\sigma$, where π is a representation of V_r and σ is a representation of W_s. Suppose $r+s=n$. First we compute $\alpha^*_{H,G}(\alpha_{H,G}(\pi\otimes\sigma))$ using Mackey's Theorem. Recall that the definitions give

$$\alpha^*_{H,G}(\alpha_{H,G}(\pi\otimes\sigma)) \quad = \quad \sum_{p+q=n} \mathrm{Res}^{V_p\times W_q}_{W_n}(\mathrm{Ind}^{W_n}_{V_r\times W_s}(\pi\otimes\sigma)) \; .$$

Let N be the representative of the double-coset $V_p\times W_q\backslash W_n/V_r\times W_s$ as in the previous proposition parameterized by the tuple $(a_1,\dots,a_t,b,c,d)$ subject to the same constraints, and for each $u\le t$ set

$$J_u = \sum_{i\le u} a_i.$$

Let W_n act on G^n by multiplication of matrices on column vectors. Clearly this is a faithful action. For the remainder of this proof, given a subset $S\subset\{1,\dots,n\}$ let $G(S)$ be the subgroup of G^n with the identity element occurring in those slots not indexed by S. Then $N(V_r\times W_s)N^{-1}$ is the subgroup of W_n stabilizing

$$G([1,J_t]\cup[p+1,p+b])$$

and

$$G([J_t+1,p]\cup[p+b+1,n])$$

with the additional constraint that the G-action on entries with indices in the interval $[J_{v-1}+1,J_v]$ sent into the interval $[J_{u-1}+1,J_u]$ is restricted to $g_uHg_v^{-1}$. Therefore, as H is a normal subgroup it follows that

$$(N(V_r\times W_s)N^{-1})\cap V_p\times W_q = V_{(a_1,\dots,a_t,c,b)}\times W_d$$

(note the transposition of b and c).

Let ρ be the representation of the group $V_{(a_1,\dots,a_t,c,b)}\times W_d$ given by $\rho(x)=(\pi\otimes\sigma)(N^{-1}xN)$. Identifying $V_{(a_1,\dots,a_t,c,b)}\times W_d$ with $V_{(a_1,\dots,a_t,b,c)}\times W_d$ via conjugation by N^{-1}, the representation ρ is equivalent to the representation $\mathrm{Res}^{V_{(a_1,\dots,a_t,b)}}_{V_r}(\pi)\otimes\mathrm{Res}^{V_c\times W_d}_{W_s}(\sigma)$ of $V_{(a_1,\dots,a_t,b,c)}\times W_d$ (using the inner automorphism hypothesis). Identify $V_p\times W_q$ with the subgroup of W_n permuting and left-multiplying by H elements of

$$G([1,J_t]\cup[r+1,r+c])$$

and permuting and left-multiplying by G elements of

$$G([J_t+1,r]\cup[r+c+1,n]).$$

Then, by Mackey's theorem $\alpha^*_{H,G}(\alpha_{H,G}(\pi\otimes\sigma))$ equals

$$\sum_{a_1+\cdots a_t+b=r,c+d=s} \mathrm{Ind}^{V_p\times W_q}_{V_{(a_1,\dots,a_t,b,c)}\times W_d}(\mathrm{Res}^{V_{(a_1,\dots,a_t,b)}}_{V_r}(\pi)\otimes\mathrm{Res}^{V_c\times W_d}_{W_s}(\sigma)).$$

Now write a for J_t to simplify the notation slightly. By comparison to going around the diagram the other way, the above calculation shows that it suffices to prove that if γ is a representation of V_a and δ is a representation of V_c then

$$\alpha_H(\Psi^t(\gamma)\otimes\delta) = \sum_{a_1+\cdots+a_k=a} \mathrm{Ind}^{V_p}_{V_{(a_1,\dots,a_t,c)}}(\mathrm{Res}^{V_{(a_1,\dots,a_t)}}_{V_a}(\gamma)\otimes\delta).$$

We have

$$\Psi^t(\gamma) = \sum_{a_1+\cdots+a_t=a} \mathrm{Ind}_{V_{(a_1,\ldots,a_t)}}^{V_a}(\mathrm{Res}_{V_a}^{V_{(a_1,\ldots,a_t)}}(\gamma)),$$

so the left-hand side then becomes a sum over $a_1+\cdots+a_t = a$ of terms of the form

$$\begin{aligned} & \mathrm{Ind}_{V_{(a,c)}}^{V_p}(\mathrm{Ind}_{V_{(a_1,\ldots,a_t)}}^{V_a}(\mathrm{Res}_{V_a}^{V_{(a_1,\ldots,a_t)}}(\gamma))\otimes\delta) \\ = \; & \mathrm{Ind}_{V_{(a,c)}}^{V_a}(\mathrm{Ind}_{V_{(a_1,\ldots,a_t,c)}}^{V_{(a,c)}}(\mathrm{Res}_{V_a}^{V_{(a_1,\ldots,a_t)}}(\gamma)\otimes\delta)) \\ = \; & \mathrm{Ind}_{V_{(a_1,\ldots,a_t,c)}}^{V_p}(\mathrm{Res}_{V_a}^{V_{(a_1,\ldots,a_t)}}(\gamma)\otimes\delta), \end{aligned}$$

precisely as seen on the right-hand side, as desired. □

Note that the previous proof reaffirms that $M(G)$ is indeed a PSH-algebra over itself (since $\Psi^1 = id$).

Definition 7. *Using the notation of the preceding proposition, the compatibility relation between $\alpha_{H,G}$ and $\alpha_{H,G}^*$ will be called the t-compatibility axiom for $M(G)$ as a Hopf module over $M(H)$, and such a Hopf module will be called t-compatible.*

Let us now endow $M(G)$ with notions of positivity and inner product analogous with the structures on R. Let $\Omega(M_0(G)) = \{1\}$, for $n > 0$ let $\Omega(M_n(G))$ be the set of (equivalence classes of) irreducible representations of W_n, and let

$$\Omega(M(G)) = \bigcup_{n\geq 0} \Omega(M_n(G)).$$

Call the elements of $\Omega(M(G))$ the *irreducible* elements of $M(G)$. Clearly $\Omega(M(G))$ and $\Omega(M(H))\times\Omega(M(G))$ form $\mathbb{Z}$-bases of $M(G)$ and $M(H)\otimes M(G)$, respectively, and in this way we again have inner products on $M(G)$ and $M(H)\otimes M(G)$ in which $\Omega(M(G))$ and $\Omega(M(H))\times\Omega(M(G))$ are orthonormal bases. These inner products will again be denoted using angle brackets. Also, define the resulting partial orders on $M(G)$ and $M(H)\otimes M(G)$ analogously with those on R and $R\otimes R$, and hence we have the analogous notions of positivity of both elements and maps between these spaces. We immediately have the following propositions.

Proposition 26. *$\alpha_{H,G}$ and $\alpha_{H,G}^*$ are positive maps.*

Proof. This is simply the statement that restriction and induction send representations to representations. □

Proposition 27. *$\alpha_{H,G}$ and $\alpha_{H,G}^*$ are adjoint maps with respect to the inner products mentioned above.*

Proof. This is a restatement of Frobenius reciprocity. □

$M(G)$ also interacts well with the unit and counit of $M(H)$, the latter considered as a PSH-algebra. Recall that there are graded maps $e_H\colon \mathbb{Z}\to M(H)$ and $e_H^*\colon M(H)\to\mathbb{Z}$ such that the diagram

$$\begin{array}{ccccc} \mathbb{Z}\otimes M(H) & \xrightarrow{e_H\otimes 1} & M(H)\otimes H & \xleftarrow{1\otimes e_H} & M(H)\otimes\mathbb{Z} \\ & \searrow & \downarrow{\scriptstyle \alpha_H} & \swarrow & \\ & & M(H) & & \end{array}$$

and the diagram obtained from the above diagram by reversing all arrows and replacing each map with its adjoint both commute (the unlabeled maps are the canonical isomorphisms). Existence of such maps are called the axioms of unit and counit, respectively. In the case of $M(H)$, $e_H : \mathbb{Z} \to M(H)_0 = \mathbb{Z}$ and $e_H^*|_{M(H)_0} : M(H)_0 \to \mathbb{Z}$ are mutually inverse isomorphisms (this fact is called the axiom of connectivity) and $e^*|_{M(H)_n} = 0$ for $n > 0$. As an abelian group, there is precisely one nontrivial action of $\mathbb{Z}$ on $M(G)$. In particular, it is clear that both the diagram

$$\begin{array}{ccc} \mathbb{Z} \otimes M(H) & \xrightarrow{e_H \otimes 1} & M(H) \otimes M(G) \\ & \searrow & \downarrow{\scriptstyle \alpha_{H,G}} \\ & & M \end{array}$$

as well as the diagram obtained by reversing all arrows and replacing all maps with their adjoints commute.

Definition 8. *The commutativity of the above diagram and the diagram obtained by reversing all arrows and replacing all maps with their adjoints will be referred to as the properties of unit and counit, respectively, for the module M.*

4. Hopf Modules over PSH Algebras

In this section we axiomatize the properties of $M(G)$ seen in the previous section using a particular type of modules over PSH-algebras. Recall that Ψ^t is the Hopf t^{th}-power map. We first establish some of its properties.

Proposition 28. *Ψ^t is a Hopf algebra endomorphism for any commutative cocommutative Hopf algebra (A, m, m^*), i.e. Ψ^t is an endomorphism with respect to both the algebra structure and the coalgebra structure. Furthermore, if A is a PSH-algebra then Ψ^t is a PSH-algebra endomorphism, i.e. a positive Hopf algebra endomorphism.*

Proof. By definition, Ψ^t is the composition

$$A \xrightarrow{m^{*(t)}} A^{\otimes t} \xrightarrow{m^{(t)}} A.$$

$A^{\otimes t}$ has the structure of a Hopf algebra with component-wise multiplication and comultiplication given by

$$a_1 \otimes \cdots \otimes a_t \mapsto a_1^{(1)} \otimes \cdots \otimes a_t^{(1)} \otimes a_1^{(2)} \otimes \cdots \otimes a_t^{(2)},$$

where the above expression uses sumless Sweedler notation. It is immediate from the Hopf axiom that the first map is a ring homomorphism, since it is a composition of ring homomorphisms. Since A is cocommutative m^* is clearly a coalgebra morphism, so it follows that the first map is in fact a Hopf algebra morphism. Since H is commutative, it also follows similarly that the second map is also a Hopf algebra morphism, so the composition Ψ^t is as well, as needed. Positivity in the case that A is a PSH-algebra is immediate from the positivity of m and m^*. $\square$

Proposition 29. *Ψ^t is self-adjoint in any PSH-algebra A. In the case $A = R$, given a partition $\lambda = (l_1, \dots, l_r)$, the element*

$$z_\lambda = z_{l_1} \cdots z_{l_r}$$

is an eigenvector of Ψ^t *with eigenvalue* $t^r = t^{l(\lambda)}$. *In particular, the collection of all* z_λ *as* λ *varies over all partitions forms an orthogonal basis of* R *consisting of eigenvectors of* Ψ^t.

Proof. Self-adjointness of Ψ^t follows immediately from the definition $\Psi^t = m^{(t)} \circ m^{*(t)}$, the adjointness of m and m^*, and the contravariance of the adjoint. Since z_n is primitive, we have

$$\Psi^t(z_n) = tz_n.$$

Therefore, since Ψ^t is a ring homomorphism the proposition follows. □

Definition 9. *A Hopf module* M *over a Hopf algebra* A *is a module which is simultaneously a comodule.*

Definition 10. *A* t*-compatible Hopf module* (M, α, α^*) *over the Hopf algebra algebra* (A, m, m^*), *i.e. for which the diagram*

$$\begin{array}{ccccc} A \otimes M & \xrightarrow{m^* \otimes \alpha^*} & A \otimes A \otimes A \otimes M & \xrightarrow{\Psi^t \otimes \tau \otimes 1} & A \otimes A \otimes A \otimes M \\ \downarrow \alpha & & & & \downarrow m \otimes \alpha \\ M & & \xrightarrow{\qquad \alpha^* \qquad} & & A \otimes M \end{array}$$

commutes, will be called a t*-Hopf module over* A.

The following axiomatizes the key properties of $M(G)$ as a module over $M(H)$.

Definition 11. *A* t*-PSH module (positive self-adjoint* t*-Hopf module)* (M, α, α^*) *over a PSH-algebra* (A, m, m^*) *is a* $\mathbb{Z}^{\geq 0}$*-graded* t*-Hopf module*

$$M = \bigoplus_{n \geq 0} M_n$$

over a PSH-algebra A *with graded associative multiplication and graded coassociative comultiplication* α *and* α^*, *respectively, with a designated basis* Ω_n *of its* n^{th} *homogeneous part (the irreducible elements), for which* α *and* α^* *are positive and adjoint with respect to the inner product induced by the irreducible elements, and satisfying the axioms of unit and counit.*

Similar to the construction for PSH-algebras, we now show the existence of the adjoint map to the map on a t-PSH module M given by multiplication by an element $r \in A$.

Proposition 30. *Let* M *be a* t*-PSH module over the PSH-algebra* A *as in the previous definition. For every* $r \in A$ *there is a linear map* $\widetilde{r}\colon M \to M$ *such that*

$$\langle rm, n \rangle = \langle m, \widetilde{r}(n) \rangle$$

for all $m, n \in M$. *This map is clearly uniquely determined. Furthermore, this association satisfies the following properties:*

$$(1) \quad r \in A_p \Rightarrow \widetilde{r}(M_q) \subset M_{q-p}$$

$$(2) \quad \widetilde{r} \circ \widetilde{s} = \widetilde{sr} = \widetilde{rs} = \widetilde{s} \circ \widetilde{r}$$

$$(3) \quad m^*(r) = \sum_i r_i' \otimes r_i'' \Rightarrow \widetilde{r}(sy) = \sum_i \Psi^t(r_i')^*(s)\widetilde{r_i''}(y),$$

for all $s \in A$.

Positivity of r *implies positivity of* $\widetilde{r}$.

