Gelfand Pairs, Representation Theory of the Symmetric Group, and the Theory of Spherical Functions

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Abstract

This thesis gives an introduction to the study of Gelfand pairs and their applications. We begin with a brief introduction to the notion of a Gelfand pair and then move to some of the foundational results concerning Gelfand pairs. Next, we explore specific examples of Gelfand pairs, developing tools of independent interest as we progress. We find that consideration of a specific example of a Gelfand pair and of the tools used in our study naturally leads us to a discussion of the representation theory of the symmetric group. We then conclude our study by developing the theory of spherical functions on groups, which gives us a glimpse of the relevance of Gelfand pairs to areas of mathematics outside of representation theory.

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1 Introduction

The aim of this thesis is to give an introduction to the theory of Gelfand pairs and to explore related topics that emerge naturally from a discussion of Gelfand pairs. Attempts have been made to make this work as self-contained and accessible as possible. Very little background is necessary to understand the results and arguments here beside command of basic linear algebra, group theory, and elementary notions from representation theory.

There are two natural and equivalent definitions of a Gelfand pair that we give here (equivalence of the definitions will be proved later).

The first definition is natural and easy to define. Let G be a finite group and $H \leq G$ a subgroup. Then we say that (G,H) is a *Gelfand pair* if for every irreducible representation (π, V) of G, the subspace $V^H \subseteq V$ of H-fixed vectors is at most one dimensional, or equivalently, the restriction $\pi|_H$ contains the trivial representation 1_H with multiplicity at most 1.

The second (equivalent) definition of a Gelfand pair requires the notion of an *induced representation*, which we recall here:

Let G be a finite group, $H \leq G$ a subgroup of G and $\pi : H \to GL(V)$ a representation of H. We define the induced representation $Ind_H^G \pi$ to be the vector space of all functions $f : G \to V$ such that $f(hx) = \pi(h)f(x)$ for $h \in H$ and $x \in G$. Now we define, for $g \in G$:

$$(\pi^G(g)f)(x) = f(xg).$$

Under this definition, π^G is a representation defined by g acting on $Ind_H^G \pi$ by right translation. We will usually abuse notation and write π^G simply as π when it is clear that we are talking about the induced representation.

Now if G is any group with a representation (π, V), then it is an elementary fact that π can be written uniquely as a direct sum of irreducible representations

$$\pi = \bigoplus_{i=1}^{n} d_i \pi_i$$

where $\{\pi_1, \ldots, \pi_n\}$ are the distinct irreducible representations of G and the d_i are non-negative integers.

The second and equivalent definition is: a *Gelfand pair* is a pair (G,H) where G is a group with subgroup $H \leq G$ that satisfies a certain property. This property is that when the representation $Ind_H^G 1_H$ constructed by inducing the trivial representation of H, is decomposed as a direct sum of irreducible representations, each irreducible occurs with multiplicity less than or equal to one, i.e., if we write

$$Ind_H^G 1_H = \bigoplus_{i=1}^{n} d_i \pi_i$$

then $d_i \leq 1$ for $i = 1, \ldots, n$.

It turns out that this "multiplicity-free" property can be detected by a certain ring of functions on G. Consider the set of H-bi-invariant functions,

$$\mathcal{H} = \{\varphi : G \to \mathbb{C} : \varphi(hgh') = \varphi(g) \text{ for } h, h' \in H, g \in G\}.$$

We can consider $\mathcal{H}$ as an algebra over the complex numbers in which the ring structure is endowed by the multiplication rule

$$(\varphi_1 * \varphi_2)(g) = \frac{1}{|H|} \sum_{s \in G} \varphi_1(s) \varphi_2(s^{-1} g).$$

This algebra is known as the *Hecke algebra* of the pair (G, H). The first major result of this paper will be to show that it is equivalent to say that (G, H) is a Gelfand pair and that the Hecke algebra of (G, H) is a commutative ring.

We will then explore some interesting examples of multiplicity free induced representations including the so-called *Gelfand-Graev Representation* of the general linear group over a finite field $GL_n(\mathbb{F}_q)$ and the representation of the symmetric group induced by the trivial representation of the subgroup $S_m \times S_n \leq S_{m+n}$ where m and n are positive integers. In exploring these examples, we develop general tools which are useful for detecting Gelfand pairs.

This second example, in which we prove that $(S_{m+n}, S_m \times S_n)$ is a Gelfand pair, leads us naturally to generalize the subgroup $S_m \times S_n \leq S_{m+n}$. That is, we consider the subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_k} \leq S_n$ where $(\lambda_1, \ldots, \lambda_k)$ is a partition of a positive integer n. Although $(S_n, S_{\lambda_1} \times \cdots \times S_{\lambda_k})$ is not in general a Gelfand pair, we may utilize some of the same tools we used to study Gelfand pairs to deduce an interesting connection between the study of partitions of n and the irreducible representations of S_n, demonstrating the power of some of the tools used to study Gelfand pairs. What we will show is that there exists a natural bijection between the partitions of n and irreducible representations of S_n.

