AN IMPROVED BOUND FOR CHARACTERIZING INTEGER-VALUED FACTORIAL RATIO SEQUENCES

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Abstract. We study the nonnegativity of a certain class of step functions, associated with the integrality of sequences of ratios of factorial products. In particular, we extend the work of previous authors Bell and Bober [1], obtaining tighter lower bounds on the mean square of such step functions, allowing us to find better asymptotic and general restrictions on when the factorial ratio sequences can be integer-valued.

1. Introduction

Given integer parameters $a_1, \ldots, a_K, b_1, \ldots, b_L \in \{1, 2, 3, \ldots\}$, we may define the factorial ratio sequence $\{u_n\}_{n=1}^{\infty}$ by

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!} \ \forall n \in \mathbb{N}.$$

An open question in mathematics is to determine which selections of these parameters $\mathbf{a} = \{a_1, \ldots, a_K\}$ and $\mathbf{b} = \{b_1, \ldots, b_L\}$ ensure that u_n is an integer for all $n \in \mathbb{N}$. In approaching this factorial ratio problem, there are a few standard definitions and assumptions to be made, as outlined below.

First, it is natural to restrict our attention to $\mathbf{a}$ and $\mathbf{b}$ such that

$$\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l, \tag{1}$$

A simple application of Stirling's approximation rules out the possibility that the left-hand sum could be greater than the right-hand sum; in that case $0 < u_n < 1$ for large n. The reason for disallowing the opposite inequality comes from considering another problem in mathematics equivalent to this factorial ratio problem. In particular, a theorem of Landau [3] implies that a factorial ratio sequence $u_n(\mathbf{a}, \mathbf{b})$ is integer-valued for all n if and only if the step function

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

is always nonnegative. Assumption (1) ensures that $f(x; \mathbf{a}, \mathbf{b})$ is a bounded periodic function with period 1. Indeed, it can be shown (see [1]) that if a step function of the form $f(x; \mathbf{a}, \mathbf{b})$ is to be both bounded and nonnegative, the condition (1) must be satisfied and the function must take values only in the range $\{0, \ldots, L - K\}$. In addition, upon considering this alternate formulation of the factorial ratio problem, more simplifying assumptions become clear. We can assume without loss of generality that

$$\gcd(a_1, a_2, \ldots, a_K, b_1, b_2, \ldots, b_{K+1}) = 1 \tag{2}$$

for $(\mathbf{a}, \mathbf{b})$ is a solution to the step function problem if and only if $(\frac{1}{d}\mathbf{a}, \frac{1}{d}\mathbf{b})$ is a solution, where d is the above gcd. It is obvious from either formulation of the problem that we may assume that

$$a_k \neq b_l \text{ for all } k, l. \tag{3}$$

In studying this problem it is also useful to define the terms *height* $h(L, K) = L - K$ and *length* $l(L, K) = L + K$ associated with any particular factorial ratio sequence. For the case $h(L, K) = 1$, the factorial ratio problem has been completely solved (see [2]). That is, given the three assumptions above, a classification of exactly which height-1 parameter pairs $(\mathbf{a}, \mathbf{b})$ ensure that $u_n(\mathbf{a}, \mathbf{b})$ is an integer for all $n > 0$ has been achieved. Such parameter pairs vary in length yet none have length greater than 9 (9 is achievable, as in the pair $\mathbf{a} = \{24, 9, 6, 4\}$, $\mathbf{b} = \{18, 12, 8, 3, 2\}$). As noted by Bell and Bober in [1], there is no "simple" reason why parameter pairs of length 11 or greater cannot produce integer-valued factorial ratio sequences; no quick observation allows us to conclude that all functions of the form

$$f(x) = \sum_{k=1}^{5} \lfloor a_k x \rfloor - \sum_{l=1}^{6} \lfloor b_l x \rfloor$$

cannot be positive for all x if $\sum a_k = \sum b_l$ and $a_k \neq b_l$ for all k, l.

Even without an explanation for this apparent length-limit, it seems reasonable to ask whether maximum lengths exist for factorial ratio sequences at other heights. If we fix $h(L, K) = D$, might there be a length-limit on $l(L, K)$ (depending on D) past which no integer-valued factorial ratio sequences exist? This is the problem addressed in [1]; within that paper Bell and Bober achieve the following positive result.