Proof. Let $r \in A$ and let ϕ_r be the linear functional on A defined by taking the inner product with r, i.e. $\phi_r(s) = \langle r, s\rangle$. Let $\widetilde{\widetilde{r}}\colon M \to M$ be the following composition:

$$M \xrightarrow{\alpha^*} A \otimes M \xrightarrow{\phi_r \otimes 1} \mathbb{Z} \otimes M \cong M.$$

Clearly $\widetilde{\widetilde{r}}$ is a linear map. Let $m, n \in M$. Since α and α^* are adjoint, we have $\langle rm, n\rangle = \langle r \otimes m, \alpha^*(n)\rangle$. Using sumless Sweedler notation, we have

$$\begin{aligned} \langle r \otimes m, \alpha^*(n)\rangle &= \langle r, n^{(1)}\rangle\langle m, n^{(2)}\rangle \\ &= \langle m, \phi_r(n^{(1)})n^{(2)}\rangle \\ &= \langle m, \widetilde{\widetilde{r}}(n)\rangle. \end{aligned}$$

Therefore, $\widetilde{\widetilde{r}}$ satisfies the defining property for $\widetilde{r}$, so the map exists for all r and is given by the composition above. Furthermore, it is immediate from the definition that the inner product $\langle m, \widetilde{r}(n)\rangle$ is determined for all m, and this clearly forces the uniqueness of $\widetilde{r}$.

The first property is immediate from the grading of A and M, and the second property follows from the contravariance of the adjoint and the commutativity of A. Let $r, s \in A$ and $y, z \in M$. It is clear that the adjoint map to multiplication on $A \otimes M$ by $a \otimes b \in A \otimes A$ is $a^* \otimes \widetilde{b}$, so we have we have

$$\begin{aligned} \langle \widetilde{r}(sy), z\rangle &= \langle sy, rz\rangle \\ &= \langle s \otimes y, \alpha^*(\alpha(r \otimes z))\rangle \\ &= \langle s \otimes y, (m \otimes \alpha) \circ (\Psi^t \otimes \tau \otimes 1) \circ (m^* \otimes \alpha^*)(r \otimes z)\rangle \\ &= \langle s \otimes y, \textstyle\sum_i (m \otimes \alpha) \circ (\Psi^t \otimes \tau \otimes 1)(r'_i \otimes r''_i \otimes \alpha^*(z)) \\ &= \textstyle\sum_i \langle s \otimes y, (\Psi^t(r'_i) \otimes r''_i)\alpha^*(z)\rangle \\ &= \textstyle\sum_i \langle \Psi^t(r'_i)^*(s) \otimes \widetilde{r''_i}(y), \alpha^*(z)\rangle \\ &= \langle \textstyle\sum_i \Psi^t(r'_i)^*(s)\widetilde{r''_i}(y), z\rangle. \end{aligned}$$

Since z is arbitrary the third claim follows.

Positivity of $\widetilde{r}$ when r is positive follows immediately from the explicit formula for $\widetilde{r}$ and the fact that the composition of positive maps is a positive map. □

A similar construction can also be made by considering the multiplication α as inducing, for each $x \in M$, a linear map $A \to M$. It is natural to consider whether there is a well-defined adjoint map in this case as well, which instead would be a map $M \to A$. We have the following proposition.

Proposition 31. *Let M be a t-PSH module over the PSH-algebra A with multiplication and comultiplication α and α^*, respectively. For every $x \in M$ there is a unique linear map $\widetilde{x}\colon M \to A$ such that*

$$\langle rx, y\rangle = \langle r, \widetilde{x}(y)\rangle$$

for all $r \in A$, $y \in M$, and this unique map is given by the composition

$$M \xrightarrow{\alpha^*} A \otimes M \xrightarrow{1 \otimes \phi_x} A \otimes \mathbb{Z} \cong A,$$

where $\phi_x\colon M \to \mathbb{Z}$ is the linear functional given by $\phi_x(y) = \langle x, y\rangle$. This association of maps with elements satisfies the following properties:

$$(1) \quad x \in M_p \Rightarrow \widetilde{x}(M_q) \subset A_{q-p}$$

$$(2) \quad \widetilde{rx} = r^* \circ \widetilde{x} = \widetilde{x} \circ \widetilde{r}$$

$$\text{(3)}\quad \alpha^*(x) = \sum_i a_i \otimes b_i \Rightarrow \widetilde{x}(ry) = m((\Psi^t \otimes 1)(\widetilde{\alpha^*(x)}(r \otimes y))) = \sum_i \Psi^t(a_i^*(r))\widetilde{b_i}(y)$$

$$\text{(4)}\quad m^*(r) = \sum_i r_i' \otimes r_i'' \Rightarrow \widetilde{x}(ry) = \sum_i \Psi^t(r_i')\widetilde{\widetilde{r_i''}(x)}(y)$$

$$\text{(5)}\quad \widetilde{x+y} = \widetilde{x} + \widetilde{y}.$$

Again, positivity of x implies positivity of $\widetilde{x}$.

Proof. Again, we use sumless Sweedler notation. We have

$$\begin{aligned}\langle rx, y\rangle &= \langle r \otimes x, \alpha^*(y)\rangle \\ &= \langle r, y^{(1)}\rangle\langle x, y^{(2)}\rangle \\ &= \langle r, \phi_x(y^{(2)})y^{(1)}\rangle,\end{aligned}$$

so the composition above has the desired property of $\widetilde{x}$. Uniqueness is clear.

The first property is immediate from the compatibility of the grading structure on A and M, and the second property is clear from the contravariance of the adjoint and the calculation

$$\begin{aligned}\langle \widetilde{rx}(y), s\rangle &= \langle y, srx\rangle \\ &= \langle \widetilde{x}(y), sr\rangle \\ &= \langle \widetilde{x}(y), rs\rangle \\ &= \langle (r^* \circ \widetilde{x})(y), s\rangle\end{aligned}$$

for all $s \in A$.

Using t-Hopf compatibility and the self-adjointness of Ψ^t, we see

$$\begin{aligned}\langle \widetilde{x}(ry), s\rangle &= \langle ry, sx\rangle \\ &= \langle r \otimes y, (\alpha^* \circ \alpha)(s \otimes x)\rangle \\ &= \langle r \otimes y, (\Psi^t \otimes 1)(m^*(s))\alpha^*(x)\rangle \\ &= \langle \widetilde{\alpha^*(x)}(r \otimes y), (\Psi^t \otimes 1)(m^*(s))\rangle \\ &= \langle m((\Psi^t \otimes 1)(\widetilde{\alpha^*(x)}(r \otimes y))), s\rangle,\end{aligned}$$

giving the first equality of the third property. The second equality follows immediately from the fact that the adjoint map to right-multiplication on $A \otimes A$ by $a \otimes b \in A \otimes M$ is $a^* \otimes \widetilde{b}$.

By the previous proposition we have

$$\begin{aligned}\langle \widetilde{x}(ry), s\rangle &= \langle ry, sx\rangle \\ &= \langle y, \widetilde{r}(sx)\rangle \\ &= \langle y, \sum_i \Psi^t(r_i')^*(s)\widetilde{r_i''}(x)\rangle \\ &= \langle \widetilde{\widetilde{r_i''}(x)}(y), \sum_i \Psi^t(r_i')^*(s)\rangle \\ &= \langle \sum_i \Psi^t(r_i')\widetilde{\widetilde{r_i''}(x)}(y), s\rangle,\end{aligned}$$

from which the fourth property follows.

The fifth property is clear from the composition above and linearity. Positivity of $\widetilde{x}$ when x is positive follows immediately from the explicit formula for $\widetilde{x}$ and the fact that the composition of positive maps is a positive map. ∎

For an application of the previous proposition, note that for $1 \in M(G)$ Frobenius reciprocity says that $\widetilde{1}$ acts on $M_n(G)$ as restriction from W_n to V_n. In particular, if $\pi \in M(H)_n$ is a representation of V_n then $\widetilde{1}(r \cdot 1) = \mathrm{Res}^{V_n}_{W_n}(\mathrm{Ind}^{W_n}_{V_n}(\pi))$. The previous proposition gives $\widetilde{1}(r \cdot 1) = \Psi^t(1^*(r))\widetilde{1}(1) = \Psi^t(r)$, confirming the identity

$$\mathrm{Res}^{V_n}_{W_n}(\mathrm{Ind}^{W_n}_{V_n}(\pi)) = \bigoplus_{a_1+\cdots a_t=n} \mathrm{Ind}^{V_n}_{V_{(a_1,\ldots,a_t)}}(\mathrm{Res}^{V_{(a_1,\ldots,a_t)}}_{V_n}(\pi)).$$

Recall that the group P of *primitive* elements in a PSH-algebra A is the orthogonal complement in I of the subgroup I^2, where

$$I = \bigoplus_{n>0} A_n.$$

Observe that if $\pi \in R_n$ is a representation then π is primitive if and only if the restriction of π to all proper subgroups $S_r \times S_s$ of S_n is zero. We now formulate a parallel definition of primitivity for t-PSH modules over a PSH-algebra A.

Definition 12. *Let (M, α, α^*) be a t-PSH module over A. Define the subgroup Q of primitive elements in M to be the orthogonal complement of the subgroup IM in M.*

From the adjointness of α and α^* and the counit axiom, it follows that $q \in Q$ if and only if $\alpha^*(q) = 1 \otimes q$.

Proposition 32. *Let (M, α, α^*) be a t-PSH module over the PSH-algebra A, and let $p_i, p'_j \in P$, $1 \le i \le r$, $1 \le j \le s$, and $x, y \in M$ be primitive, each pair either equal or orthogonal (e.g. primitive irreducibles). Set $\pi = p_1 \cdots p_r x$ and $\pi' = p'_1 \cdots p'_s y$. Then we have*

$$\langle \pi, \pi' \rangle = 0$$

unless $x = y$ and the p_i's and p'_j's are equal up to rearrangement, in which case we have

$$\langle \pi, \pi' \rangle = t^r n_1! \cdots n_t! \langle p_1, p_1 \rangle \cdots \langle p_r, p_r \rangle \langle x, x \rangle,$$

where n_i is the number of appearances of the i^{th} distinct element in the list $p_1, \ldots, p_r$.

Proof. Recall that if $p \in A$ is primitive then p^* is a derivation. We first obtain a similar fact about $\widetilde{p}$ using the preceding proposition. Suppose $p \in P$. Then $m^*(p) = p \otimes 1 + 1 \otimes p$, so given $r \in A$, $x \in M$ we have

$$\widetilde{p}(rm) = \Psi^t(p)^*(r)m + r\widetilde{p}(m) = tp^*(r)m + r\widetilde{p}(m).$$

Therefore, we calculate

$$\begin{aligned}
\langle \pi, \pi' \rangle &= \langle (\Pi_{i \le r} p_i)x, (\Pi_{j \le s} p'_j)y \rangle \\
&= \langle (\Pi_{2 \le i \le r} p_i)x, \widetilde{p_1}((\Pi_{j \le s} p'_j)y) \rangle \\
&= \langle (\Pi_{2 \le i \le r} p_i)x, t(p_1)^*(\Pi_{j \le s} p'_j)y + (\Pi_{j \le s} p'_j)\widetilde{p_1}(y) \rangle \\
&= \sum_{1 \le l \le s} t \langle (\Pi_{2 \le i \le r} p_i)x, (\Pi_{j \le s, j \ne l} p'_j) p_1^*(p_l) y \rangle \\
&\quad + \langle (\Pi_{2 \le i \le r} p_i)x, (\Pi_{j \le s} p'_j)\widetilde{p_1}(y) \rangle
\end{aligned}$$

Next, suppose $p \in A_n$ is a primitive element. By the definition of primitivity in M it follows immediately that $p^*(y) = 0$ (for this all that is needed is that $p \in I$. So the final term above is zero, leaving only the sum. From Zelevinsky we know that if $r \in A_l$, $l > 0$, then $r^*(p) = 0$ unless $n = l$, in which case $r^*(p) = \langle r, p \rangle$. The pairwise orthogonality condition thus implies that if $p_i \ne p'_j$ then $p_i^*(p'_j) = 0$.

Therefore, if n_1 is the number of appearances of p_1 in the list $p'_1, \ldots, p'_s$ then the above expression becomes

$$kn_1\langle(\Pi_{2\leq i\leq r}p_i)x, (\Pi_{2\leq j\leq s}p'_j)y\rangle,$$

and the proposition follows by induction on r. □

Keep for now the notation that (M, α, α^*) is a t-PSH module over A. Let $\mathcal{C} = \Omega(A)\cap P$ be the set of primitive irreducible elements of A and let $\mathcal{D} = \Omega(M)\cap Q$ be the set of primitive irreducible elements of M. Borrowing yet more notation from Zelevinsky, let $S(\mathcal{C}, \mathbb{Z}^{\geq 0})$ denote the additive monoid of functions $\mathcal{C} \to \mathbb{Z}^{\geq 0}$ of finite support. Given $d \in \mathcal{D}$ and $\phi \in S(\mathcal{C}, \mathbb{Z}^{\geq 0})$, define

$$\pi_\phi = \prod_{c\in\mathcal{C}} c^{\phi(c)} \in A, \quad \pi_{d,\phi} = \pi_\phi d \in M.$$

Let $\Omega(\phi)$ be the set of irreducibles in $\omega \in \Omega(A)$ such that $\omega \leq \pi_\phi$, let $\Omega(d, \phi)$ be the set of irreducibles $\omega' \in \Omega(M)$ such that $\omega' \leq \pi_{d,\phi}$. Finally, set

$$A(\phi) = \bigoplus_{\omega\in\Omega(\phi)} \mathbb{Z}\omega, \quad M(d, \phi) = \bigoplus_{\omega\in\Omega(d,\phi)} \mathbb{Z}\omega$$

and

$$M(d) = \bigoplus_{\phi\in S(\mathcal{C},\mathbb{Z}^{\geq 0})} M(d, \phi).$$

Proposition 33. *Let $x, x' \in \mathcal{D}$ and $\phi, \phi' \in S(\mathcal{C}, \mathbb{Z}^{\geq 0})$. Then $\Omega(x, \phi)$ and $\Omega(x', \phi')$ are disjoint unless $(x, \phi) = (x', \phi')$. Also, we have the following equality of abelian groups:*

$$M = \bigoplus_{d\in\mathcal{D},\phi\in S(\mathcal{C},\mathbb{Z}^{\geq 0})} M(d, \phi).$$

Furthermore, M is graded with respect to ϕ in the sense that

$$\alpha(A(\phi') \otimes M(x, \phi'')) \subset M(x, \phi' + \phi'')$$

and

$$\alpha^*(M(x, \phi)) \subset \bigoplus_{\phi'+\phi''=\phi} A(\phi) \otimes M(x, \phi'').$$

In particular, $M(d)$ is a t-PSH submodule of M and

$$M = \bigoplus_{d\in\mathcal{D}} M(d).$$

Proof. By the previous proposition we have

$$\langle\pi_{x,\phi}, \pi_{x',\phi'}\rangle = 0$$

unless $x = x'$ and $\phi = \phi'$. Disjointness of $\Omega(x, \phi)$ and $\Omega(x', \phi')$ then follows from the positivity of $\pi_{x,\phi}$ and $\pi_{x',\phi'}$.