The final major result of this paper comes from a discussion of spherical functions, a special class of functions from a group G into the complex numbers. We will see that the discussion of these spherical functions leads to an interesting theory that can be viewed as a generalization of Fourier analysis. It is a classical result from Fourier analysis that if we let

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad -\pi < x < \pi$$

then $\{e_n\}_{-\infty}^{\infty}$ forms an orthonormal basis for $L^2([-\pi, \pi])$. As a result, if we let

$$f_m = \sum_{n=-m}^{m} \langle f, e_n \rangle e_n$$

then $f_m \to f$ as $m \to \infty$ for a reasonably nice function f. This fact has the following generalization, namely the Schur Orthogonality Relation: If G is a compact group, then the irreducible characters of G form an orthonormal basis of the space of class functions on G.

Thus, Schur Orthogonality amounts to Fourier Analysis on finite or compact groups. This fact has a further significant generalization to Gelfand pairs, which is as follows. Given a Gelfand pair (G, H), if (π, V) is a representation of G with a unique H-fixed vector $v \in V$, the associated *spherical function* will be given by $\sigma(g) = \langle \pi(g)v, v \rangle$. We begin our consideration of spherical functions by studying the example of diagonally embedding a group G into the direct product $G \times G$. As it turns out, $(G \times G, G)$ is a Gelfand pair, and the study of spherical functions in this important example allows us to recover classical character theory, i.e., we find that the spherical functions are the irreducible characters of G, which form an orthonormal basis of the space of class functions on G.

Generalizing this example to the more abstract setting of spherical functions on arbitrary Gelfand pairs (G, H) will allow us to construct an orthonormal basis of the space $C(H \backslash G / H)$ consisting of H-bi-invariant functions from G into the complex numbers. This result, a generalization of the classical results from Fourier analysis and character theory, cuts to the core of why Gelfand pairs are important not just to representation theory but to other areas of mathematics.

2 Foundations

We lay some of the foundations here that are necessary for the study of Gelfand pairs. We begin our theory with a proposition:

Proposition 1. *Let $H \leq G$ be a subgroup of a finite group G. Let (ψ_1, V_1) and (ψ_2, V_2) be representations of H. Let $\mathcal{M} = Hom_G(Ind_H^G \psi_1, Ind_H^G \psi_2)$ and*

$$\mathcal{H} = \{\varphi : G \to End_{\mathbb{C}}(V_1, V_2) : \varphi(hgh') = \psi_2(h)\varphi(g)\psi_1(h') \text{ for } h, h' \in H, g \in G\}.$$

Then $\mathcal{M}$ and $\mathcal{H}$ are isomorphic $\mathbb{C}$-algebras, where the multiplicative structure in $\mathcal{M}$ is composition of homomorphisms and that in $\mathcal{H}$ is convolution:

$$(\varphi_1 * \varphi_2)(g) = \frac{1}{|H|} \sum_{s \in G} \varphi_1(s) \circ \varphi_2(s^{-1}g).$$

Proof. For $M \in \mathcal{M}$, define $\varphi_M : G \to End_{\mathbb{C}}(V_1, V_2)$ by $\varphi_M(g)v = (Mf_v)(g)$ for $v \in V$, where $f_v(h) = \psi_1(h)v$ for $h \in H$ and $f_v(g) = 0$ for $g \notin H$. It is clear that under this definition $f_v \in Ind_H^G \psi_1$.

That $\varphi_M \in \mathcal{H}$ is a computation. Let $h, h' \in H$ and $g \in G$. Then:

$$\begin{aligned}
\varphi_M(hgh')v &= (Mf_v)(hgh') \\
&= \psi_2(h)(Mf_v)(gh') \\
&= \psi_2(h)(M\psi_1(h')f_v)(g) \\
&= \psi_2(h)(Mf_{\psi_1(h')v})(g) \\
&= \psi_2(h)\varphi_M(g)\psi_1(h')v
\end{aligned}$$

so that indeed this map is well-defined.

To see that it is a vector space homomorphism, let $c \in \mathbb{C}$ and $M, M' \in \mathcal{M}$. Then for $v \in V$, we simply use the definition of addition of functions to attain:

$$\begin{aligned}
\varphi_{M+cM'}(g)v &= ((M + cM')f_v)(g) \\
&= (Mf_v + (cM')f_v)(g) \\
&= (Mf_v)(g) + c(M'f_v)(g) \\
&= \varphi_M(g)v + c\varphi_{M'}(g)v \\
&= (\varphi_M(g) + c\varphi_{M'}(g))v.
\end{aligned}$$

So indeed the map $M \mapsto \varphi_M$ is a vector space homomorphism.

To show it is isomorphism, we construct its two sided inverse. Let $\varphi \in \mathcal{H}$. Then define M_φ by $(M_\varphi f)(g) = \sum_r \varphi(gr^{-1})f(r)$ where r runs over a set of coset representatives of G/H and $f \in Ind_H^G \psi_1$. Now let $h \in H$ and $g \in G$. Then

$$\begin{aligned}(M_\varphi f)(hg) &= \sum_r \varphi(hgr^{-1})f(r)\\ &= \psi_1(h)\sum_r \varphi(gr^{-1})f(r)\\ &= \psi_1(h)(M_\varphi f)(g)\end{aligned}$$

so that indeed $M_\varphi f \in Ind_H^G \psi_2$

Now for $g, g' \in G$ it holds:

$$\begin{aligned}(M_\varphi \psi_1^G(g')f)(g) &= \sum_r \varphi(gr^{-1})(\psi_1^G(g')f)(r)\\ &= \sum_r \varphi(gr^{-1})f(rg') \text{ since } f \in Ind_H^G\psi_1\\ &= \sum_r \varphi(gg'r^{-1})f(r)\\ &= (M_\varphi f)(gg')\\ &= (\psi_2^G(g')M_\varphi f)(g)\end{aligned}$$

proving that $M_\varphi \in \mathcal{M}$. Now we check that the maps are indeed inverses, i.e., that $M_{\varphi_M} = M$ for $M \in \mathcal{M}$ and $\varphi_{M_\varphi} = \varphi$ for $\varphi \in \mathcal{H}$. Now for $v \in V$ and $g \in G$, we have:

$$\begin{aligned}\varphi_{M_\varphi}(g)v &= (M_\varphi f_v)(g)\\ &= \sum_r \varphi(gr^{-1})f_v(r)\\ &= \varphi(g)v\end{aligned}$$

where the last equality holds because $f_v(r) = 0$ for all $r \notin H$ and for the one coset representative $h \in H$, we have

$$\varphi(gh^{-1})f_v(h) = \varphi(g)\psi_1(h^{-1})\psi_1(h)v = \varphi(g)v.$$

That $M_{\varphi_M} = M$ for $M \in \mathcal{M}$ is a similarly straightforward calculation and is thus omitted.

To complete our proof we only need to check that this map respects the multiplicative structure, i.e., $\varphi_{M\circ M'} = \varphi_M * \varphi_{M'}$. This is another computation. For any $g \in G$ and any $v \in V$ we have:

$$
\begin{aligned}
\varphi_M * \varphi_{M'}(g)v &= \frac{1}{|H|}\sum_{s\in G}\varphi_M(s)\circ\varphi_{M'}(s^{-1}g)v \\
&= \frac{1}{|H|}\sum_{s\in G}\varphi_M(s)\circ(M'f_v)(s^{-1}g) \\
&= \frac{1}{|H|}\sum_{\substack{s\in G\\ g\in sH}}\varphi_M(s)\circ(M'\psi_1(s^{-1}g))v \\
&= \frac{1}{|H|}\sum_{\substack{s\in H\\ g\in sH}}M\psi_1(s)(M'\psi_1(s^{-1}g))v \\
&= \begin{cases} (M\circ M')f_v(g) & \text{if } g\in H \\ 0 & \text{otherwise} \end{cases} \\
&= \varphi_{M\circ M'}v
\end{aligned}
$$

and the $\mathbb{C}$-algebra isomorphism is established. □

Proposition 2. *Let $H \leq G$ be a subgroup of a finite group G and $\psi : H \to GL(V)$ a representation of H. Let $\mathcal{M} = End_G(Ind_H^G\psi)$ and*

$$\mathcal{H} = \{\varphi : G \to End_{\mathbb{C}}(V) : \varphi(hgh') = \psi(h)\varphi(g)\psi(h') \text{ for } h, h' \in H, g \in G\}.$$

Then $\mathcal{M} \cong \mathcal{H}$ as $\mathbb{C}$-algebras.

Proof. This is an immediate consequence of proposition 1. □

Definition 1. *$\mathcal{H}$ as defined in the proposition is called the Hecke algebra of the pair (G, H).*

As noted in the introduction, the Hecke algebra of a pair (G, H) will be useful in detecting whether or not the (G, H) is a Gelfand pair. In order to uncover this connection, we must first prove some foundational facts:

Proposition 3. *Suppose G is a finite group and $H \leq G$ a subgroup. Suppose further that (π_1, V_1) and (π_2, V_2) are representation of H and $\mathcal{H}$ is the associated Hecke algebra. Now let $H^g = H \cap gHg^{-1}$ for $g \in G$. Then the subspace of $\mathcal{H}$ consisting of all functions supported on the coset HgH is isomorphic as a vector space to $Hom_{H^g}(\pi_1^g, \pi_2|_{H^g})$ where $\pi_1^g, \pi_2|_{H^g}$ are the two representations of H^g defined by $\pi_1^g(h) = \pi_1(g^{-1}hg)$ and $\pi_2|_{H^g}$ is just the restriction of π_2 to H^g.*

Proof. Let $g \in G$ be fixed and $\varphi \in \mathcal{H}$ supported on HgH be given. Define a map $M_\varphi : V_1 \to V_2$ by $M_\varphi(v) = \varphi(g)v$ for $v \in V_1$. Then for $h \in H^g$ it is true that:

$$M_\varphi(\pi_1^g(h)v) = \varphi(g)(\pi_1(g^{-1}hg)v) = \varphi(hg)v = \pi_2(h)\varphi(g)v = \pi_2(h)|_{H^g} M_\varphi(v).$$

So M_φ is a H^g-module homomorphism, i.e., $M_\varphi \in Hom_{H^g}(\pi_1^g, \pi_2|_{H^g})$.