Theorem 1.1. *Fix $L - K = D$. If*

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

where

$$\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$$

and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \ldots\}$ for all k, l, and if

$$f(x) \geq 0$$

for all x, then asymptotically for $D \to \infty$, we must have

$$K + L \ll D^2 (\log D)^2.$$

Also derived by Bell and Bober are some explicit bounds on $K + L$ for small D. Defining $\mathbf{B}(D)$ to be the greatest factorial ratio length possible for a given height D, the paper gives $\mathbf{B}(1) < 112371$ and $\mathbf{B}(2) < 502827$. Note that this bound for $\mathbf{B}(1)$ is quite far from the actual value $\mathbf{B}(1) = 9$ proven in [2]; similarly, computer searches focused on height-2 factorial ratio sequences suggest that the given bound for $\mathbf{B}(2)$ is far from optimal.

In this paper we provide improvements in both the asymptotic bound (for large D) and explicit bounds (for small D) for $\mathbf{B}(D)$, as well as give a general bound that, despite being weaker in both extremes, applies for all D. The methods of analysis used in this paper are similar to, and indeed based off of, those used by Bell and Bober in [1].

1.1. Notation.

$\lfloor x \rfloor$ denotes the *floor* of x, which is the largest integer less than or equal to x. $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. Also, define $e(x) = e^{2\pi i x}$.

2. Statement and Proof of the Main Theorem

Theorem 2.1. *Suppose that*

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

where

$$\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$$

and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \ldots\}$ *for all* k, l, *and suppose that*

$$f(x) \geq 0$$

for all x. *Denote* $L - K = D$. *Then the following statements must be true.*

(1) *For all* $D \in \mathbb{N}$, $K + L \leq 287D^{3.44}$. *This result allows us to define the function* $\mathbf{B}(D)$ *to be the greatest factorial ratio length* $K + L$ *possible for a given height* D.
(2) *In particular, we have the following results for small* D*:*
$\mathbf{B}(1) < 43$, $\mathbf{B}(2) < 202$, $\mathbf{B}(1) < 495$, *and* $\mathbf{B}(1) < 926$.
(3) *Asymptotically for* $D \to \infty$ *we must have*

$$\mathbf{B}(D) \ll D^2(\log(\log D))^2.$$

As our approach to proving this theorem initially follows the work done by Bell and Bober in [1], we first reproduce here some of the results and discussion from that paper.

2.1. Problem Approach and Previous Results.

Generally, we wish to arrive at bounds for $\mathbf{B}(D)$ by considering

$$\int_0^1 \left| f(x) - \frac{D}{2} \right|^2 \mathrm{d}x. \tag{4}$$

Since $f(x)$, given our assumptions, takes on values only in the range $[0, D]$, this integral is clearly bounded above by $D^2/4$. Thus, if we are able to derive a lower bound on (4) as a function of $K+L$, we may look to when these two bounds conflict in order to see which values of $K+L$ for a fixed D are possible. In order to estimate (4), we first compute the Fourier coefficients of $f(x)$.

Lemma 2.2. *Suppose that* $f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$. *Then the Fourier expansion of* f *is*

$$f(x) = \frac{L-K}{2} + \frac{1}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \left[\sum_{a_k | n} a_k - \sum_{b_l | n} b_l \right] e(nx).$$

That is, $\hat{f}(0) = (L-K)/2$ *and for* $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi i n}\left[\sum_{a_k|n} a_k - \sum_{b_l|n} b_l\right].$$

Proof. First note that we can rewrite $f(x)$ as:

$$\begin{aligned} f(x) &= \sum_{k=1}^{K}(a_k x - \{a_k x\}) - \sum_{l=1}^{L}(b_l x - \{b_l x\}) \\ &= \left(\sum_{k=1}^{K} a_k - \sum_{l=1}^{L} b_l\right) x + \sum_{l=1}^{L}\{b_l x\} - \sum_{k=1}^{K}\{a_k x\} \quad = \sum_{l=1}^{L}\{b_l x\} - \sum_{k=1}^{K}\{a_k x\} \end{aligned}$$

where assumption (1) is used to equate $\left(\sum_{k=1}^{K} a_k - \sum_{l=1}^{L} b_l\right) = 0$. Now, the Fourier expansion for the fractional part of x is well known as

$$\{x\} = \frac{1}{2} - \frac{1}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e(nx)}{n}.$$

which allows us to write

$$\begin{aligned} f(x) &= \frac{L-K}{2} - \frac{1}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\left[\sum_{l=1}^{L} e(nb_l x) - \sum_{k=1}^{K} e(na_k x)\right] \\ &= \frac{L-K}{2} + \frac{1}{2\pi i}\left[\sum_{k=1}^{K}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e(na_k x)}{n} - \sum_{l=1}^{L}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e(nb_l x)}{n}\right] \end{aligned}$$