Next, let $\omega \in \Omega_M$. It remains to show that $\omega \in \pi_{x,\phi}$ for some x, ϕ. This is trivial if $\omega \in \mathcal{D}$ (take $\phi = 0$). Then since $M_0 \subset Q$ it suffices to consider $\omega \in \oplus_{n>0}M_n$ and ω not primitive. But then ω is not in the orthogonal complement of IM, so there exists $r \in I$, $x \in M$ such that $\omega \leq rx$. By positivity we may assume $r \in \Omega(A)$, $x \in \Omega(M)$, and $r \neq 1$. Zelevinsky showed $r \leq \pi_\phi$ for some $\phi \in S(\mathcal{C}, \mathbb{Z}^{\geq 0})$, and by induction on $\deg\omega$, we can assume $x \leq \pi_{y,\phi'}$ for some ϕ'. So, by positivity we see

$$\omega \leq rx \leq \pi_\phi\pi_{y,\phi'} = \pi_{y,\phi+\phi'}.$$

This proves the direct sum decomposition for M into abelian groups.

The grading statement for α follows immediately from the definitions and positivity. The grading statement for α^* then follows by the adjointness of α and α^*. It follows from these that $M(d)$ is a t-PSH submodule for all $d \in \mathcal{D}$ and that we have the final direct sum decomposition. □

Proposition 34. *Let $\phi, \phi' \in S(\mathcal{C}, \mathbb{Z}^{\geq 0})$ have disjoint supports, let $x \in \mathcal{D}$, let $r_1, r_2 \in \Omega(\phi)$, and let $\omega_1, \omega_2 \in \Omega(x, \phi')$. Then*

$$\langle r_1\omega_1, r_2\omega_2\rangle = \langle \Psi^t(r_1), r_2\rangle \delta_{\omega_1,\omega_2}.$$

Furthermore, if $a, b \in A(\phi)$ and then

$$\langle a\omega_1, b\omega_2\rangle = \langle \Psi^t(a), b\rangle \delta_{\omega_1,\omega_2} = \langle a, \Psi^t(b)\rangle \delta_{\omega_1,\omega_2} = \langle b\omega_1, a\omega_2\rangle.$$

Proof. Let $m^*(r_1) = \sum_i s_i' \otimes s_i''$ and $\alpha^*(\omega_1) = \sum_j u_j \otimes m_j$. Then we have

$$\begin{aligned} \langle r_1\omega_1, r_2\omega_2\rangle &= \langle \alpha^*(\alpha(r_1 \otimes \omega_1)), r_2 \otimes \omega_2\rangle \\ &= \textstyle\sum_{i,j} \langle \Psi^t(s_i')u_j \otimes s_i'' m_j, r_2 \otimes \omega_2\rangle. \end{aligned}$$

Since $r_1 \in \Omega(\phi)$ and $\omega_1 \in \Omega(x, \phi')$ it follows from the previous proposition that $s_i' \in A(\phi_0^{(i)})$ and $s_i'' \in A(\phi_1^{(i)})$ with $\phi_0^{(i)} + \phi_1^{(i)} = \phi$ and $u_j \in A(\phi_0^{(j)\prime})$ and $m_j \in M(x, \phi_1^{(j)\prime})$ with $\phi_0^{(j)\prime} + \phi_1^{(j)\prime} = \phi'$. Now it also follows from the previous proposition that Ψ^t restricts to a map $A(\sigma) \to A(\sigma)$ for all $\sigma \in S(\mathcal{C}, \mathbb{Z}^{\geq 0})$. In particular, $\Psi^t(s_i') \in A(\phi_0^{(i)})$, so

$$\Psi^t(s_i')u_j \otimes s_i'' m_j \in A(\phi_0^{(i)} + \phi_0^{(j)\prime}) \otimes M(x, \phi_1^{(i)} + \phi_1^{(j)\prime}).$$

Therefore, the orthogonality relations proved in the last proposition imply that this term is orthogonal to $r_2 \otimes \omega_2$ unless $\phi_0^{(i)} + \phi_0^{(j)\prime} = \phi$ and $\phi_1^{(i)} + \phi_1^{(j)\prime} = \phi'$. In particular, this term can contribute to inner product only if there exist $\phi_0^{(i)}$, $\phi_0^{(j)\prime}$, $\phi_1^{(i)}$, and $\phi_1^{(j)\prime}$ satisfying the following system of equations:

$$\phi_0^{(i)} + \phi_0^{(j)\prime} = \phi = \phi_0^{(i)} + \phi_1^{(i)}$$
$$\phi_1^{(i)} + \phi_1^{(j)\prime} = \phi' = \phi_0^{(j)\prime} + \phi_1^{(j)\prime}.$$

Since ϕ and ϕ' have disjoint supports it follows that this system has the unique solution

$$\phi_0^{(j)\prime} = \phi_1^{(i)} = 0, \quad \phi_0^{(i)} = \phi, \quad \phi_1^{(j)\prime} = \phi'.$$

In other words, only the term $\Psi^t(r_1) \otimes \omega_1$ contributes. It follows that

$$\begin{aligned} \langle r_1 \otimes \omega_1, r_2 \otimes \omega_2\rangle &= \langle \Psi^t(r_1) \otimes \omega_1\rangle\langle r_2 \otimes \omega_2\rangle \\ &= \langle \Psi^t(r_1), r_2\rangle\langle \omega_1, \omega_2\rangle \\ &= \langle \Psi^t(r_1), r_2\rangle \delta_{\omega_1,\omega_2}. \end{aligned}$$

The final string of identities then follows from the self-adjointness of Ψ^t. □

We have seen that a t-PSH module can always be written as the direct sum of t-PSH modules with a single primitive irreducible element. Therefore, to understand the structure of t-PSH modules it suffices to understand the structure of those with one primitive irreducible element. In particular, we may in fact assume that this primitive irreducible element has degree 0. To see this, let M be a t-PSH module over A with a single primitive irreducible element δ. Observe that the decomposition theorem for t-PSH modules implies that each homogeneous part of

M of grading lower than δ is the trivial group, and a constant shift of grading of M respects all of the defining properties of M, so without loss of generality $\deg(\delta) = 0$. By the definition of primitivity in M and the grading of α^*, it follows that M_0 is a subgroup of the group of primitive elements of M. In particular, since δ is the unique primitive irreducible it follows that $\Omega(M_0) = \{\delta\}$ so $M_0 \cong \mathbb{Z}$. Therefore, we may identify M_0 with $\mathbb{Z}$ and by positivity we may identify this unique primitive irreducible element with $1 \in \mathbb{Z}$. It will be clear from context whether $\mathbb{Z}$ is identified with A_0, M_0, or otherwise.

Proposition 35. *Let (M, μ, μ^*) be a Hopf module with the s-compatibility axiom over a Hopf algebra (A, α, α^*). Let (B, β, β^*) be another Hopf algebra, and suppose we have Hopf algebra morphisms $\delta\colon B \to A$ and $\delta^*\colon A \to B$ (i.e. δ and δ^* are both simultaneously algebra and coalgebra morphisms) which satisfy the identity*

$$\delta^* \circ \delta = \Psi^t.$$

Define the action $\nu\colon B \otimes M \to M$ as the composition

$$\nu\colon B \otimes M \xrightarrow{\delta\otimes 1} A \otimes M \xrightarrow{\mu} M$$

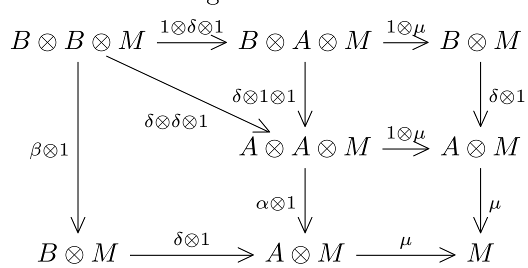
and the coaction $\nu^*\colon M \to B \otimes M$ as the composition

$$\nu^*\colon M \xrightarrow{\mu^*} A \otimes M \xrightarrow{\delta^*\otimes 1} B \otimes M.$$

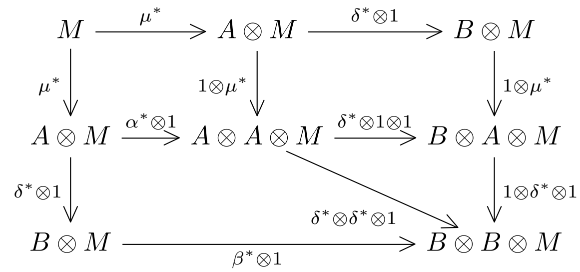
Then (M, ν, ν^*) is a Hopf module with the st-compatibility axiom.

In particular, if we have the further hypotheses that (M, μ, μ^*) is a s-PSH module, that (A, α, α^*) and (B, β, β^*) are PSH-algebras, and that δ and δ^* are positive and adjoint, then (M, ν, ν^*) is a st-PSH module over B.

Proof. The unit and counit axioms for (M, ν, ν^*) follow from the corresponding properties of the maps involved in the above compositions. To show associativity of ν, it suffices to show that the diagram:



commutes. The upper-left triangle commutes trivially, and the trapezoid below it commutes because δ is a morphism of algebras. The upper-right square commutes trivially, and the lower-left square commutes because μ is associative. Since δ^* is a coalgebra morphism, identical reasoning shows that the diagram



also commutes, so ν^* is coassociative.

Only the *st*-compatibility axiom for ν and ν^* remains. For this, it suffices to show commutativity of the diagram

$$\begin{array}{ccccc}
B\otimes M & \xrightarrow{\beta^*\otimes\mu^*} & B\otimes B\otimes A\otimes M & \xrightarrow{1^{\otimes 2}\otimes\delta^*\otimes 1} & B\otimes B\otimes B\otimes M \\
\downarrow{\scriptstyle \delta\otimes 1} & & \downarrow{\scriptstyle (\delta)^{\otimes 2}\otimes 1\otimes 1} & & \downarrow{\scriptstyle \Psi^{st}\otimes\tau\otimes 1} \\
A\otimes M & \xrightarrow[\alpha^*\otimes\mu^*]{} & A\otimes A\otimes A\otimes M & & B\otimes B\otimes B\otimes M \\
\Big\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \Psi^{s}\otimes\tau\otimes 1} & & \downarrow{\scriptstyle 1^{\otimes 2}\otimes\delta\otimes 1} \\
 & & A\otimes A\otimes A\otimes M & \xrightarrow{(\delta^*)^{\otimes 2}\otimes 1^{\otimes 2}} & B\otimes B\otimes A\otimes M \\
\Big\downarrow & & \downarrow{\scriptstyle \alpha\otimes\mu} & & \downarrow{\scriptstyle \beta\otimes\mu} \\
M & \xrightarrow{\mu^*} & A\otimes M & \xrightarrow{\delta^*\otimes 1} & B\otimes M.
\end{array}$$

The lower-left square commutes by the s-Hopf axiom for M, the upper-left square commutes since δ is a coalgebra morphism, and the lower-right square commutes since δ^* is an algebra morphism. Finally, since δ^* is an algebra morphism and δ is a coalgebra morphism we have

$$\begin{aligned}
\delta^*\circ\Psi^s\circ\delta &= \delta^*\circ\alpha^{(s)}\circ\alpha^{*(s)}\circ\delta \\
&= \beta^{(s)}\circ(\delta^*)^{\otimes s}\circ\delta^{\otimes s}\circ\beta^{*(s)} \\
&= \beta^{(s)}\circ(\Psi^t)^{\otimes s}\circ\beta^{*(s)} \\
&= \beta^{(st)}\circ\beta^{*(st)} \\
&= \Psi^{st},
\end{aligned}$$

so the upper-right square commutes, as needed.

The additional statement in the PSH setting follows immediately. □

If $M = A$ is a PSH-algebra viewed as a module over itself with the typical action and coaction, the previous proposition shows how to build a t-compatible PSH module out of such maps δ and δ^*. The next proposition shows that the k-PSH module $M(G)$ ($k = |G|$) over R arises in this manner.

Proposition 36. *Recall that as a PSH-algebra $M(G)$ has the structure*

$$M(G) \cong \bigotimes_{\omega\in\Omega(G)} R,$$

where $\Omega(G) = \{\omega_1, \ldots, \omega_p\}$ is a complete set of the irreducible representations of G. Let $\delta\colon R \to M(G)$ be the graded linear map defined on the n^{th} homogeneous part by

$$\delta(\pi) = \mathrm{Ind}_{S_n}^{W_n}(\pi)$$

for representations π of S_n, and let $\delta^*\colon M(G) \to R$ be the graded linear map defined by

$$\delta^*(\sigma) = \mathrm{Res}_{W_n}^{S_n}(\sigma)$$

for representations σ of W_n. If $\epsilon \in M_0(G)$ is the identity element of $M(G)$ as a PSH-algebra, then δ is right-multiplication by ϵ with respect to the module product and δ^* is its adjoint, the map $\widetilde{\epsilon}$. δ and δ^* are mutually adjoint PSH-algebra morphisms and are given by

$$\delta = (m^{(d_1)}\otimes\cdots\otimes m^{(d_p)})\circ m^{*(d)}$$

and

$$\delta^* = m^{(d)} \circ (m^{*(d_1)} \otimes \cdots \otimes m^{*(d_p)}),$$

where $d_i = \dim(\omega_i)$ *and* $d = \sum d_i$. *Furthermore,* δ *and* δ^* *satisfy the relation*

$$\delta^* \circ \delta = \Psi^k,$$

and the resulting k*-PSH module structure of* $M(G)$ *over* R *given by the previous proposition is precisely the* k*-PSH module structure of* $M(G)$ *over* R *described in the previous section.*

Proof. Let μ and μ^* be the multiplication and comultiplication in $M(G)$ as a PSH-algebra and let ν and ν^* be the multiplication and comultiplication in $M(G)$ as a k-PSH module over R. Then by the definition of ν, δ is given by right-multiplication by ϵ, as claimed. The corresponding claim about δ^* follows immediately from Frobenius reciprocity. The fact that these maps are in fact PSH-algebra morphisms was proven by Zelevinsky (see Zelevinsky, Chapter II, Proposition 7a.