Now given $M \in Hom_{H^g}(\pi_1^g, \pi_2|_{H^g})$, define $\varphi_M : G \to End_{\mathbb{C}}(V_1, V_2)$ by $\varphi_M(hgh')v = \pi_2(h)\varphi(g)\pi_1(h')v$ if $h, h' \in H$ and $\varphi_M(\hat{g})v = 0$ if $\hat{g} \notin HgH$. It is easy to see that under this definition, φ_M is an element of $\mathcal{H}$ supported on HgH.

It is also easy to see that the maps $\varphi \mapsto M_\varphi$ and $M \mapsto \varphi_M$ are mutual inverses. This is sufficient to prove the vector space isomorphism. □

Proposition 4 (Mackey's Theorem)**.** *Suppose G is a finite group and $H \leq G$ a subgroup. Suppose further that (π_1, V_1) and (π_2, V_2) are representations of H. For $g \in G$, let H^g be defined as in the previous proposition. Similarly, let $\pi_1^g, \pi_2|_{H^g}$ be defined as in the previous proposition. Then*

$$Hom_G(Ind_H^G \pi_1, Ind_H^G \pi_2) \cong \bigoplus_{g \in HgH} Hom_{H^g}(\pi_1^g, \pi_2|_{H^g})$$

and as a consequence

$$dim\ Hom_G(Ind_H^G \pi_1, Ind_H^G \pi_2) = \sum_{g \in HgH} dim\ Hom_{H^g}(\pi_1^g, \pi_2|_{H^g}).$$

Proof. This is clear from what we have shown so far. □

The following result, known as Schur's Lemma, is important in group representation theory and will be used here throughout.

Proposition 5 (Schur's Lemma)**.** *Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible representations of a group G. Then any element of $Hom_G(V_1, V_2)$ is either the zero map or an isomorphism. As a corollary, $Hom_G(\pi, \pi) \cong \mathbb{C}$ for any irreducible representation π of G.*

Proof. Suppose $A \in Hom_G(V_1, V_2)$ is nonzero. Since A is a G-module homomorphism, $image(A)$ is a G-invariant subspace of V_2. But that V_2 is irreducible forces that either $image(A) = 0$ or $image(A) = V_2$. The first case is false since A is nonzero. Thus, the second case holds, i.e., A is surjective.

Similarly, $ker(A)$ is a G-invariant subspace of V_1, and, thus, either $ker(A) = 0$ or $ker(A) = V_1$. The second case is false since A is nonzero; thus the fist case holds, i.e., A is injective. This proves that A is a bijective homomorphism, i.e., an isomorphism, which proves the fist statement.

To see that the second statement holds, we let $\psi \in Hom_G(\pi, \pi)$ be nonzero. Let λ be an eigenvalue of ψ (we are assured that such an eigenvalue exists since π is finite dimensional and since $\mathbb{C}$ is an algebraically closed field). Then $\psi - \lambda I \in Hom_G(\pi, \pi)$ is not invertible so by what we have just proven it is zero, i.e. $\psi = \lambda I$. □

Proposition 6. *Let ρ be a finite dimensional representation of a group G. Then ρ is multiplicity free if and only if $Hom_G(\rho, \rho)$ is commutative. In this case, dim $Hom_G(\rho, \rho)$ is equal to the number of irreducible constituents in the direct sum decomposition of ρ.*

Proof. Let $\rho = \bigoplus_{i=1}^{m} \rho_i$ be the direct sum decomposition of ρ into irreducible representations ρ_i with the ρ_i's ordered so that equivalent representations occur consecutively. Now any $\sigma \in Hom_G(\rho, \rho)$ can be written as a block diagonal matrix $A = (a_{ij})$, where $a_{ij} \in Hom_G(\rho_j, \rho_i)$. But Schur's lemma gives us that

$$Hom_G(\rho_j, \rho_i) \cong \begin{cases} \mathbb{C} & \text{if } \rho_i \cong \rho_j \\ \{0\} & \text{otherwise.} \end{cases}$$

Thus, we can see that $Hom_G(\rho, \rho)$ is isomorphic to an algebra of block diagonal matrices over the complex numbers where the size of the blocks corresponds to the multiplicity of each irreducible representation ρ_i. Composition of homomorphisms corresponds to matrix multiplication so that $Hom_G(\rho, \rho)$ will indeed be commutative if and only if each block is of size one, i.e., if and only if ρ is multiplicity free as claimed.

If this is the case, then $Hom_G(\rho, \rho)$ is isomorphic to the algebra of $m \times m$ diagonal matrices over the complex numbers so that indeed $\dim \mathrm{Hom}_G(\rho, \rho) = m$, which completes the proof. □

Proposition 7. *Let $H \leq G$ be a subgroup of the finite group G and let $\psi : H \to V$ be a finite dimensional representation. Then $Ind_H^G \psi$ is multiplicity free if and only if the associated Hecke algebra $\mathcal{H}$ is commutative.*

Proof. This follows from Propositions 2 and 6. □

Definition 2. *Suppose G is a group, $H \leq G$ a subgroup and (ψ, V) a representation of H. If $Ind_H^G \psi$ is multiplicity free, we say that (G, H, ψ) is a Gelfand triple. In the case that $\psi = 1_H$ we say that (G, H) is a Gelfand pair or that H is a Gelfand subgroup of G.*

From these results, we can see that the study of Gelfand pairs is intimately linked with the study of Hecke algebras. In the following section, we will explore specific examples of Gelfand pairs and will produce general tools that allow us to exploit this fundamental connection.