Rearranging, we have

$$\begin{aligned} f(x) &= \frac{L-K}{2} + \frac{1}{2\pi i}\left[\sum_{k=1}^{K}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e(na_k x)}{n} - \sum_{l=1}^{L}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{e(nb_l x)}{n}\right] \\ &= \frac{L-K}{2} + \frac{1}{2\pi i}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\left[\sum_{\substack{n,a_k\\ na_k=m}} \frac{1}{n} - \sum_{\substack{n,b_l\\ nb_l=m}} \frac{1}{n}\right] e(mx) \end{aligned}$$

and replacing n in the sum with $n = m/a_k$ yields the desired result. □

On considering this Fourier expansion of f, we are now able to see from Parseval's theorem that

$$\begin{aligned} \int_0^1 |f(x) - D/2|^2 \,\mathrm{d}x &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left|\hat{f}(n)\right|^2. \\ &= 2\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2. \end{aligned} \tag{5}$$

where the last line follows from the fact that $\left|\hat{f}(n)\right| = \left|\hat{f}(-n)\right|$.

Now we use the following theorem of Carlson (see [4]) along with a Möbius inversion-type relation to derive a workable expression for $\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2$.

Proposition 2.3. *Let $f(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ in some half-plane. Then if $f(s)$ is analytic and of finite order for $\sigma \geq \alpha$, and*

$$\frac{1}{2T}\int_{-T}^{T}|f(\alpha+it)|^2\,\mathrm{d}t$$

is bounded as $T\to\infty$, then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2\,\mathrm{d}t=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}.$$

If we let

$$g(n)=g(n;\mathbf{a},\mathbf{b})=\#\{a_k : a_k=n\}-\#\{b_l : b_l=n\},$$

we may note from Lemma 2.2 that for $n\geq 1$ we have

$$\hat{f}(n)=\frac{1}{2\pi i}\sum_{d|n}\frac{dg(d)}{n}.$$

Then forming the Dirichlet series

$$G(s)=D(g,s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s} \tag{6}$$

and

$$F(s)=D(\hat{f},s)=\sum_{n=1}^{\infty}\frac{\hat{f}(n)}{n^s}, \tag{7}$$

we have the relation

$$G(s)\zeta(s+1)=2\pi i F(s), \tag{8}$$

where $\zeta(s)=D(1,s)=\sum_{n=1}^{\infty}n^{-s}$ is the Riemann ζ-function. This follows from the following standard result on Dirichlet series (see [7, page 62]).

Proposition 2.4. *If $A(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ and $B(s)=\sum_{n=1}^{\infty}\frac{b(n)}{n^s}$ are Dirichlet series connected by the relation*

$$a(n)=\sum_{d|n}b(d)\ \text{for}\ n\geq 1$$

then

$$A(s)=B(s)\zeta(s).$$

Now, applying Proposition 2.3 with $F(s)$ in place of $f(s)$, we can see immediately from (8) that

$$\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|=\lim_{T\to\infty}\frac{1}{4\pi^2}\frac{1}{2T}\int_{-T}^{T}|G(it)\zeta(1+it)|^2\,\mathrm{d}t. \tag{9}$$

This is the expression that we analyze in order to find a lower bound for $\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2$, and this is the place at which our discussion diverges from that of Bell and Bober.

2.2. **An Improved Integral Bound.** Bell and Bober estimate the integral in equation (9) by substituting a truncated Euler product

$$\zeta_M(s) = \prod_{p \le M} \left(1 - \frac{1}{p^s}\right)^{-1},$$

for the ζ-function. Essentially, they split the integral up into a main component and an error component, the first of which they bound using a limit on how small $\zeta_M(1+it)$ can be, and the second of which they bound using a limit on how large $\zeta_M(1+it)$ can be. Their rationale for doing this is that the actual ζ-function ($\zeta(s) = \zeta_\infty(s)$) and its inverse are unbounded on the line $1+it$, making straightforward estimation of the integral in (9) quite difficult. By using the tamer ζ_M instead, standard integral-estimation tools such as L^∞ bounds became available.