The relation

$$\delta^* \circ \delta = \Psi^k$$

follows from the formulas derived earlier for the action on products of the adjoint map to right-multiplication by elements of a k-PSH module; in particular, since $\nu^*(\epsilon) = 1 \otimes \epsilon$, for any $r \in R$ we have

$$\begin{aligned} \delta^*(\delta(r)) &= \widetilde{\epsilon}(r \cdot \epsilon) \\ &= \Psi^k(1^*(r))\widetilde{\epsilon}(\epsilon) \\ &= \Psi^k(r), \end{aligned}$$

as needed.

To verify the coordinate-wise expressions for δ and δ^* we need first to make the coordinate expression of $M(G)$ above precise. Consider $M(G)$ as

$$M(G) = \bigotimes_{i=1}^{p} R$$

where the i^{th} tensor factor corresponds to the irreducible representation ω_i of G. Zelevinsky proved (see Chapter II, Section 7) an explicit such isomorphism as follows. Let X_i be the space of ω_i. Let $\phi_i \colon R \to M(G)$ be the map sending a representation (π, Y) of S_n to the representation $\phi_i(\pi)$ of W_n acting on the space $Y \otimes \bigotimes^n X_i$ by

$$\phi_i(\pi)(g_1, \dots, g_n)(y \otimes v_1 \otimes \cdots v_n) = y \otimes \omega_i(g_1)(v_1) \otimes \cdots \otimes \omega_n(g_n)(v_n)$$

for $(g_1, \dots, g_n) \in G^n$ and

$$\phi_i(\pi)(\sigma)(y \otimes v_1 \otimes \cdots \otimes v_n) = \pi(\sigma)(y) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

for $\sigma \in S_n$. Zelevinsky proved that the map

$$\phi \colon \bigotimes_{i=1}^{p} R \to M(G), \quad r_1 \otimes \cdots \otimes r_p \mapsto \phi_1(r_1) \cdots \phi_p(r_p)$$

is an isomorphism of PSH-algebras. Since δ and δ^* are PSH-algebra morphisms, it suffices to check the desired expression on a set of algebra generators of $\bigotimes^p R$. We will check this for δ^* first.

Since the set $\{x_1, x_2 \dots\}$ generates R as an algebra, the elements

$$1 \otimes \cdots \otimes x_n \otimes \cdots \otimes 1$$

generate $\bigotimes^p R$. Note that

$$m^{(d)}((m^{*(d_1)} \otimes \cdots \otimes m^{*(d_p)})(1 \otimes \cdots \otimes x_n \otimes \cdots \otimes 1)) = \Psi^{d_i}(x_n).$$

Let $\sigma \in S_n$. Recall that the character of this representation is given by

$$\sigma \mapsto d_i^{cycles(\sigma)}.$$

It is now sufficient to show that this is also the character of the representation

$$\phi(1 \otimes \cdots \otimes x_n \otimes \cdots \otimes 1) = \phi_i(x_n)$$

restricted to S_n. Since the space of x_n is $\mathbb{C}$ with the trivial action, the space of $\phi_i(x_n)$ is $\mathbb{C} \otimes \bigotimes^n X_i = \bigotimes^n X_i$. Choose a basis $v_1, \dots v_{d_i}$ of X_i. Then the set of all simple tensors of the form

$$v_{j_1} \otimes \cdots \otimes v_{j_n}$$

is a basis of $\bigotimes^n X_i$. Since $\phi(\sigma)$ acts by permuting these elements, to compute the character we need only count the number of fixed elements. Any such basis vector can be fixed by σ only if $j_r = j_s$ for all r and s in the same cycle of σ. Conversely, any basis vector which has this property for all r and s in the same cycle is fixed by $\phi(\sigma)$. Therefore if $cycles(\sigma) = c$ then the set of such basis vectors is indexed by c-tuples of integers $j \leq d_i$, and there are d_i^c such c-tuples, so we have the needed formula for δ^*. The formula for δ is then obtained by taking adjoints. □

The previous proposition allows the structures of $M(G)$ as a k-PSH module over R to be computed entirely in terms of the PSH-algebra structure of R. The following propositions make this description of $M(G)$ in terms of R even more precise.

Proposition 37. *Let (M, μ, μ^*) be a s-compatible Hopf module and let (N, ν, ν^*) be a t-compatible Hopf module, both over the commutative, cocommutative Hopf algebra (A, α, α^*). Then with the multiplication*

$$\beta\colon A \otimes M \otimes N \to M \otimes N, \quad a \otimes m \otimes n \mapsto a^{(1)}m \otimes a^{(2)}n$$

and the comultiplication

$$\beta^*\colon M \otimes N \to A \otimes M \otimes N, \quad m \otimes n \mapsto m^{(1)}n^{(1)} \otimes m^{(2)} \otimes n^{(2)}$$

$(M \otimes N, \beta, \beta^*)$ is a $(s+t)$-compatible Hopf module over A. If all objects involved are of the PSH type, then so is the module $M \otimes N$.

Proof. The result that $M \otimes N$ is a Hopf module is standard. We need only check the $(s+t)$-compatibility axiom. Recall that this amounts to checking the equality

$$\beta^* \circ \beta = (\alpha \otimes \beta) \circ (\Psi^{s+t} \otimes \tau \otimes 1) \circ (\alpha^* \otimes \beta^*).$$

Indeed, for $a \otimes m \otimes n \in A \otimes M \otimes N$, we see

$$\begin{aligned}
& (\beta^* \circ \beta)(a \otimes m \otimes n)) \\
= \; & \beta^*(\mu(a^{(1)} \otimes m) \otimes \nu(a^{(2)} \otimes n)) \\
= \; & ((\alpha \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1))(\mu^*(\mu(a^{(1)} \otimes m)) \otimes \nu^*(\nu(a^{(2)} \otimes n))) \\
= \; & ((\alpha \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1))(\Psi^s(a^{(1)(1)})m^{(1)} \otimes a^{(1)(2)}m^{(2)} \otimes \\
& \qquad \Psi^t(a^{(2)(1)})n^{(1)} \otimes a^{(2)(2)}n^{(2)}) \\
= \; & \Psi^s(a^{(1)(1)})m^{(1)}\Psi^t(a^{(2)(1)})n^{(1)} \otimes a^{(1)(2)}m^{(2)} \otimes a^{(2)(2)}n^{(2)} \\
= \; & \Psi^s(a^{(1)(1)})\Psi^t(a^{(1)(2)})m^{(1)}n^{(1)} \otimes a^{(2)(1)}m^{(2)} \otimes a^{(2)(2)}n^{(2)} \\
= \; & (\alpha \circ (\Psi^s \otimes \Psi^t) \circ \alpha^*)(a^{(1)})m^{(1)}n^{(1)} \otimes a^{(2)(1)}m^{(2)} \otimes a^{(2)(2)}n^{(2)}
\end{aligned}$$

$$\begin{aligned} &= (\alpha^{(s+t)} \circ \alpha^{*(s+t)})(a^{(1)})m^{(1)}n^{(1)} \otimes a^{(2)(1)}m^{(2)} \otimes a^{(2)(2)}n^{(2)} \\ &= \Psi^{s+t}(a^{(1)})m^{(1)}n^{(1)} \otimes a^{(2)(1)}m^{(2)} \otimes a^{(2)(2)}n^{(2)} \\ &= ((\alpha \otimes \beta) \circ (\Psi^{s+t} \otimes \tau \otimes 1) \circ (\alpha^* \otimes \beta^*))(a \otimes m \otimes n), \end{aligned}$$

as needed. The statement in the PSH case is clear. □

Proposition 38. *Let A be a commutative, cocommutative Hopf algebra and for $j \leq n$ let M_i be a t_i-compatible Hopf module over A. Let $M = M_1 \otimes \cdots \otimes M_n$ and let $t = \sum t_i$. With the multiplication*

$$a \otimes m_1 \otimes \cdots \otimes m_n \mapsto a^{(1)}m_1 \otimes \cdots \otimes a^{(n)}m_n$$

and comultiplication

$$m_1 \otimes \cdots \otimes m_n \mapsto m_1^{(1)} \cdots m_n^{(1)} \otimes m_1^{(2)} \otimes \cdots \otimes m_n^{(2)}$$

M is a t-compatible Hopf module over A. Furthermore, permuting the order of the M_i preserves the isomorphism type of M; in particular, the map between these modules that permutes the order of tensor factors according to the same permutation of the ordering of the modules is an isomorphism of Hopf modules.

Proof. It is clear that the module M is equal to the module

$$M_1 \otimes (M_2 \otimes (\cdots \otimes M_n) \cdots),$$

so the t-compatibility axiom follows immediately from the previous proposition and induction. The statement regarding permuting the order of the M_i follows from the definition of the multiplication and comultiplication in M and the commutativity and cocommutativity of A. □

The previous propositions, along with the following definition, provide a convenient expression for the module $M(G)$ over R.

Definition 13. *Let A be a Hopf algebra. If the maps $\delta, \delta^* \colon A \to A$ are simply $\delta = \delta^* = \Psi^t$, then since $\Psi^t \circ \Psi^t = \Psi^{t^2}$ this choice of maps induces a t^2-compatible Hopf module structure of A over itself. A with this module structure will be denoted $A^{(t)}$.*

Proposition 39. *In the notation of the previous definition, if $\Omega(G)$ is the set of irreducible representations of G, then we have the following isomorphism of $|G|$-compatible Hopf modules:*

$$M(G) \cong \bigotimes_{\omega \in \Omega(G)} R^{(\dim \omega)}.$$

Observe that this isomorphism along with the previous proposition implies the following standard fact:

$$\sum_{\omega \in \Omega(G)} (\dim \omega)^2 = |G|.$$

Proof. This follows immediately from the observation that the multiplication and comultiplication maps for the above tensor product are precisely the same multiplication and comultiplication maps which we saw for $M(G)$ earlier. □

The previous proposition completely describes the modules $M(G)$ in the sense that their structure is now completely understood in terms of the structure of the PSH-algebra R, which is itself very well understood thanks to the work of other authors. This is a particularly powerful result since R is isomorphic as a PSH-algebra to the Hopf algebra of symmetric functions on countably many variables, and a great deal is known about this ring of symmetric functions.

5. A Note on Grothendieck Groups of Categories of k-Compatible Modules

It is the goal of this section to demonstrate a method for constructing, from any commutative, cocommutative Hopf algebra (H, m, m^*) over a commutative ring Z, a graded ring with the k^{th} homogeneous component consisting of the isomorphism classes of k-compatible Hopf modules over H. The main missing ingredient is a notion of a 0-compatible Hopf module. In particular we need a map $\Psi^0 \colon H \to H$ which shares the appropriate properties of the maps Ψ^k discussed earlier for $k \geq 1$, in particular the relation $m \circ (\Psi^k \otimes \Psi^l) \circ m^* = \Psi^{k+l}$ (it is this relation from which the graded structure arises). Such a map is given by

Definition 14. *Set $\Psi^0 = e \circ e^* \colon H \to H$, where e and e^* are the unit and counit, respectively, of H.*

Proposition 40. *The map Ψ^0 satisfies the relation*

$$m \circ (\Psi^k \otimes \Psi^0) \circ m^* = m \circ (\Psi^0 \otimes \Psi^k) \circ m^* = \Psi^k$$

for all $k \geq 0$. It follows from previous results that

$$m \circ (\Psi^k \otimes \Psi^l) \circ m^* = m \circ (\Psi^l \otimes \Psi^k) \circ m^* = \Psi^{k+l}$$

for all $k, l \geq 0$.

Proof. This is immediate from the following diagram, which commutes as a result of the unit and counit axioms (all tensor products are over the underlying ring Z, and in the case of $k = 0$ we may interpret $m^{(0)} = e, m^{*(0)} = e^*$):

$$\begin{array}{ccccc} H & \xrightarrow{m^{*(k)}} & H^{\otimes k} & \xrightarrow{m^{(k)}} & H \\ {\scriptstyle m^*}\downarrow & & \downarrow\uparrow & & \uparrow{\scriptstyle m} \\ H \otimes H & \xrightarrow{e^* \otimes m^{*(k)}} & Z \otimes H^{\otimes k} & \xrightarrow{e \otimes m^{(k)}} & H \otimes H. \end{array}$$

In particular, the composition of the upper maps is Ψ^k, and the middle double arrows are the natural isomorphisms between $H^{\otimes k}$ and $Z \otimes H^{\otimes k}$. The result then follows by running around the diagram the other way. □

Definition 15. *Given a commutative, cocommutative Hopf algebra H, a 0-compatible Hopf module over H is a Hopf module over H satisfying the same compatibility relation defining k-compatible Hopf modules for $k \geq 1$, with Ψ^0 taking the place of Ψ^k.*

It is clear that, given a Hopf algebra H, the collection $\mathcal{C}_k(H)$ of k-compatible Hopf modules over H forms a monoidal category under the direct sum; this is true for Hopf modules over H, and we saw earlier that the collection of k-compatible Hopf modules is closed under the direct sum (this was never actually proved but is completely obvious since the multiplication and comultiplication are defined component-wise), so the result descends to the k-compatible case. So, we may form the Grothendieck group $\mathcal{G}_k(H)$ of the category $\mathcal{C}_k(H)$ for any $k \geq 0$, which consists of isomorphism classes of k-compatible Hopf modules over H with the sum given by the direct sum of class representatives. The previous proposition along with the proof of Proposition 37 implies that the tensor product of k- and l-compatible Hopf modules over H gives a $(k+l)$-compatible Hopf module over H for $k, l \geq 0$. In particular, this hence descends to a bilinear map

$$\otimes \colon \mathcal{G}_k(H) \times \mathcal{G}_l(H) \to \mathcal{G}_{k+l}(H).$$

It follows that the graded module

$$\mathcal{G}(H) = \bigoplus_{k \geq 0} \mathcal{G}_k(H)$$

(with the obvious grading) in turn inherits a graded, commutative, and associative multiplication

$$\mu_H \colon \mathcal{G}(H) \otimes \mathcal{G}(H) \to \mathcal{G}(H).$$

Definition 16. *This resulting graded commutative ring $\mathcal{G}(H)$ will be called the compatibility ring associated with the commutative, cocommutative Hopf module H.*

Some interesting relations among these compatibility rings begin to emerge when we consider changing the base Hopf algebra. First, we need a notion of this change of base.