3 Examples

We now consider some specific examples of Gelfand triples/pairs.

Let $\mathbb{F}_q$ be any finite field, $n \in \mathbb{N}$ and let $G = GL_n(\mathbb{F}_q)$. Now let $N \leq G$ consist of the subgroup of all upper triangular matrices with 1's on the diagonal and let $\psi : \mathbb{F}_q \to \mathbb{C}^*$ be a nontrivial linear character. Define $\psi_0 : N \to \mathbb{C}$ by

$$\psi_0(a) = \psi(a_{12} + a_{23} + \cdots + a_{n-1,n}) \text{ for } a = (a_{ij}) \in N.$$

Our goal will be to show that the induced representation $Ind_N^G \psi_0$ is multiplicity free. To this end, we will show that the associated Hecke algebra $\mathcal{H}$ is commutative. First we need a proposition:

Proposition 8 (Bruhat Decomposition). *$G = GL_n(\mathbb{F}_q)$ can be decomposed as a disjoint union of double cosets with representatives of the following form:*

$$GL_n(\mathbb{F}_q) = \amalg_{w \in W} BwB$$

where $B \leq G$ is the Borel subgroup of upper triangular matrices and W is the group of $n \times n$ permutation matrices.

Proof. I will first prove that G can indeed be decomposed as such a union. Then I will prove that the union is disjoint. I will proceed by induction on n. For $n = 1$, $B = G$ so the result holds trivially. Now let $n > 1$ and let $g \in G$ be arbitrary. I will prove that there exists a permutation matrix w that is an element of BgB, which suffices to prove our assertion. We consider two cases:

Case 1: $g_{n,1} \neq 0$. In this case, we may multiply g by appropriate elements $b_0, b_1 \in B$ on the left and right so that $b_0 g b_1$ is 0 in every entry in the first column and last row besides the $n,1$th entry which is equal to $g_{n,1}$. We may then normalize this to 1 by dividing by $g_{n,1}$. Now by disregarding the first column and last row, we have an $n-1 \times n-1$ matrix g^* to which we may apply our induction hypothesis to find 2 $n-1 \times n-1$ upper triangular matrices b', b'' such that $b' g^* b''$ is an $n-1 \times n-1$ permutation matrix. By expanding these matrices $b' b_0 g b_1 b''$ will give an $n \times n$ permutation matrix.

Case 2: $g_{n,1} = 0$. Then pick i as great as possible and j as least as possible such that $g_{i,1} \neq 0$ and $g_{n,j} \neq 0$. By multiplying by elements of B on the right and left, we can clear the first and jth columns and the ith and last rows, except for the entries $g_{i,1}$ and $g_{n,j}$, which we can normalize to 1. We can then apply the induction hypothesis to the matrix obtained by removing these rows and columns, to create a permutation matrix. This completes the induction.

Now I demonstrate that the union is indeed disjoint. Let $w_1, w_2 \in W$ be representatives of the same double coset. Then there exists $b \in B$ such that $w_1 b w_2^{-1} \in B$. Now since $w_1 b w_2^{-1}$ is just b with some rows and columns permuted, if we changed some of the nonzero entries of b to any arbitrary field element, the result would still be upper triangular, i.e., still in B. So let us then replace b with the identity matrix. Then: $w_1 w_2^{-1} \in B$. But since $w_1, w_2 \in W$, it follows that $w_1 w_2^{-1} \in B \cap W = \{I_n\}$, where I_n is the n x n identity matrix. Thus $w_1 = w_2$ and the result is proven. □

Proposition 9 (Modified Bruhat Decomposition)**.** *$GL_n(\mathbb{F}_q)$ may be decomposed as a disjoint union of double cosets in the following fashion:*

$$GL_n(\mathbb{F}_q) = \sqcup_{m \in M} NmN$$

where N is the subgroup of upper triangular matrices with 1's on the diagonal and M is the subgroup of monomial matrices (matrices with exactly one nonzero entry in every row and column).

Proof. Let $D \leq G$ be the subgroup of diagonal matrices. The result, ignoring disjointness of the union, follows from the Bruhat decomposition because $B = DN = ND$ and $M = DW = WD$. That the union is disjoint can be deduced as before. □

Now we introduce a powerful tool for proving that a Hecke algebra $\mathcal{H}$ is commutative. We define an involution of a group G to be a map $\iota : G \to G$ that satisfies $\iota^2 = id$ and $\iota(g_1 g_2) = \iota(g_2)\iota(g_1)$ for $g_1, g_2 \in G$. Similarly, a map $\bar{\iota} : \mathcal{H} \to \mathcal{H}$ is called an involution of $\mathcal{H}$ if $\bar{\iota}^2 = id$ and $\bar{\iota}(\varphi_1 \varphi_2) = \bar{\iota}(\varphi_2)\bar{\iota}(\varphi_1)$ for any $\varphi_1, \varphi_2 \in \mathcal{H}$.