In this paper, we take a more direct approach to the problem of $\zeta(1+it)$'s varying size. Define the set

$$A_\epsilon = \{t \in \mathbb{R} : |\zeta(1+it)| > \epsilon\}$$

containing exactly those points where $\zeta(1+it)$ is "large." Then we can write

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t \\ &\qquad \ge \lim_{T\to\infty} \left[\frac{1}{2T}\left(\int_{[-T,T]\cap A_\epsilon} \epsilon^2 |G(it)|^2 \,\mathrm{d}t + \int_{[-T,T]\setminus A_\epsilon} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t\right)\right] \\ &\qquad = \lim_{T\to\infty} \frac{1}{2T}\left[\epsilon^2 \int_{-T}^{T} |G(it)|^2 \,\mathrm{d}t - \int_{[-T,T]\setminus A_\epsilon} \left(\epsilon^2 - |\zeta(1+it)|^2\right) |G(it)|^2 \,\mathrm{d}t\right] \end{aligned} \tag{10}$$

Recalling our definition of $G(s)$, we see that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)|^2 \,\mathrm{d}t \ge K + L \tag{11}$$

(equality holds iff all of the a_k and b_l are distinct) and also that

$$\max_{t\in\mathbb{R}} |G(it)|^2 \le (K+L)^2. \tag{12}$$

Using these facts to simplify (10), we arrive at

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t \\ &\qquad \ge \epsilon^2(K+L) - (K+L)^2 \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\setminus A_\epsilon} (\epsilon^2 - |\zeta(1+it)|^2)\mathrm{d}t. \end{aligned} \tag{13}$$

At this point we make use of the following lemma to deal with the last remaining integral.

Lemma 2.5. *For all $f : [a,b] \to \mathbb{R}$ such that $f(x) \ge 0 \ \forall\, x \in [a,b]$*

$$\int_{B_\epsilon} (\epsilon^2 - (f(t))^2)\mathrm{d}t \le \int_0^\epsilon 2t \cdot \mu(B_t)\mathrm{d}t$$

where we have defined

$$B_\epsilon = \{t \in [a,b] : f(t) \le \epsilon\}$$

and μ is the standard Lebesgue measure.

Proof. Let $\{\epsilon_k\}_{k=0}^N$ be a strictly decreasing sequence of real numbers with $\epsilon_0 = \epsilon$ and $\epsilon_N = 0$. We can write

$$\begin{aligned}\int_{B_\epsilon} (\epsilon^2 - (f(t))^2)\mathrm{d}t &\le \int_{B_\epsilon \setminus B_{\epsilon_1}} (\epsilon^2 - \epsilon_1^2)\mathrm{d}t + \int_{B_{\epsilon_1}} (\epsilon^2 - (f(t))^2)\mathrm{d}t \\ &= \int_{B_\epsilon} (\epsilon^2 - \epsilon_1^2)\mathrm{d}t + \int_{B_{\epsilon_1}} (\epsilon_1^2 - (f(t))^2)\mathrm{d}t \\ &= (\epsilon^2 - \epsilon_1^2)\mu(B_\epsilon) + \int_{B_{\epsilon_1}} (\epsilon_1^2 - (f(t))^2)\mathrm{d}t\end{aligned}$$

and continuing to expand in this way, eventually determine that

$$\begin{aligned}\int_{B_\epsilon} (\epsilon^2 - (f(t))^2)\mathrm{d}t &\le \sum_{k=0}^{N-1} (\epsilon_k^2 - \epsilon_{k+1}^2)\mu(B_{\epsilon_k}) + \int_{B_{\epsilon_N}} (\epsilon_N^2 - (f(t))^2)\mathrm{d}t \\ &= \sum_{k=0}^{N-1} (\epsilon_k^2 - \epsilon_{k+1}^2)\mu(B_{\epsilon_k})\end{aligned}$$