Definition 17. *A change of base Hopf algebra of degree k ($k \geq 0$) from the commutative, cocommutative Hopf algebra $(A, \alpha, \alpha^*, e_A, e_A^*)$ to the commutative, cocommutative Hopf algebra $(B, \beta, \beta^*, e_B, e_B^*)$ is a pair of Hopf algebra maps*

$$\Delta = \{\delta \colon B \to A, \quad \delta^* \colon A \to B\}$$

which together satisfy the relations

$$\delta^* \circ \delta = \Psi^k, \quad \delta^* \circ e_A \circ e_A^* \circ \delta = e_B \circ e_B^*.$$

Proposition 41. *Let*

$$\Delta = \{\delta \colon B \to A, \quad \delta^* \colon A \to B\}$$

be a change of base Hopf algebra of degree $k \geq 0$ as above. Then for any $l \geq 0$ we have

$$\delta^* \circ \Psi^l \circ \delta = \Psi^{kl}.$$

Proof. This was proven in the previous section (proof of Proposition 35) in the case of $k, l \geq 1$, and the case where $l = 0$ is true by definition of Δ. For the case $k = 0$, $l \geq 1$, first note (as in the proof of Proposition 35) that since δ and δ^* are Hopf algebra morphisms we have

$$\begin{aligned} \delta^* \circ \Psi^l \circ \delta &= \delta^* \circ \alpha^{(l)} \circ \alpha^{*(l)} \circ \delta^* \\ &= \beta^{(l)} \circ (\delta^*)^{\otimes l} \circ \delta^{\otimes l} \circ \beta^{*(l)} \\ &= \beta^{(l)} \circ (e_B \circ e_B^*)^{\otimes l} \circ \beta^{*(l)}. \end{aligned}$$

This final expression is equal to Ψ^0 as a result of the following commutative diagram:

$$\begin{array}{ccccc} B & \xrightarrow{e_B^*} & Z & \xrightarrow{e_B} & B \\ {\scriptstyle \beta^{*(l)}}\downarrow & & \downarrow\uparrow & & \uparrow{\scriptstyle \beta^{\otimes l}} \\ B^{\otimes l} & \xrightarrow{(e_B^*)^{\otimes l}} & Z^{\otimes l} & \xrightarrow{e_B^{\otimes l}} & B^{\otimes l}. \end{array}$$

This diagram commutes as a result of the unit and counit axioms (consider tensoring all objects by B and restricting to the identity element in that factor). The middle vertical maps are the natural isomorphisms, so traversing the diagram below yields the expression derived before, while the upper maps of course give Ψ^0. $\square$

Given Δ as above, the previous proposition along with the proof of Proposition 35 shows that if (M, μ, μ^*) is any l-compatible Hopf module over the commutative, cocommutative Hopf algebra A, then the module (M, ν, ν^*) is a kl-compatible Hopf module over the commutative, cocommutative Hopf algebra B, with ν and ν^* given as before:

$$\nu = \mu \circ (\delta \otimes 1)\colon B \otimes M \to M$$
$$\nu^* = (\delta^* \otimes 1) \circ \mu^*\colon M \to B \otimes M.$$

For now denote this Hopf module (M, ν, ν^*) by $F_\Delta(M)$, and consider $F_\Delta(\cdot)$ as an operation in this way. If N is another Hopf module over A, using the natural isomorphism

$$B \otimes (M \oplus N) \cong (B \otimes M) \oplus (B \otimes N)$$

it is clear that we have the isomorphism

$$F_\Delta(M \oplus N) \cong F_\Delta(M) \oplus F_\Delta(N).$$

Furthermore, if we have a map of A-Hopf modules $h\colon M \to N$ then setting

$$F_\Delta(h) = h$$

trivially yields a function $M \to N$.

Proposition 42. *The operation F_Δ defined above is a functor from the monoidal category $\mathcal{C}(A)$ (under the direct sum OR the tensor product!) of Hopf modules over A to the monoidal category $\mathcal{C}(B)$ of Hopf modules over B. This relies only on the existence of the maps δ and δ^* without the relations specified to be a change of base of Hopf algebra of degree k. However, with those additional constraints, for each $l \geq 0$ F_Δ descends to a functor*

$$F_{\Delta,l}\colon \mathcal{C}_l(A) \to \mathcal{C}_{kl}(B).$$

Proof. By the remarks preceding this proposition, it suffices to show F_Δ respects the tensor product and that given a morphism of A-Hopf modules $h\colon M \to N$ with $M, N \in \mathcal{C}(A)$, the resulting map

$$F_\Delta(h)\colon F_\Delta(M) \to F_\Delta(N)$$

is a morphism of B-Hopf modules. For the latter, $F_\Delta(h)$ is clearly additive because F_Δ does not change the underlying abelian group structures. However, $F_\Delta(h)$ is also homogeneous, as the following calculation shows:

$$\begin{aligned} F_\Delta(h)(bm) &= h(\delta(b)m) \\ &= \delta(b)h(m) \\ &= bF_\Delta(m). \end{aligned}$$

Similarly, using the co-homogeneity of h, $F_\Delta(h)$ is also co-homogeneous:

$$\begin{aligned} \nu_N^*(F_\Delta(h)(m)) &= (\delta^* \otimes 1)(\mu_N^*(h(m))) \\ &= (\delta^* \otimes 1)(1 \otimes h)(\mu_M^*(m)) \\ &= (1 \otimes h)(\delta^* \otimes 1)(\mu_M^*(m)) \\ &= (1 \otimes F_\Delta(h))(\nu_M^*(m)). \end{aligned}$$

Only the desired interaction with the tensor product remains. In particular, if (M, μ, μ^*) and $(\widetilde{M}, \tilde{\mu}, \tilde{\mu}^*)$ are two Hopf modules over (A, α, α^*), we wish to show that

$$F_\Delta(M \otimes \widetilde{M}) \cong F_\Delta(M) \otimes F_\Delta(\widetilde{M}).$$

This is a bit tedious. The obvious isomorphism is the natural map

$$F_\Delta(M \otimes \widetilde{M}) \to F_\Delta(M) \otimes F_\Delta(\widetilde{M}), \quad m \otimes \tilde{m} \mapsto m \otimes \tilde{m}.$$

The comultiplication of $M \otimes \widetilde{M}$ is the map

$$(\alpha \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ (\mu^* \otimes \tilde{\mu}^*),$$

so the comultiplication of $F_\Delta(M \otimes \widetilde{M})$ is the map

$$(\delta^* \otimes 1 \otimes 1) \circ (\alpha \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ (\mu^* \otimes \tilde{\mu}^*).$$

Similarly, the comultiplication of $F_\Delta(M) \otimes F_\Delta(\widetilde{M})$ is

$$(\beta \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ [[(\delta^* \otimes 1) \circ \mu^*] \otimes [(\delta^* \otimes 1) \circ \tilde{\mu}^*]].$$

Using a basic property of the tensor-transposing map τ and the fact that δ^* is a morphism of coalgebras, the previous two comultiplication expressions are in fact equal:

$$\begin{aligned}
& (\beta \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ [[(\delta^* \otimes 1) \circ \mu^*] \otimes [(\delta^* \otimes 1) \circ \tilde{\mu}^*]] \\
= \;& (\beta \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ [\delta^* \otimes 1 \otimes \delta^* \otimes 1] \circ (\mu^* \otimes \tilde{\mu}^*) \\
= \;& (\beta \otimes 1 \otimes 1) \circ [\delta^* \otimes \delta^* \otimes 1 \otimes 1] \circ (1 \otimes \tau \otimes 1) \circ (\mu^* \otimes \tilde{\mu}^*) \\
= \;& (\delta^* \otimes 1 \otimes 1) \circ (\alpha \otimes 1 \otimes 1) \circ (1 \otimes \tau \otimes 1) \circ (\mu^* \otimes \tilde{\mu}^*).
\end{aligned}$$

So the two Hopf modules have the same comultiplication. The argument for the equality of the multiplication maps is nearly identical, with all the maps occurring in the opposite order (recall all these constructions are actually adjoint in the PSH case, and everything mentioned in this section holds for PHS algebras/modules too). □

By passing to Grothendieck groups, it follows that, for each $l \geq 0$, $F_{\Delta,l}$ gives rise to a map of abelian groups

$$f_{\Delta,l} \colon \mathcal{G}_l(A) \to \mathcal{G}_{kl}(B)$$

and hence a k-graded map of abelian groups

$$f_\Delta \colon \mathcal{G}(A) \to \mathcal{G}(B).$$

The fact that F_Δ respects the tensor product means that f_Δ is in fact a k-graded ring homomorphism!

6. A Review of Parabolic Induction

Throughout this section, fix a finite field $\mathbb{F}_q$ with q elements. For each positive integer n, let $G_n = GL_n(\mathbb{F}_q)$. Following Bump's notation, for $n \geq 1$, let $R_n(q)$ denote the Grothendieck group of the category of finite-dimensional complex representations of G_n, let $R_0(q) = \mathbb{Z}$, and as before set

$$R(q) = \bigoplus_{n \geq 0} R_n(q).$$

Zelevinsky showed that this group can be endowed with the structure of a PSH-algebra. In the place of typical induction, the multiplication will be given by parabolic induction, defined below, and, in the place of typical restriction, the comultiplication will be given by a related procedure. Naturally, similarly to the

PSH-algebra R considered in the first section, $R(q)$ has a natural $\mathbb{Z}$-basis consisting of the irreducible representations, and hence the positive elements of $R(q)$ may be defined as earlier as those elements with all nonnegative coefficients when written in terms of irreducibles; equivalently, the positive elements of $R(q)$ correspond to sums of isomorphism classes of actual representations. There is also an inner product induced on $R(q)$ by the usual inner product on the space of class functions (it is clear that $R(q)$ can be viewed as the $\mathbb{Z}$-span of the class functions on the groups G_n, and we will freely make this association and treat characters and representations interchangeably). Before specifying the multiplication and comultiplication on $R(q)$, we first need some new terms.

For a fixed positive integer n, let $s = (s_1, \ldots, s_l)$ be a sequence of positive integers such that $\sum s_i = n$, and let $t_k = \sum_{i \leq k} s_i$. Let $V = \mathbb{F}_q^n$, and for a nonnegative integer $m \leq n$ let $V_m \subset V$ be the subspace of vectors whose last $n - m$ coordinates are equal to zero.

Definition 18. *Let s be as above. An increasing sequence of subspaces of the form*

$$\mathcal{F} = \{0 = W_0 < W_1 < \cdots < W_l = V\}$$

such that $\dim W_i/W_{i-1} = s_i$ *is called a flag of type s. There is a particular flag of type s, which will be denoted $\mathcal{F}_s$ and called the standard flag of type s, defined as*

$$\mathcal{F}_s = \{0 = V_0 < V_{t_1} < \cdots < V_{t_l} = V\},$$

where the V_i are as above.

There is an obvious action of G_n on the set of flags of V of type s in which $g \in G_n$ acts by

$$\{0 = W_0 < W_1 < \cdots < W_l = V\} \mapsto \{0 = gW_0 < gW_1 < \cdots < gW_l = V\}.$$

This action is transitive, since for each flag of type s as above there exists a basis $w_1, \ldots, w_n$ of V such that $w_1, \ldots, w_{t_i}$ is a basis of W_i for each i, and the $g \in G_n$ sending one of these bases corresponding to $\mathcal{F}$ to an analogous basis corresponding to another flag clearly maps $\mathcal{F}$ to this other flag.

Definition 19. *Let s be as above. The stabilizer $P_{\mathcal{F}} < G_n$ of a flag of type s will be called a parabolic subgroup of type s, and the stabilizer $P_s = P_{\mathcal{F}_s}$ of the standard flag $\mathcal{F}_s$ will be called the standard parabolic subgroup of type s. In the case $l = |s| = 2$, a parabolic subgroup of type s will be called a maximal (proper) parabolic subgroup, for obvious reasons.*

If $\mathcal{F}$ is a flag, it is clear that $P_{g\mathcal{F}} = gP_{\mathcal{F}}g^{-1}$, so the parabolic subgroups of type s are simply the conjugates of P_s. Furthermore, by the definition of $\mathcal{F}_s$, P_s consists of those matrices $p \in G_n$ which have a block upper-triangular form with square blocks along the diagonal of lengths $s_1, \ldots, s_l$. There is a convenient semidirect product decomposition of P_s, and hence by conjugacy for all $P_{\mathcal{F}}$. This is the Levi decomposition. First, two definitions.

Definition 20. *Let s be as above. The subgroup $M_s < P_s$ consisting of all invertible block diagonal matrices of sizes $s_1, \ldots, s_l$ will be called the Levi factor corresponding to P_s, or the standard Levi factor of type s.*

Definition 21. *Let s be as above. The subgroup $U_s < P_s$ consisting of all matrices with the upper-triangular block form described above such that each block on the* *diagonal is an identity matrix will be called the unipotent radical of P_s, or the standard unipotent radical of type s.*

It is clear that $M_s \cap U_s = 1$, $M_s U_s = P_s$, and $U_s \lhd P_s$, so we have the semidirect product decomposition $P_s = M_s \rtimes U_s$, which will be written $P_s = M_s U_s$, or, when s is understood, $P = MU$. In particular, we have $P_s/U_s \cong M_s$. This will be important shortly.