Proposition 10. *Let H be a subgroup of a finite group G and (ψ, V) a one-dimensional representation of H. Let $\mathcal{H}$ be the associated Hecke algebra. Now suppose that $\iota : G \to G$ is an involution satisfying $\psi(\iota(h)) = \psi(h)$ for all $h \in H$. Then the map $\bar{\iota} : \mathcal{H} \to \mathcal{H}$ defined by $\bar{\iota}(\varphi(g)) = \varphi(\iota(g))$ where $g \in G$ is an involution of $\mathcal{H}$.*

Proof. Observe that if $\varphi \in \mathcal{H}$, $h_1, h_2 \in \mathcal{H}$, and $g \in G$, then

$$\begin{aligned}\bar{\iota}(\varphi(h_1 g h_2) &= \varphi(\iota(h_1 g h_2)) \\ &= \varphi(\iota(h_2)\iota(g)\iota(h_1)) \text{ since } \iota \text{ reverses multiplication in } G \\ &= \psi(h_2)\varphi(\iota(g))\psi(h_1) \text{ since } \varphi \in \mathcal{H} \text{ and } \psi(\iota(h)) = \psi(h) \text{ for all } h \in H \\ &= \psi(h_1)\bar{\iota}(\varphi(g))\psi(h_2)\end{aligned}$$

where the last equality holds since V is 1-dimensional implies that $End_{\mathbb{C}}V$ is commutative. This proves that $\bar{\iota}(\varphi) \in \mathcal{H}$.

Further, note that:

$$\begin{aligned}\bar{\iota}^2(\varphi(g)) &= \bar{\iota}(\varphi(\iota(g))) \\ &= \varphi(\iota^2(g)) \\ &= \varphi(g) \text{ since } \iota \text{ is an involution.}\end{aligned}$$

Thus, $\bar{\iota}^2 = id$. Finally, that $\bar{\iota}$ reverses multiplication follows easily from that fact that $End_{\mathbb{C}}V$ is commutative, which completes the proof. □

Proposition 11. *Suppose that there is an involution $\iota : G \to G$ such that $HgH = H\iota(g)H$ for all $g \in G$. Then (G, H) is a Gelfand pair.*

Proof. We wish to show that $Ind_N^G \psi$ is multiplicity-free, where $\psi = 1_H$. We construct $\bar{\iota}$ according to Proposition 10 and note that since ψ is the trivial representation that the requirement that $\psi(\iota(h)) = \psi(h)$ for all $h \in H$ holds trivially.

Now $\mathcal{H}$ consists of all function $\varphi : G \to \mathbb{C}$ such that φ is constant on $H - H$ double cosets (this is because ψ is trivial). Thus, $\bar{\iota}$ acts by the identity on $\mathcal{H}$. Finally, observe that $\bar{\iota} = id$ being an involution implies that $\mathcal{H}$ is commutative. □

We now return to our study of the example introduced. We will apply the involution method to attain our desired result in this example, but first we will need this basic fact:

Proposition 12. *Let $\mathbb{F}$ be any field and suppose $m = (m_{ij}) \in \mathbb{F}^{n\times n}$ is a monomial matrix and that m_{ij} and $m_{i+1,k}$ are nonzero implies that $k \le j + 1$. Then m is of the form*

$$m = \begin{pmatrix} & & & D_1 \\ & & D_2 & \\ & \iddots & & \\ D_p & & & \end{pmatrix}$$

where p is a positive integer and the D_i are diagonal matrices for $i = 1, \dots, p$.

Proof. The proof is by induction on n. For $n = 1$, the conclusion follows trivially, so suppose $n > 1$ and that our conclusion holds for all matrices that satisfy the above conditions and of are size less than n. Now since m is monomial, there exists a unique $i \in \{1, \dots, n\}$ such that $m_{1i} \neq 0$. If $i = n$, then we note that the $n-1 \times n-1$ matrix constructed by removing the first row and last column of m is of the desired form by the induction hypothesis. So suppose that $i < n$. We claim then that $m_{2,k} \neq 0$ forces $k = i+1$. If not, then $k < i$. But then for any $j \geq 2$, we have that $m_{jl} \neq 0$ implies that $l < i$ by the assumption that m_{ij} and $m_{i+1,k}$ both nonzero implies that $k \leq j+1$. But now this implies that column $i+1$ is composed purely of zeros, a contradiction. This proves now that $m_{2,i+1}$ is nonzero. Repeating this argument $n-i$ times proves that $m_{1+k,i+k}$ is nonzero for $k = 1, \dots, n-i$ so that indeed m is of the form:

$$m = \begin{pmatrix} 0 & D \\ * & 0 \end{pmatrix}$$

where D is an $n-i+1 \times n-i+1$ diagonal matrix. Now applying the induction hypothesis to $*$, our conclusion follows. □

Proposition 13. *Let $G = GL_n(\mathbb{F}_q)$, and let N and ψ_0 be as above. Then $Ind_N^G \psi_0$ is multiplicity free.*

Proof. Let $g \in G$ and consider the double coset NgN. By the modified Bruhat decomposition proposition, we may find a monomial matrix $m = (m_{ij})$ such that $NgN = NmN$. Assume that there is some $\varphi \in \mathcal{H}$ and some $g \in NmN$ with $\varphi(g) \neq 0$. Now I claim that if m_{ij} and $m_{i+1,k}$ are both nonzero entries of m, then $k \leq j+1$ so that m is of the form

$$m = \begin{pmatrix} & & & D_1 \\ & & D_2 & \\ & \iddots & & \\ D_p & & & \end{pmatrix}$$

where the D_i are diagonal matrices for $i = 1, \dots, p$.