(last line follows from the fact that $\epsilon_N = 0 \Rightarrow B_{\epsilon_N} = \phi$). Now taking $N \to \infty$ while ensuring that the sequence $\{\epsilon_k\}_{k=0}^N$ becomes dense in $[0, \epsilon]$, the right hand side expression becomes a Riemann integral. That is, we get

$$\begin{aligned}\int_{B_\epsilon} (\epsilon^2 - (f(t))^2)\mathrm{d}t &\le \int_0^\epsilon \mu(B_t)\mathrm{d}(t^2) \\ &= \int_0^\epsilon 2t \cdot \mu(B_t)\mathrm{d}t,\end{aligned}$$

the desired result. □

Applying this lemma to (13), our bound becomes

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t \ge \epsilon^2(K+L) - 2(K+L)^2 \int_0^\epsilon x \cdot m(|\zeta| \le x)\mathrm{d}x, \tag{14}$$

having defined the shorthand

$$m(|\zeta| \le x) = \lim_{T\to\infty} \frac{1}{2T} \mu(\{t \in [-T, T] : |\zeta(1+it)| \le x\}).$$

We use Markov's inequality to estimate $m(|\zeta| \le x)$. In particular, if we define

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(1+it)|^{2k} \,\mathrm{d}t = M_k$$

then we may write

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(1+it)|^{-2k} \,\mathrm{d}t &= M_{-k} \\ \Rightarrow m(|\zeta|^{-1} \ge x) &\le \frac{M_{-k}}{x^{2k}} \\ \Rightarrow m(|\zeta| \le x) &\le x^{2k} M_{-k} \quad \forall\, x > 0.\end{aligned}$$

Plugging this into (14), we see that for any fixed k and ϵ

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}&|G(it)\zeta(1+it)|^2\,\mathrm{d}t\\ &\geq \epsilon^2(K+L)-2(K+L)^2\int_0^\epsilon x(x^{2k}M_{-k})\mathrm{d}x\\ &=\epsilon^2(K+L)-\frac{1}{k+1}\epsilon^{2k+2}M_{-k}(K+L)^2.\end{aligned}$$

To achieve the tightest bound for a fixed factorial ratio sequence length $(K+L)$, we must maximize that last expression over all $\epsilon>0$ and $k\in\mathbb{R}$. We first compute the maximum over ϵ; this is simple to do using standard single-variable calculus techniques, with the result

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}&|G(it)\zeta(1+it)|^2\,\mathrm{d}t\\ &\geq \max_{\epsilon>0}\left\{\epsilon^2(K+L)-\frac{1}{k+1}\epsilon^{2k+2}M_{-k}(K+L)^2\right\}\\ &=\frac{k}{1+k}\left(M_{-k}\right)^{-\frac{1}{k}}(K+L)^{1-1/k}\text{ for all }k\in\mathbb{R}\end{aligned}\tag{15}$$

Now, before we maximize the right-hand side over k, we must learn more about M_{-k}.

2.3. **Computing and Bounding M_k.** It is well known (see [5]) that M_k can be expressed in terms of $d_z(n)$, the n-th Dirichlet series coefficient of $\zeta(s)^z$:

$$M_k=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(1+it)|^{2k}\,\mathrm{d}t=\sum_{n=1}^{\infty}\frac{d_k(n)^2}{n^2}$$

Note that $d_z(n)$ is a multiplicative function, and for any prime power p^a, we have

$$\begin{aligned}d_{-k}(p^a)&=\text{ coefficient of }p^{-as}\text{ in }(1-p^{-s})^k\\ &=\binom{k}{a}(-1)^a\end{aligned}$$

Then we can write

$$M_{-k}=\prod_{p\text{ prime}}\left[\sum_{i=0}^{\infty}\binom{k}{i}^2p^{-2i}\right]\text{ for all }k\in\mathbb{R}\tag{16}$$

Using the fact that $\sum_{i=0}^{\infty}\binom{k}{i}^2x^{-i}\leq\sum_{i=0}^{\infty}\binom{k^2}{i}x^{-i}$ for all $k\geq1$ and $x>0$, we may bound M_{-k} above:

$$\begin{aligned}M_{-k}&\leq\prod_{p\text{ prime}}\left[\sum_{i=0}^{\infty}\binom{k^2}{i}p^{-2i}\right]\\ &=\prod_{p\text{ prime}}(1+p^{-2})^{k^2}\\ &=(\zeta(2)/\zeta(4))^{k^2}\\ &=\left(\frac{15}{\pi^2}\right)^{k^2}\quad\text{for all }k\geq1\end{aligned}\tag{17}$$

Inserting this bound into (15), we have for all fixed $k \geq 1$:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t \geq \frac{k}{1+k} \left(\frac{\pi^2}{15}\right)^k (K+L)^{1-1/k}.$$