The construction of the Hopf algebra at the level of the type-A Weyl groups utilized the inclusions $S_k \times S_l \longrightarrow S_{k+l}$. Similarly, there is an inclusion

$$i\colon G_k \times G_l \longrightarrow G_{k+l}$$

defined by

$$i(g,h) = \begin{bmatrix} g & \\ & h \end{bmatrix}.$$

In fact, i is an isomorphism $G_k \times G_l \longrightarrow M_{k,l}$. Let P be the standard parabolic subgroup of type (k,l) and let M be its Levi factor and let U be its unipotent radical. Given characters π of G_k and σ of G_l, $\pi \otimes \sigma$ is a character of $G_k \times G_l \cong M$. Then, since $P/U \cong M$, we may extend $\pi \otimes \sigma$ to the character $(\pi \otimes \sigma)'$ of P by composing with the projection map

$$P \longrightarrow P/U \cong M,$$

In other words, $(\pi \otimes \sigma)'$ acts by

$$(\pi \otimes \sigma)' \left(\begin{bmatrix} A & B \\ & C \end{bmatrix} \right) = \pi(A)\sigma(B).$$

Then, define the character $\pi \circ \sigma$ of $G = G_{k+l}$ by

$$\pi \circ \sigma = \mathrm{Ind}_P^G((\pi \otimes \sigma)').$$

The operation $\circ$ is called parabolic induction. It is clear that it is bilinear, and without too much trouble it is seen to be associative as well. Say π_i is a character of G_{s_i} for $1 \le i \le k$, let $s = (s_1, \dots, s_k)$, and suppose $\sum_{i \le k} s_i = n$. One way to see associativity is to show that for any bracketing of the product $\pi_1 \circ \cdots \circ \pi_k$, the resulting character is given by first pulling the character $\pi_1 \otimes \cdots \otimes \pi_k$ back from M_s to P_s by the projection

$$P \longrightarrow P_s/U_s \cong M_s$$

and then inducing the result to G_n. This is proven by Green, and a similar proof will be given shortly in the case of interest of this paper. With these properties of $\circ$ established, it follows that extending the operation to a map

$$m\colon R(q) \otimes R(q) \longrightarrow R(q)$$

defines an associative multiplication on $R(q)$, and this is our multiplication of interest. It is obvious that m is a positive map with respect to the positive structure on $R(q)$ mentioned earlier.

For the resulting structure on $R(q)$ to be a PSH-algebra, there must be a comultiplication, and this operation must be adjoint to parabolic induction. Zelevinsky shows that this operation is given by the following construction. Let (π, V) be a representation of G_n, and suppose $k+l = n$. Then let U be the standard unipotent radical of type (k,l), and let

$$V^U = \{v \in V \colon \pi(u)v = v \ \forall u \in U\}$$

be the subspace of vectors of V invariant under U. Using again the central fact $P/U \cong M$, it follows that V^U is a representation of M, since if $m \in M$ and $u \in U$, then since M normalizes U (U is normal) there exists $u' \in U$ such that $um = mu'$ and hence

$$\pi(u)\pi(m)v = \pi(m)\pi(u')v = \pi(m)v$$

so $\pi(m)v \in V^U$. Let $m^*_{k,l}(\pi)$ be this representation of M. Since $M \cong G_k \times G_l$, we have $m^*_{k,l}(\pi) \in R_k(q) \otimes R_l(q)$. Then, the comultiplication

$$m^* \colon R(q) \longrightarrow R(q) \otimes R(q)$$

is defined on the n^{th} homogeneous part $R_n(q)$ by

$$m^*|_{R_n(q)} = \sum_{k+l=n} m^*_{k,l}$$

and extended to $R(q)$ by linearity. This is valid since the above construction respects the direct sum.

By considering the block shape of the standard unipotent radicals, it is obvious that m^* is coassociative, and by construction it is a positive map. Zelevinsky shows that the operations m and m^* are in fact adjoints with respect to the inner product on $R(q)$. Furthermore, and quite remarkably, m and m^* satisfy the Hopf axiom, so $R(q)$ is a PSH-algebra. Recall the result of Zelevinsky, mentioned in an earlier section, which states that any PSH-algebra decomposes as a tensor product of PSH-algebras isomorphic to the elementary PSH-algebra R (but perhaps differing by a scaling of the grading), indexed by the set of primitive irreducible elements. In the case of $R(q)$, these primitive irreducible elements are quite interesting and important. They are known as the cuspidal representations.

In the case of the elementary PSH-algebra R, there is a unique primitive irreducible element, the element $x_1 \in R_1$. Recall that this element corresponds to the trivial representation of S_1, and that applying the multiplication n times yields the element

$$x_1^n = \mathrm{Ind}_{S_1 \times \cdots \times S_1}^{S_n} 1 = \mathbb{C}S_n,$$

the regular representation of S_n, which includes each irreducible element of S_n as a component. Thus the element x_1 generates the entire PSH-algebra, not as an algebra, but in the sense that every irreducible element occurs in powers of x_1. In this sense, x_1 is the unique building block of R.

Returning now to $R(q)$, let $\mathcal{C}(q)$ be the set of cuspidal representations of $R(q)$. An element $\omega \in \mathcal{C}(q)$ is characterized by the two properties of being irreducible and being primitive, i.e. that $m^*(\omega) = \omega \otimes 1 + 1 \otimes \omega$. ω is thus the isomorphism class of an irreducible representation π of G_n satisfying

$$m^*_{k,l}(\pi) = 0$$

for all $k \neq 0, n$, or, equivalently, that (π, V) is a cuspidal representation of G_n if and only if π is irreducible and V^U is the trivial representation for all nontrivial standard unipotent radicals $U < G_n$.

7. Hopf Modules for the Finite Sympectic and Odd Orthogonal Groups

Let all notation be as in the previous section. In addition, let $J_n \in G_n$ be the element

$$J_n = \begin{bmatrix} & & & & 1 \\ & & & \cdots & \\ & & 1 & & \\ & \cdots & & & \\ 1 & & & & \end{bmatrix} \in G_n$$

and let $K_{2n} \in G_{2n}$ be the element

$$K_{2n} = \begin{bmatrix} & J_n \\ -J_n & \end{bmatrix} \in G_{2n}.$$

When the meaning is clear, the subscripts in J_n and K_{2n} may be omitted. The n^{th} symplectic group, which will be denoted Sp_{2n}, is the subgroup of G_{2n} defined by

$$Sp_{2n} = \{X \in G_{2n} \colon X^T K X = K\}$$

and the n^{th} orthogonal group, which will be denoted O_n, is the subgroup of G_n defined by

$$O_n = \{X \in G_n \colon X^T J X = J\}.$$

The groups G_n, Sp_{2n}, and O_n are called finite groups of Lie type, and G_n is referred to as a type-A group, O_{2n+1} as a type-B group, Sp_{2n} as a type C group, and O_{2n} as a type-D group. Analogous to the construction of the Hopf algebra constructed by Zelevinsky associated with the type-A groups reviewed in the previous chapter, this section will introduce a Hopf module construction for the groups of types B and C. This approach is in parallel with the earlier construction of Hopf modules associated with the Weyl groups of types B and C (which are isomorphic to the wreath products $S_n \rtimes Z_2^n$) defined over Zelevinsky's Hopf algebra associated with the type-A Weyl groups (which are isomorphic to S_n).

For $n \geq 1$, let $M_n(q)$ denote the Grothendieck group of the category of finite-dimensional complex representations of Sp_{2n}, and let $N_n(q)$ denote the Grothendieck group of the category of finite-dimensional complex representations of O_{2n+1}. As before, let $M_0(q) = N_0(q) = \mathbb{Z}$, and set

$$M(q) = \bigoplus_{n\geq 0} M_n(q), \qquad N(q) = \bigoplus_{n\geq 0} N_n(q).$$

Consider these as graded abelian groups with the n^{th} homogeneous parts given by $M_n(q)$ and $N_n(q)$ respectively. For $n \geq 1$, $M_n(q)$ clearly has a $\mathbb{Z}$-basis $\Omega(M_n(q))$ consisting of the isomorphism classes of the irreducible representations of Sp_{2n}, and $\Omega(M_0(q)) = \{1\}$ is a basis of $M_0(q)$. Hence, $M(q)$ has a basis $\Omega(M(q))$ given by

$$\Omega(M(q)) = \bigcup_{n\geq 0} \Omega(M_n(q)).$$

Correspondingly, $N_n(q)$ has a similarly defined basis $\Omega(N_n(q))$, and their union $\Omega(N(q))$ forms a basis for $N(q)$. These are defined to be the irreducible elements of $M(q)$ and $N(q)$. From these bases, define the positive elements $p \geq 0$ to be those elements with all nonnegative coefficients when written in these bases, and let $\leq_{M(q)}$ and $\leq_{N(q)}$ be the resulting partial orderings on $M(q)$ and $N(q)$. Of course, the positive elements are linear combinations, with nonnegative integer coefficients, of isomorphism classes of irreducible representations, and the notion of a positive map between these various spaces will remain the same. In addition, endow $M(q)$ and $N(q)$ each with an inner product in which $\Omega(M(q))$ and $\Omega(N(q))$ form orthonormal bases. As seen before, the restriction of these inner products to the n^{th} homogeneous part obviously agrees with the standard inner product defined on the corresponding representations.

Suppose $n \geq 1$ and $k + l = n$. In order to define multiplications on $M(q)$ and $N(q)$ analogous to that on $R(q)$, we need maps

$$G_k \times Sp_{2l} \longrightarrow Sp_{2n}, \quad G_k \times O_{2l+1} \longrightarrow O_{2n+1}.$$

To obtain these maps, it is convenient to introduce some terms and basic facts.

Definition 22. *Let $a = (a_1, \ldots, a_n)$ be a sequence of nonnegative integers. The odd palindrome of type a is the sequence*

$$(a_1, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_1),$$

which will be denoted $\mathrm{OPal}(a)$.

Definition 23. *Define a standard parabolic subgroup of Sp_{2n} (or O_n) of type* $\mathrm{OPal}(s)$ *to be the intersection of a standard parabolic subgroup of G_{2n} (or G_n) of type* $\mathrm{OPal}(s)$ *with Sp_{2n} (or O_n). When the context is clear, standard parabolic subgroups of G_n, Sp_{2n}, and O_n will all be referred to simply as parabolic subgroups.*

It should be noted that there is a general definition of parabolic subgroups, which in the cases at hand are simply the conjugates in the appropriate groups of the standard parabolic subgroups. However, in this paper we will use strictly standard parabolic subgroups, which leads to no loss of generality for representation theory in the view of characters as class functions, and the term "parabolic subgroup" should always be taken to mean "standard parabolic subgroup."

Proposition 43. *Let $a = (a_1, \ldots, a_l)$ satisfy $a_l + 2\sum_{i<l} a_i = 2n$, and let P be the standard parabolic subgroup of G_{2n} of type* $\mathrm{OPal}(a)$. *If $P = MU$ is the Levi decomposition of P, then the standard symplectic parabolic subgroup $P \cap Sp_{2n}$ has Levi decomposition*

$$P \cap Sp_{2n} = (M \cap Sp_{2n})(U \cap Sp_{2n}).$$

Similarly, if $b = (b_1, \ldots, b_k)$ satisfies $b_l + 2\sum_{i<k} b_i = n$ and if Q is the standard parabolic subgroup of G_n of type $OPal(b)$ then the standard orthogonal parabolic subgroup $Q \cap O_n$ has Levi decomposition

$$Q \cap O_n = (M \cap O_n)(U \cap O_n).$$

Proof. Let $i_k \colon G_k \longrightarrow G_k$ be the involution

$$i_k(g) = J(g^{-1})^T J.$$

When k is understood, let $i = i_k$. A simple computation shows that if a matrix $X \in P$ is symplectic then its diagonal is of the block form

$$D = \mathrm{diag}(D_1, \ldots, D_{l-1}, D_l, i(D_{l-1}), \ldots, i(D_1)).$$

Actually, any such diagonal matrix where the D_i, $i < l$, are arbitrary invertible matrices and D_l is symplectic is a symplectic matrix, and these are precisely the symplectic matrices with that block diagonal structure. Furthermore, multiplication of two block upper-triangular matrices yields a block upper-triangular matrix with block diagonal equal to the product of the block diagonals of the original matrices. It follows that $D^{-1}X$ is an element of U, and since D lies in $M \cap Sp_{2n}$ and $X \in P \cap Sp_{2n}$, it follows that $D^{-1}X \in U \cap Sp_{2n}$, and we have the decomposition

$$X = D(D^{-1}X) \in (M \cap Sp_{2n})(U \cap Sp_{2n}).$$

The reverse inclusion is clear, and the desired Levi decomposition for $P \cap Sp_{2n}$ follows.

The initial statement about the diagonal of a block upper-triangular symplectic matrix holds true for similar block upper-triangular orthogonal matrices where the middle block diagonal entry is orthogonal. Then it is clear that the above argument goes through in the orthogonal case nearly identically, so the second result follows as well. □

The previous proposition is useful because it allows parabolic induction to be defined for the symplectic and orthogonal groups as was done for the general linear groups. In particular, recall that in the case of the general linear groups the semidirect decomposition

$$P = MU, \ U \lhd P$$

allowed a representation of M to be pulled back to a representation of P, which in turn may then be induced up to the entire general linear group. There is an analogous process, defined as follows.

Let π be a representation of G_k and let σ be a representation of Sp_{2l}, where $k + l = n$. The proof of the previous proposition shows that the symplectic Levi factor $M \cap Sp_{2n}$ of type $\mathrm{OPal}(k, 2l)$ is isomorphic to the direct product

$$M \cap Sp_{2n} \cong G_k \times Sp_{2l}$$

with an isomorphism given by the mapping

$$(g, s) \mapsto \begin{bmatrix} g & & \\ & s & \\ & & i(g) \end{bmatrix}.$$

In particular, $\pi \otimes \sigma$ is a representation of M with character given by

$$\begin{bmatrix} g & & \\ & s & \\ & & i(g) \end{bmatrix} \mapsto \chi_\pi(g)\chi_\sigma(s).$$

This representation may then be pulled back to a representation of $P \cap Sp_{2n}$ by the projection map

$$P \cap Sp_{2n} \longrightarrow (P \cap Sp_{2n})/(U \cap Sp_{2n}) \cong M \cap Sp_{2n},$$

which may in turn be induced to Sp_{2n}. The resulting representation will be again be denoted $\pi \circ \sigma$.

As before, it is clear that this map extends to a graded positive linear map

$$\alpha \colon R(q) \otimes M(q) \longrightarrow M(q).$$

As will be shown shortly, this map is associative and hence endows $M(q)$ with the structure of a module over $R(q)$. The map alpha will often be denoted by juxtaposition or by the symbol $\circ$:

$$\alpha(\pi \otimes \sigma) = \pi \circ \sigma.$$

$M(q)$ can also be given a coaction which interacts with this action well. Let π now be a representation of Sp_{2n} acting on the vector space V. Calculations identical to those in the previous section show that π restricts to a representation of the Levi factor $M \cap Sp_{2n}$ of type $\mathrm{OPal}(k, 2l)$ (again with $k + l = n$) on the subspace

$$V' = \{v \in V : uv = v \ \forall u \in U \cap Sp_{2n}\}$$

of unipotent invariants - this follows from the fact that the Levi factor normalizes the unipotent radical. By the isomorphism mentioned above, the result is a representation $\pi_{k,l}$ of $G_k \times Sp_{2l}$. The coaction is then defined on positive elements in the n^{th} homogeneous part by

$$\alpha^*(\pi) = \sum_{k+l=n} \pi_{k,l}$$

and extended by linearity. It is then clear that the resulting map is a graded positive linear map.