Now suppose for a contradiction that $m_{ij} \neq 0$ and $m_{i+1,k} \neq 0$ with $k > j+1$. Since $m_{ij} \neq 0$ and since ψ is non-trivial, we may select $t \in \mathbb{F}_q$ such that $tm_{ij} \notin \ker \psi$, i.e., $\psi(tm_{ij}) \neq 1$. (This is because $\mathbb{F}_q$ is a field). Now let

$$x = I_n + tm_{ij} e_{i,i+1},$$

$$y = I_n + tm_{i+1,k}e_{jk}$$

where e_{jk} is an $n \times n$ matrix that is 1 at position j, k and zero elsewhere.

It is easy to see that $xm = (I_n + tm_{ij}e_{i,i+1})m = m + tm_{ij}m_{i,i+1}e_{ik} = my$.

Now, it is a simple computation that $\psi_0(x) = \psi(tm_{ij}) \neq 1$ and $\psi_0(y) = \psi(0) = 1$.

Now since $x, y \in N$ and $\varphi \in \mathcal{H}$, it holds that

$$\psi_0(x)\varphi(m) = \varphi(xm) = \varphi(my) = \varphi(m)\psi_0(y).$$

But then

$$(\psi_0(x) - \psi_0(y))\varphi(m) = 0$$

which forces $\varphi(m) = 0$ since our above computation demonstrates that $\psi_0(x) - \psi_0(y)$ is nonzero. But then $\varphi(NmN) = 0$, a contradiction. This shows that m does indeed adopt the given form.

Next, I claim that each D_i is a homothety, i.e., a scalar matrix, for each $i = 1, \ldots, p$. This is equivalent to the statement that $m_{ij} \neq 0$ and $m_{i+1,j+1} \neq 0$ implies $m_{ij} = m_{i+1,j+1}$. So suppose that m_{ij} and $m_{i+1,j+1}$ are nonzero. Further, suppose for a contradiction that $m_{ij} \neq m_{i+1,j+1}$. Then $m_{ij} - m_{i+1,j+1} \neq 0$ so that we may select $s \in \mathbb{F}_q$ such that $s(m_{ij} - m_{i+1,j+1}) \notin \ker\psi$. So if we alter x and y from above to be

$$x = I_n + sm_{ij}e_{i,i+1},$$

$$y = I_n + sm_{i+1,j+1}e_{j,j+1}$$

then as above a simple calculation gives

$$(\psi_0(x) - \psi_0(y))\varphi(m) = 0.$$

Now it is easy to see from the formulation of x and y that $\psi_0(x) = \psi(sm_{ij})$ and $\psi_0(y) = \psi(sm_{i+1,j+1})$. But now since

$$\psi_0(x) - \psi_0(y) = \psi(sm_{ij}) - \psi(sm_{i+1,j+1}) = \psi(s(m_{ij} - m_{i+1,j+1})) \neq 0$$

it follows that $\varphi(m) = 0$, a contradiction. Thus $m_{ij} = m_{i+1,j+1}$ and indeed each D_i is a homothety.

Finally, we consider the involution $\iota : G \to G$ defined by $\iota(g) = hg^T h$ where

$$h = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

It is a simple computation to see that ι thus defined fixes matrices m of the form described above (for this, note that right multiplication by h reverses the order of columns, while left multiplication by h reverses the order of rows) and that $\psi_0(\iota(n)) = \psi_0(n)$ for all $n \in N$. Thus, we find that $\bar{\iota}$ as defined earlier is an involution of $\mathcal{H}$. Now we claim that $\bar{\iota}$ is the identity map on $\mathcal{H}$. Once we have substantiated this claim, we will be done proving that $\mathcal{H}$ is commutative because indeed this implies that for any $\varphi_1, \varphi_2 \in \mathcal{H}$, it holds:

$$\varphi_1 * \varphi_2 = \bar{\iota}(\varphi_1 * \varphi_2) = \bar{\iota}(\varphi_2) * \bar{\iota}(\varphi_1) = \varphi_2 * \varphi_1$$

where the second equality holds since $\bar{\iota}$ is an antihomomorphism.

But the claim follows trivially from what we have already proven: that φ is determined by its values on the monomial matrices m and that ι fixes these matrices, i.e., for m a matrix of the above form it holds:

$$\bar{\iota}(\varphi(m)) = \varphi(\iota(m)) = \varphi(m)$$

and our proof is complete. □

Thus, we have seen in action the power of the involution method as a tool to study Gelfand pairs. We will again apply this method to another interesting example: that of the symmetric group S_n. There is a natural embedding $S_n \times S_m \hookrightarrow S_{n+m}$ in which S_n acts on the first n elements of $\{1, \ldots, n+m\}$ and S_m acts on the last m. We will again use the involution method to prove that $S_n \times S_m \hookrightarrow S_{n+m}$ is a Gelfand subgroup.

Proposition 14. *$S_n \times S_m$ is a Gelfand subgroup of S_{n+m}.*

Proof. Take $H = S_n \times S_m$ and $G = S_{n+m}$. Define the map $\iota : G \to G$ to be the involution $g \mapsto g^{-1}$. We must check that each double coset HgH is fixed by ι.