Finally, if we pick $k = \log(K+L)$ (note that $(K+L) \geq 3$ for all nontrivial factorial ratio sequences, so we are assured that $k \geq 1$), this expression simplifies into the manageable bound

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t &\geq \frac{1}{2} \left[(K+L)^{\log\left(\frac{\pi^2}{15}\right)} \right] \left[\frac{(K+L)}{e} \right] \\ &= \frac{1}{2e} (K+L)^{1-\log\left(\frac{15}{\pi^2}\right)} \quad \left(= \frac{1}{2e}(K+L)^{.581} \right) \end{aligned} \tag{18}$$

2.4. **Bounding $\mathbf{B}(D)$.** This result, combined with equations (5) and (9), gives us the following condition on step-functions $f(x; \mathbf{a}, \mathbf{b})$ associated with a factorial ratio sequences $u_n(\mathbf{a}, \mathbf{b})$:

$$\int_0^1 \left| f(x) - \frac{(L-K)}{2} \right|^2 \mathrm{d}x \geq \frac{1}{4\pi^2 e} (K+L)^{1-\log\left(\frac{15}{\pi^2}\right)}.$$

Now, since for an integer-valued factorial ratio sequence $f(x; \mathbf{a}, \mathbf{b})$ takes on only values from $\{0, \ldots, L-K\}$, the left-hand side of this inequality can itself be no greater than $(L-K)^2/4$. Then we must have

$$\begin{aligned} \frac{(L-K)^2}{4} &\geq \frac{1}{4\pi^2 e} (K+L)^{1-\log\left(\frac{15}{\pi^2}\right)} \\ \Rightarrow (K+L) &\leq \left(\pi^2 e (L-K)^2\right)^{\frac{1}{1-\log(15/\pi^2)}} \\ &\leq 287 (L-K)^{3.44} \end{aligned}$$

true for any integer-valued factorial ratio sequence provided $(K+L) \geq 3$. Of course, since we know empirically that $\mathbf{B}(D) > 3$ for all $D = (L-K) \in \mathbb{N}$, this establishes the general bound $\mathbf{B}(D) \leq 287 D^{3.44}$ for every $D \in \mathbb{N}$.

3. Explicit bounds on $\mathbf{B}(D)$ for small D

Already, the above discussion gives bounds $\mathbf{B}(1) < 287$ and $\mathbf{B}(2) < 3110$, which are superior to those derived in [1]. But that result is the end product of a series of progressively-worse estimates designed to simplify expressions and calculations. If we retreat to the inequality in (15), our first usable bound on the size of the main integral, we may manually optimize our bound on $\mathbf{B}(D)$ for a small fixed D. We require

$$\frac{(L-K)^2}{4} \geq \max_{k\in\mathbb{R}} \left\{ \frac{k}{1+k} (M_{-k})^{-\frac{1}{k}} (K+L)^{1-1/k} \right\}$$

Computing M_{-k} to high accuracy using equation (16), we find that for $(K+L) = 43$ and $k = 7.346$, the right-hand side above is $\approx 0.2550 > 1/4$. Thus we must have that $\mathbf{B}(1) < 43$.

Similarly, we find that for $(K+L) = 202$ and $k = 9.95092$, the right-hand side above is $\approx 1.00004 > 1$. Thus we must have that $\mathbf{B}(2) < 202$. Additional computations yield $\mathbf{B}(3) < 495$ and $\mathbf{B}(4) < 926$.

4. Asymptotic bounds for $\mathbf{B}(D)$

In the previous sections, we derived a universal bound on $\mathbf{B}(D) \leq 287D^{3.44}$ that applies for all $D \in \mathbb{N}$. Already we have shown that we can do better for small D; likewise we can construct a better asymptotic bound for large D, as we outline below.