It is clear that the construction above has an analogue in the case of the odd orthogonal groups. The above discussion will not be given again, but the corresponding maps are defined in complete analogy and are again graded, positive, and linear.

Below is a proof of the associativity of α. A proof using the definition of induction can also be given, which would work outside the case of finite fields. This proof is an adaptation of Green's original proof of associativity of parabolic induction, and is interesting in that it makes use of a symplectic (or orthogonal) analogue of the Hall polynomials. The following terminology will be of use.

Definition 24. *Let $a_1, \ldots, a_l$ be a sequence of nonnegative integers with sum $\sum a_i = n$, let c be a conjugacy class of G_n, and let c_i be a conjugacy class of G_{a_i} for $1 \le i \le l$. Given a matrix (or a conjugacy class of matrices) X, the resulting isomorphism class of $\mathbb{F}_q[x]$-modules which has a representative in which x acts as matrix multiplication by that matrix (or by a representative element of the conjugacy class) will be called the isomorphism class of $\mathbb{F}_q[x]$-modules of type X, and denoted V_X. A flag of type $(c; c_1, \ldots c_l)$ is an increasing sequence*

$$\{0 = W_0 < W_1 < \cdots < W_l = \mathbb{F}_q^n\}$$

of submodules of $\mathbb{F}_q^n$, as an $\mathbb{F}_q[x]$-module of type c, for which each W_i/W_{i-1} is an $\mathbb{F}_q[x]$-module of type c_i.

In Green's notation, let $g^c_{c_1,\ldots,c_l}$ be the number of flags of type $(c; c_1, \ldots, c_l)$.

The following is a useful variant of the above terminology in the symplectic and orthogonal cases.

Definition 25. *Suppose now that $b_1, \ldots, b_l$ is a sequence of nonnegative integers such that $b_l + 2\sum_{i<l} b_i = 2n$, that d is a conjugacy class of Sp_{2n}, that d_l is a conjugacy class of Sp_{b_l}, and that, for $i < l$, d_i is a conjugacy class of G_{b_i}. A symplectic flag of type $(d; d_1, \ldots, d_l)$ is an increasing sequence*

$$\{0 = W_0 < W_1 < \cdots < W_{2l-1} = \mathbb{F}_q^{2n}\},$$

obtained by left-multiplication of a standard flag by a symplectic matrix, of submodules of $\mathbb{F}_q^{2n}$, as an $\mathbb{F}_q[x]$-module of type d, for which each W_i/W_{i-1} is an $\mathbb{F}_q[x]$-module of type d_i for $i \le l$ and for which each W_{2l-i}/W_{2l-i-1} is an $\mathbb{F}_q[x]$-module of type $i(d_i)$ for $i < l$ (recall that i is the involution defined earlier).

$s^d_{d_1,\dots,d_l}$ *will denote the number of such symplectic flags of type* $(d; d_1, \dots, d_l)$.

The notion of orthogonal flags is defined in complete analogy, with orthogonal matrices taking the place of symplectic ones and with the involution j replacing the involution i. In fact, a "sympectic flag" is simply a flag in the orbit of a standard flag under the action of Sp_{2n} whose factors are the block diagonal of an element of a block upper-triangular symplectic matrix and lying in the conjugacy classes specified in the "type."

The following is a proposition of Green.

Proposition 44. *For* $1 \le i \le l$ *let* χ_i *be a character of* G_{a_i}, *with* $\sum a_i = n$. $\chi_1 \otimes \cdots \otimes \chi_l$ *is a character of the standard Levi subgroup of type* $(a_1, \dots, a_l)$, *and hence may be simultaneously parabolically induced to the character* $\chi_1 \circ \cdots \circ \chi_l$ *of* G_n. *The value of this character on a conjugacy class* c *of* G_n *is given by*

$$(\chi_1 \circ \cdots \circ \chi_l)(c) = \sum_{c_1,\dots,c_l} g^c_{c_1,\dots,c_l} \chi_1(c_1) \cdots \chi_l(c_l),$$

where the sum is over all l*-tuples of conjugacy classes* c_i *of* G_{a_i}, $1 \le i \le l$.

Proof. Green proves this in his paper, with a proof quite similar to the proof of the following proposition. □

The following is a modification of the previous proposition in the symplectic case.

Proposition 45. *For* $1 \le i \le l$ *let* χ_i *be a character of* G_{a_i}, *let* σ *be a character of* Sp_{2k}, *and suppose* $k + \sum a_i = n$. *Then* $\chi_1 \otimes \cdots \otimes \chi_l \otimes \sigma$ *is a character of the standard symplectic Levi subgroup of type* $\mathrm{OPal}(a_1, \dots, a_l, 2k)$, *and hence may be simultaneously parabolically induced to the character* $\chi_1 \circ \cdots \circ \chi_l \circ \sigma$ *of* Sp_{2n}. *The value of this character on a conjugacy class* e *of* Sp_{2n} *is given by*

$$(\chi_1 \circ \cdots \circ \chi_l \circ \sigma)(e) = \sum_{c_1,\dots,c_l,d} s^e_{c_1,\dots,c_l,d} \chi_1(c_1) \dots \chi_l(c_l)\sigma(d),$$

where the sum is over all conjugacy classes d *of* Sp_{2k} *and, for* $1 \le i \le l$, *all conjugacy classes* c_i *of* G_{a_i}.

Proof. Let P denote the type $\mathrm{OPal}(a_1, \dots, a_l, 2k)$ standard symplectic parabolic subgroup, and let M and U denote the associated standard symplectic Levi subgroup and unipotent radical. Then $\chi_1 \otimes \cdots \otimes \chi_l \otimes \sigma$ is a character of M via the isomorphism

$$M \cong G_{a_1} \times \cdots \times G_{a_l} \times Sp_{2k}.$$

For ease of notation, let $\chi_1 \otimes \cdots \otimes \chi_l \otimes \sigma$ also denote the character of $P = MU$, where the normal subgroup U acts trivially, as we have seen before. Let E be a representative of the conjugacy class e of Sp_{2n}, and let $X_1, \dots, X_p$ be a complete set of representatives of the coset space Sp_{2n}/P. A standard result expressing the value of induced characters gives the following formula:

$$(\chi_1 \circ \cdots \circ \chi_l \circ \sigma)(e) = \sum_X (\chi_1 \otimes \cdots \otimes \chi_l \otimes \sigma)(X^{-1}EX),$$

where the sum is over all $X = X_i$ such that $X_i^{-1}EX \in P$.

Suppose $X_i^{-1}EX_i \in P$. Then we have the equality of cosets $EX_iP = X_iP$. Recall that $Sp_{2n} < G_{2n}$ acts on the set of flags of $\mathbb{F}_q^{2n}$ by left-multiplication, and that P is the stabilizer of the standard flag $\mathcal{F}$ of type $\mathrm{OPal}(a_1, \dots, a_l, 2k)$. Let $\mathcal{O}$ be the orbit of $\mathcal{F}$ under this action of Sp_{2n}. Then $\mathcal{O} = \{X_1\mathcal{F}, \dots, X_p\mathcal{F}\}$. The relation $EX_iP = X_iP$ is equivalent to E stabilizing the flag

$$X_i\mathcal{F} = \{0 = W_0 < W_1 < \cdots < W_{2l+1} = \mathbb{F}_q^{2n}\},$$

i.e. that $X_i\mathcal{F}$ is flag of submodules of the standard $\mathbb{F}_q[\mathrm{x}]$-modules of type E. The relation $X_i^{-1}EX_i \in P$ implies that left-multiplication by X_i^{-1} induces an isomorphism $X_i\mathcal{F} \cong \mathcal{F}$ of flags of $\mathbb{F}_q[x]$-modules. By the earlier analysis of the form of the elements of P, $X_i^{-1}EX_i$ is a block upper-triangular matrix with block diagonal of the form $\mathrm{diag}(C_1, \dots, C_l, D, i(C_l), \dots, i(C_1))$, where i is the involution $i\colon g \mapsto J(g^{-1})^TJ$. Say C_i is a representative of the conjugacy class c_i of G_{a_i} and that D is a representative of the conjugacy class d of Sp_{2k}. It follows that the flag $X_i\mathcal{F}$ is a symplectic flag of type $(e; c_1, \dots, c_l, d)$ and that

$$(\chi_1 \otimes \cdots \otimes \chi_l \otimes \sigma)(X_i^{-1}EX_i) = \chi_1(c_1)\cdots\chi_l(c_l)\sigma(d).$$

Conversely, however, if $X_i\mathcal{F}$ is a symplectic flag of type $(e; c_1, \dots, c_l, d)$, it follows that $X_i\mathcal{F}$ is stabilized by E, and hence that $X_i^{-1}EX_i \in P$, and furthermore that $X_i^{-1}EX$ has a block diagonal with i^{th} block in the conjugacy class c_i for $i \leq l$ and in the conjugacy class d for $i = l+1$. Therefore the previous expression holds for such X_i, and hence for precisely $s^e_{c_1,\dots,c_l,d}$ such X_i, and the desired formula follows. □

The following is a consequence of the previous proposition.

Proposition 46. *The pairing $\alpha\colon R(q) \otimes M(q) \longrightarrow M(q)$ is associative.*

Proof. Suppose $k + l + m = n$, and let π, ρ, and σ be characters of G_k, G_l, and Sp_{2m}, respectively. Then, for a conjugacy class a of Sp_{2n}, the previous proposition yields

$$\begin{aligned}(\pi \circ (\rho \circ \sigma))(a) &= \sum_{b,c} s^a_{b,c}\pi(b)(\rho\circ\sigma)(c)\\ &= \sum_{b,c} s^a_{b,c}\pi(b)\sum_{d,e} s^c_{d,e}\rho(d)\sigma(e)\\ &= \sum_{b,d,e}\left(\sum_c s^a_{b,c}s^c_{d,e}\right)\pi(b)\rho(d)\sigma(e),\end{aligned}$$

where the sum is over conjugacy classes b, d, e, and c of G_k, G_l, Sp_{2m}, and $Sp_{2(l+m)}$ Similarly,

$$((\pi\circ\rho)\circ\sigma)(a) = \sum_{b,d,e}\left(\sum_{\tilde{c}} s^a_{\tilde{c},e}g^{\tilde{c}}_{b,d}\right)\pi(b)\rho(d)\sigma(e),$$

where the sum is over conjugacy classes $\tilde{c}$ of G_{k+l} and b, d, and e as above. Also, the previous proposition states precisely that

$$(\pi\circ\rho\circ\sigma)(a) = \sum_{b,d,e} s^a_{b,d,e}\pi(b)\rho(d)\sigma(e)$$

with b, d, and e as above. I claim, and it suffices to show, that

$$\sum_c s^a_{b,c}s^c_{d,e} = s^a_{b,d,e} = \sum_{\tilde{c}} s^a_{\tilde{c},e}g^{\tilde{c}}_{b,d}.$$

These expressions are clearly equal - they simply amount to counting the number of symplectic flags of type $(a; b, d, e)$ (the middle term) in two different ways (the outer terms). □

A nearly identical agreement shows that the multiplication of $R(q)$ on $N(q)$ is also associative. The corresponding comultiplications are immediately seen to be coassociative, so these are indeed Hopf modules over $R(q)$.

It is also interesting to note that certain relations among the s symbols can be obtained from information about the g symbols, commutativity of usual parabolic induction, etc. For instance, since we have the equality

$$g^{\tilde{c}}_{b,d} = g^{\tilde{c}}_{d,b}$$

(this was proven by Green to show the commutativity of the product $\circ$ for $R(q)$), the end of the previous proof implies the following proposition immediately:

Proposition 47. *Given conjugacy classes a, b, d, and e as above, we have*

$$\sum_c s^a_{b,c} s^c_{d,e} = \sum_{c'} s^a_{d,c'} s^{c'}_{b,e},$$

where the c and c' run over the conjugacy classes of respectively appropriate finite general linear groups.

Next we verify that the Hopf modules $M(q)$ and $N(q)$ (which we now know are indeed Hopf modules over $R(q)$ are in fact 2-compatible Hopf modules over $R(q)$. The proof is very analogous to the proof that the Hopf module $M(\mathbb{Z}/2\mathbb{Z})$ encountered in a previous section is 2-compatible, reflecting the role of the Bruhat decomposition in the proof and the central fact that $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ is the Weyl group of both Sp_{2n} and O_{2n+1}. Again, details will be given in the case of the symplectic groups, with the proof in the orthogonal case nearly identical in every way.

The diagonal subgroup D of Sp_{2n}, which consists of elements of the form

$$\mathrm{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots a_1^{-1}),$$

is a maximal torus. Embed S_n ($n \times n$ permutation matrices) in Sp_{2n} by the mapping

$$P \mapsto \begin{bmatrix} P & \\ & i(P) \end{bmatrix}$$

(recall i is the embedding $i\colon G_n \to Sp_{2n}$, $i(P) = J(P^{-1})^T J = JPJ$). Next, consider the elements ϵ_i, $1 \le i \le n$, of the form

$$\epsilon_i = \begin{bmatrix} \ddots & & & & & & & \\ & 1 & & & & & & \\ & & 0 & & & 1 & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & -1 & & & 0 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{bmatrix}$$

where the outlying 1 is in the i^{th} row and the -1 in the i^{th} column. (In the case of the orthogonal groups no negatives are needed.) While it is not true that ϵ_i is generally of order 2, it is true that $\epsilon_i^2 \in D$, so ϵ_i is of order 2 mod D. But it is easy to see that the normalizer $N(D)$ is generated by D, S_n, and the ϵ_i, and therefore that the Weyl group

$$W_n = N(D)/D$$

is generated, modulo D, by S_n and the elements ϵ_i. In particular, note that the ϵ_i generate, modulo D, a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ and normalized by S_n, so we obtain the familiar semidirect product decomposition

$$W_n = S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n.$$

The elements of the form $P\epsilon_{i_1} \cdots \epsilon_{i_k}$, $P \in S_n$, form a complete set of coset representatives of $W_n = N(D)/D$, and these elements will be identified with their image in the quotient for ease of notation.

Given parabolic subgroups $P, Q < Sp_{2n}$, set $W_P = W_n \cap P$ and $W_Q = W_n \cap Q$, where in these intersections we are again regarding all elements mod D. We recall the following standard fact:

Proposition 48. *(Bruhat Decomposition) The natural map*

$$W_Q \backslash W_n / W_P \to Q \backslash Sp_{2n} / P$$

is a bijection.

The relevant case is the following:

Proposition 49. *Let $k+l = r+s = n$, let $P < Sp_{2n}$ be the standard parabolic subgroup of type $(k, 2l, k)$, and let Q be the standard parabolic subgroup of type $(r, 2s, r)$. A complete set of representatives of $W_Q \backslash W_n / W_P$, and hence of $Q \backslash Sp_{2n}/P$ by the Bruhat decomposition, is given by the collection of matrices*

$$\begin{bmatrix}
I_{a_1} & & & & & & & & & \\
 & & & & & & & & I_{a_2} & \\
 & & & I_c & & & & & & \\
 & & I_b & & & & & & & \\
 & & & & I_d & & & & & \\
 & & & & & I_d & & & & \\
 & & & & & & & I_b & & \\
 & & & & & & I_c & & & \\
 & -I_{a_2} & & & & & & & & \\
 & & & & & & & & & I_{a_1}
\end{bmatrix}$$

subject to the conditions

$$a_1 + a_2 + b = k \quad c + d = l$$

$$a_1 + a_2 + c = r \quad b + d = s.$$

Once again, an entirely analogous statement is true in the case of the orthogonal groups, where no negatives are needed and where there is a 1 in the center of the matrix.

Proof. Remembering that W_n is isomorphic to the group $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ and noting that

$$W_P = \left\{ \begin{bmatrix} u & & \\ & v & \\ & & i(u) \end{bmatrix} : u \in S_k,\ v \in W_l \right\} \cong S_k \times W_l$$

(and similarly for W_Q), the proof is essentially the same as the proof of Proposition 24, where now the elements of the groups are presented as matrices in a different way. Specifically, we note that the elements of W_n are precisely the monomial matrices with entires in the set $\{0, 1, -1\}$ subject to the conditions that, viewing the matrix as a block matrix with four blocks each of size $n \times n$, the entires nonzero entries in the lower left block are equal to -1, the nonzero entries in the other blocks are equal to 1, and the entries are symmetric about the center of the matrix, modulo sign. In particular, multiplying on the right or left by W_Q or W_P preserves these properties, so it suffices to see that each double coset has a representative with upper left block of the form above - uniqueness of such representatives is clear. Furthermore, the top r rows and next s rows can be permuted freely among themselves by left multiplication by elements of W_Q, and similarly for the first k and next l columns. This, in combination with the action by right and left multiplication of the elements

$$\begin{bmatrix} I & & \\ & \epsilon_i & \\ & & I \end{bmatrix}$$

gives rise to an algorithm to produce a representative as above from any element $w \in W_n$. □

Using these convenient representatives and Zelevinsky's Theorem A3.1, the 2-compatibility of the modules $M(q)$ and $N(q)$ will follow quickly. Before proving that result, it will be useful to recall this theorem. The following paragraph recalls the necessary notions as introduced by Zelevinsky.

Let G be a finite group, suppose M, U are subgroups with $M \cap U = 1$, and suppose M normalizes U. Given any subgroup H, let $\mathcal{A}(H)$ denote the category of finite dimensional complex representations of H. An element $w \in G$ acts on G by conjugation, and hence induces functors

$$w\colon \mathcal{A}(H) \to \mathcal{A}(w(H))$$
$$w(\pi) = \pi \circ w^{-1}.$$

There are also functors constructed in the same way as the maps α and α^* discussed earlier. In particular, we have

$$i_U\colon \mathcal{A}(M) \to \mathcal{A}(G)$$

where $i_U(\pi)$ is the representation of G formed by first extending π to MU by precomposing with the projection $P = MU \to P/U \cong M$ and then inducing to G. The functor

$$r_U\colon \mathcal{A}(G) \to \mathcal{A}(M)$$

sends the representation (σ, V) to the representation $(\sigma|_M, V^U)$, where V^U is the subspace of elements invariant under the action of U.

Additionally, a subgroup $H < G$ is said to be decomposable with respect to the pair (M, U) if

$$H \cap MU = (H \cap M)(H \cap U).$$

The following theorem provides a formula for the composition of these functors, which in the case $U = V = 1$ is Mackey's theorem.

Proposition 50. *(Zelevinsky's Theorem A3.1, special case $\theta = \psi = 1$) Let G, M, U be as above, and let N, V be additional subgroups with $N \cap V = 1$ and for which N*

normalizes V. Set $P = MU$ and $Q = NV$. Let $\mathcal{W}$ be a complete set of double coset representatives of $Q\backslash G/P$. Suppose, for each $w \in \mathcal{W}$ that each of $w(P)$, $w(M)$, and $w(U)$ were decomposable with respect to (N, V) and that each of $w^{-1}(Q)$, $w^{-1}(N)$, and $w^{-1}(V)$ were decomposable with respect to (M, U). Then, setting

$$M'_w = M \cap w^{-1}(N) \quad N'_w = w(M') = w(M) \cap N$$
$$V'_w = M \cap w^{-1}(V) \quad U'_w = N \cap w(U)$$

it follows from the decomposability criteria that the pair (M'_w, V'_w) gives rise to a functor

$$r_{V'_w} : \mathcal{A}(M) \to \mathcal{A}(M'_w)$$

and the pair (N'_w, U'_w) gives rise to a functor

$$i_{U'_w} : \mathcal{A}(N'_w) \to \mathcal{A}(N)$$

so, for each $w \in \mathcal{W}$, noting in particular that $N'_w = w(M'_w)$, we may define the functor

$$\mathcal{F}_w = i_{U'_w} \circ w \circ r_{V'_w} : \mathcal{A}(M) \to \mathcal{A}(N).$$

Finally, under these conditions, we have the following isomorphism of functors:

$$r_V \circ i_U = \bigoplus_{w \in \mathcal{W}} \mathcal{F}_w.$$

Proposition 51. *The Hopf module $M(q)$ is a 2-compatible Hopf module over $R(q)$; specifically, the following diagram commutes:*

$$\begin{array}{ccc}
R(q) \otimes M(q) & \xrightarrow{m^* \otimes \alpha^*} & R(q) \otimes R(q) \otimes R(q) \otimes M(q) \\
 & & \downarrow{\scriptstyle \Psi^2 \otimes \tau \otimes 1} \\
\downarrow{\scriptstyle \alpha} & & R(q) \otimes R(q) \otimes R(q) \otimes M(q) \\
 & & \downarrow{\scriptstyle m \otimes \alpha} \\
M(q) & \xrightarrow{\alpha^*} & R(q) \otimes M(q).
\end{array}$$

The proof of 2-compatibility for $N(q)$ is entirely analogous, in view of the earlier mentioned fact that Sp_{2n} and O_{2n+1} have isomorphic Weyl groups.

Proof. Let $P, Q < Sp_{2n}$ be the standard parabolic subgroups of types $\mathrm{OPal}(k, 2l)$ and $\mathrm{OPal}(r, 2s)$, respectively. Let w be the representative of $W_Q\backslash W_n/W_P$ corresponding to the partitions

$$a_1 + a_2 + b = k \quad c + d = l$$
$$a_1 + a_2 + c = r \quad b + d = s.$$

Let $P = MU$ and $Q = NV$ be the Levi decompositions of these standard parabolic subgroups. Borrowing notation from Zelevinsky, for $s \in Sp_{2n}$ define $w(s) = wsw^{-1}$, and set

$$M'_w = M \cap w^{-1}(N) \quad N'_w = w(M') = w(M) \cap N$$
$$V'_w = M \cap w^{-1}(V) \quad U'_w = N \cap w(U)$$

It follows that M'_w is the standard parabolic subgroup of type $\mathrm{OPal}(a_1, a_2, b, c, 2d)$ and N'_w is the standard parabolic subgroup of type $\mathrm{OPal}(a_1, a_1, c, b, 2d)$ (note the transposition of b and c). V'_w, viewed as a block diagonal matrix, is given by

$$V'_w = \mathrm{diag}(U_{a_1,a_2,b}, U_{c,2d}, i(U_{a_1,a_2,b}))$$

where $U_{a_1,a_2,b}$ is the standard unipotent radical in G_k of type (a_1, a_2, b) and $U_{c,2d}$ is the standard unipotent radical in Sp_{2l} of type $\mathrm{OPal}(c, 2d)$. Similarly,

$$U'_w = \mathrm{diag}(U_{a_1,a_2,c}, U_{b,2d}, i(U_{a_1,a_2,c})).$$

By the Levi decomposition applied in each of these blocks, we therefore have

$$M'_w V'_w = \mathrm{diag}(P_{a_1,a_2,b}, P_{c,2d}, i(P_{a_1,a_2,b})) = M \cap w^{-1}(Q)$$

where now the blocks are the corresponding standard parabolic subgroups. Similarly,

$$N'_w U'_w = \mathrm{diag}(P_{a_1,a_2,c}, P_{b,2d}, i(P_{a_1,a_2,c})) = N \cap w(P).$$

In the language introduced by Zelevinsky in Appendix 3, this shows exactly, after applying w and w^{-1}, respectively, that $w(M)$ is decomposable with respect to (N, V) and that $w^{-1}(N)$ is decomposable with respect to (M, U). Similar reasoning shows that $w(U)$, $w(P)$ are decomposable with respect to (N, V), and $w^{-1}(V)$ and $w^{-1}(Q)$ are decomposable with respect to (M, U). These are the necessary conditions to apply Zelevinsky's Theorem A3.1, which provides a formula for the composition $\alpha^* \circ \alpha$, playing the same role Mackey's theorem did in the earlier case of wreath products.

Specifically, if π is a representation of G_k and σ is a representation of Sp_{2l}, we have

$$(r_V \circ i_U)(\pi \otimes \sigma) \cong \bigoplus_{w \in W} (i_{U'_w} \circ w \circ r_{V'_w})(\pi \otimes \sigma).$$

Let us first understand the right hand side. To simplify the notation in the next few lines, as a sort of sumless Sweedler notation write

$$m^*_{a_1,a_2,b}(\pi) = \pi_{a_1} \otimes \pi_{a_2} \otimes \pi_b, \quad \alpha^*_{c,d}(\sigma) = \sigma_c \otimes \sigma_d$$

where the sum over simple tensors is implied. By definitions, associativity, coassociativity, we have, with equality denoting isomorphism,

$$\begin{aligned}
(i_{U'_w} \circ w \circ r_{V'_w})(\pi \otimes \sigma) &= (i_{U'_w} \circ w)(m^*_{a_1,a_2,b}(\pi) \otimes \alpha^*_{c,d}(\sigma)) \\
&= i_{U'_w}(\pi_{a_1} \otimes \pi_{a_2} \otimes \sigma_c \otimes \pi_b \otimes \sigma_d) \\
&= \pi_{a_1}\pi_{a_2}\sigma_c \otimes \pi_b\sigma_d.
\end{aligned}$$

So, summing over all $r + s = n$ (recall these parameterized the pair (N, V)) and over all $w \in \mathcal{W}$, we obtain

$$\begin{aligned}
(\alpha^* \circ \alpha)(\pi \otimes \sigma) &= \sum_{r+s=n} (r_V \circ i_U)(\pi \otimes \sigma) \\
&= \sum_{r+s=n} \sum_{w \in \mathcal{W}} (i_{U'_w} \circ w \circ r_{V'_w})(\pi \otimes \sigma) \\
&= \sum_{a_1+a_2+b=k,\ c+d=l} \pi_{a_1}\pi_{a_2}\sigma_c \otimes \pi_b\sigma_d \\
&= \sum_{a+b=k,\ c+d=l} \left(\sum_{a_1+a_2=a} \pi_{a_1}\pi_{a_2} \right) \sigma_c \otimes \pi_b\sigma_d \\
&= \sum_{a+b=k,\ c+d=l} \Psi^2(\pi_a)\sigma_c \otimes \pi_b\sigma_d \\
&= (m \otimes \alpha) \circ (\Psi^2 \otimes \tau \otimes 1) \circ (m^* \otimes \alpha^*)(\pi \otimes \sigma)
\end{aligned}$$

as needed. $\square$

Proposition 52. *The functors r_U and i_U, as in Proposition 50, are adjoint in the sense that if π and σ are representations such that the following expressions make sense, then there is an isomorphism*

$$\hom_G(\pi, i_U(\sigma)) \cong \hom_M(r_U(\pi), \sigma).$$

Proof. Frobenius reciprocity states that there is an isomorphism

$$\hom_G(\pi, i_U(\sigma)) \cong \hom_P(\mathrm{Res}_G^P(\pi), \hat{\sigma}).$$

So, it suffices to check that the map

$$\hom_P(\mathrm{Res}_G^P(\pi), \hat{\sigma}) \to \hom_M(r_U(\pi), \sigma)$$

given by restriction is an isomorphism. Note $r_U(\pi)$, as a subspace of $\mathrm{Res}_G^P(\pi)$, is stabilized by both M and U, hence by $P = MU$, and the P-maps $r_U(\pi) \to \hat{\sigma}$ are precisely the M-maps $r_U(\pi) \to \sigma$ since U acts trivially. So, it suffices to check that there are no nonzero intertwining maps from an irreducible P-submodule of $\mathrm{Res}_G^P(\pi)$ containing vectors not fixed by U into $\hat{\sigma}$. But such a map would be an embedding, and $\hat{\sigma}$ is a trivial U-module, so this is impossible. □

Proposition 53. *The multiplication and comultiplication α and α^* are adjoints with respect to the inner products on $R(q)$, $M(q)$, and the relevant tensor products.*

Proof. This is an immediate consequence of the previous proposition and the mutual orthogonality of the groups $M_k(q)$ in $M(q)$ and $R_l(q)$ in $R(q)$. □

Finally, our desired result is at hand:

Proposition 54. *The Hopf modules $M(q)$ and $N(q)$ are 2-PSH modules over $R(q)$.*

Proof. This is a restatement of the results obtained in this section. □

8. References

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