The improvement comes by tightening our estimate for M_{-k} from the one given in (17). As shown by Granville and Soundararajan [5, Theorem 3], as $k \to \infty$ we have

$$M_{-k} = \left[\prod_{p \leq k} \left(1 + p^{-1}\right)^{2k}\right] \exp\left[-\frac{2k}{\log k}\left(C + O\left(\frac{1}{\log k}\right)\right)\right]$$

where $C > 0$ is a constant. We may rewrite this as

$$M_{-k} = \left[\prod_{p \leq k} \left(1 - p^{-2}\right)^{2k} \left(1 - p^{-1}\right)^{-2k}\right] \exp\left[-\frac{2k}{\log k}\left(C + O\left(\frac{1}{\log k}\right)\right)\right]$$

$$\ll \left[\prod_{p \leq k} \left(1 - p^{-1}\right)^{-1}\right]^{2k} \exp\left(-\frac{2k}{\log k} C\right) \tag{19}$$

where we have used the fact that for k sufficiently large,

$$\prod_{p \leq k} \left(1 - p^{-2}\right)^{2k} \approx \left(\frac{6}{\pi^2}\right)^{2k}$$

$$\Rightarrow \prod_{p \leq k} \left(1 - p^{-2}\right)^{2k} \exp\left[O\left(\frac{k}{(\log k)^2}\right)\right] \ll 1.$$

Now we use the following lemma to bound that truncated Euler product.

Lemma 4.1 (Mertens' bound)**.** *For all $M > 285$,*

$$\prod_{p \leq M} \left(1 - p^{-1}\right)^{-1} \leq \frac{\log M}{e^{-\gamma}} \left(1 - \frac{1}{2(\log M)^2}\right)^{-1}$$

Proof. See Rosser and Schoenfeld [6, Theorem 7]. □

Inserting this bound into (19), we get for k sufficiently large

$$M_{-k} \ll \left[\frac{\log k}{e^{-\gamma}} \left(1 - \frac{1}{2(\log k)^2}\right)^{-1}\right]^{2k} \exp\left(-\frac{2k}{\log k} C\right).$$

Using this superior asymptotic bound for M_{-k} in (15), we find for large k that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t$$

$$\gg \frac{k}{1+k} \left[\frac{\log k}{e^{-\gamma}} \left(1 - \frac{1}{2(\log k)^2}\right)^{-1}\right]^{-2} \exp\left(\frac{2}{\log k} C\right) (K+L)^{1-1/k}$$

Once again making the choice $k = \log(K+L)$, we see that in the limit $(K+L) \to \infty$ the following inequality holds true:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t \gg (K+L)(\log(\log(K+L)))^{-2}. \tag{20}$$

This asymptotic bound is clearly superior to its counterpart (18), and just as in Section 2.4 we may use it to derive a bound on $\mathbf{B}(D)$:

$$\int_0^1 \left| f(x) - \frac{(L-K)}{2} \right|^2 \mathrm{d}x = \lim_{T\to\infty} \frac{1}{2\pi^2} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t$$

$$\Rightarrow (L-K)^2 \gg (K+L)(\log(\log(K+L)))^{-2}$$

$$\text{(note that this implies } (L-K) \gg (K+L)^{1/3}\text{)}$$

$$\Rightarrow (K+L) \ll (K+L)\frac{(\log(\log(L-K)))^2}{(\log(\log(K+L)))^2} \ll (L-K)^2(\log(\log(L-K)))^2.$$

That is, for sufficiently large D, we have

$$\mathbf{B}(D) \ll D^2(\log(\log D))^2$$

which is an improvement over the Bell and Bober result $\mathbf{B}(D) \ll D^2(\log D)^2$ [1].

5. Additional Remarks

Although the $\mathbf{B}(D)$ bounds in this paper offer significant improvement over the previous reults, they are still far from optimal. As stated in the introduction, it is known that $\mathbf{B}(1) = 9$, but even our specifically-tailored approach for that case returns at best an upper limit of 43. In order for future study to achieve tighter bounds, it seems that more attention will need to be given to the $G(it)$ term in our main integral (see equation 9). Some work has been done by the author to this effect, but no general results have yet emerged. The rationale for such study is that the weakest possible estimates for $G(it)$ were used in this paper, but the terms of $G(it)$ have a great influence on the actual value of the main integral. Indeed, that value can be computed exactly as

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(it)\zeta(1+it)|^2 \,\mathrm{d}t =$$

$$\frac{\pi^2}{6}\left[\sum \frac{(a_{k_1}, a_{k_2})}{[a_{k_1}, a_{k_2}]} + \sum \frac{(b_{l_1}, b_{l_2})}{[b_{l_1}, b_{l_2}]} - 2\sum \frac{(a_k, b_l)}{[a_k, b_l]]}\right],$$

where we may recall that all of the information about $f(x; \mathbf{a}, \mathbf{b})$ in the integral is carried by $G(it)$.

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