We may identify the elements of S_{n+m} with permutation matrices. We claim that each double coset HgH has a representative of the form

$$\begin{pmatrix} I_l & 0 & 0 & 0 \\ 0 & 0_{n-l} & 0 & I_{n-l} \\ 0 & 0 & I_{m-n+l} & 0 \\ 0 & I_{n-l} & 0 & 0_{n-l} \end{pmatrix}.$$

We may represent g in block form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, and D are matrices with only 1's and 0's and at most one 1 in any row or column, i.e., as permutation matrices. A is an $n \times n$ matrix and D is a $m \times m$ matrix. Let l denote the rank of A. Then B and C have rank $n - l$ since g is a permutation matrix. Thus D has rank $m - n + l$. Multiplying on the left by elements of S_n corresponds to permuting rows, while multiplying on the right by elements of S_n corresponds to permuting columns. Thus, multiplying on the left and right by elements of S_n allows us to rearrange the nonzero elements of A so that they are in the top left hand corner. Similarly, multiplying by elements of S_m will rearrange D so that its nonzero elements are in the top left hand corner. This yields a matrix of the form

$$\begin{pmatrix} U_l & 0 & 0 & 0 \\ 0 & 0_{n-l} & 0 & W_{n-l} \\ 0 & 0 & V_{m-n+l} & 0 \\ 0 & X_{n-l} & 0 & 0_{n-l} \end{pmatrix}$$

where $U_l, V_{m-n+l}, W_{n-l}, X_{n-l}$ are all permutation matrices. Note that we may naturally identify $S_l \times S_{n-l} \times S_{m-n+l} \times S_{n-l}$ as a subgroup of $S_n \times S_m$. Multiplying our matrix on the left and right by elements of $S_l \times S_{n-l} \times S_{m-n+l} \times S_{n-l}$ will put it in the form

$$\begin{pmatrix} I_l & 0 & 0 & 0 \\ 0 & 0_{n-l} & 0 & I_{n-l} \\ 0 & 0 & I_{m-n+l} & 0 \\ 0 & I_{n-l} & 0 & 0_{n-l} \end{pmatrix}$$

which verifies the claim. It is easy to see that squaring such a matrix gives an $n + m \times n + m$ identity matrix. That is, ι fixes such matrices. Thus, $S_n \times S_m \leq S_{n+m}$ is a Gelfand subgroup. □

We have found that if $\lambda \in \mathbb{Z}^+$ and $m, n \in \mathbb{Z}^+$ satisfy $n + m = \lambda$, then the induced representation, $Ind_{S_m \times S_n}^{S_\lambda} 1_{S_m \times S_n}$ is multiplicity free. This motivates us to wonder about the behavior of a more general case. That is, if λ is a positive integer and $\{\lambda_i\}$ positive integers satisfying $\sum_i \lambda_i = \lambda$, then what does the representation of S_λ induced by the trivial character on $\prod_i S_{\lambda_i}$ look like? We explore the answer to this question in the following section, finding that such a question leads to an interesting story that classifies the irreducible representations of the symmetric group S_λ.

4 Representation Theory of the Symmetric Group

To discuss the representation theory of the symmetric group, we introduce the notion of a partition. Suppose n is a positive integer. Then we say that $\lambda = (\lambda_1, \ldots, \lambda_k)$ forms a partition of n if $n = \sum_{i=1}^{k} \lambda_i$ and if $\lambda_1 \geq \cdots \geq \lambda_k > 0$. For example, if $n = 3$, then $\lambda = (3), \mu = (2,1), \nu = (1,1,1)$ are all of the partitions of n. We write $\lambda \vdash n$ to mean λ is a partition of n.

Now it is a basic fact from group theory that the conjugacy classes of S_n are determined by cycle type so that indeed they are in bijective correspondence to the distinct partitions of n. Thus, the number of distinct irreducible representations will be equal to the number of partitions of n.

Now suppose $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n. Then define the subgroup $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k} \leq S_n$. Given such a partition λ, we will study the representations of S_n that are induced by certain one-dimensional representations of S_λ. Now consider the following one-dimensional representations of S_λ.

The trivial representation:

$$\rho_\lambda : S_\lambda \to GL_1(\mathbb{C}) \text{ given by } \sigma \mapsto 1 \text{ for all } \sigma \in S_\lambda$$

and the sign representation:

$$\pi_\lambda : S_\lambda \to GL_1(\mathbb{C}) \text{ given by } \sigma \mapsto sgn(\sigma) \text{ for all } \sigma \in S_\lambda.$$

Where the sign of σ is defined by

$$sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Now we can consider the induced representations $Ind_{S_\lambda}^{S_n} \rho_\lambda$, $Ind_{S_\lambda}^{S_n} \pi_\lambda$. Further, we let $\psi|_H$ denote the restriction of a representation ψ of G to a subgroup $H \leq G$.

So let $\lambda \vdash n, \mu \vdash n$ be partitions of n. Now by Mackey's theorem, it holds:

$$\begin{aligned}
\dim \mathrm{Hom}_{S_n}(Ind_{S_\lambda}^{S_n} \rho_\lambda, Ind_{S_\mu}^{S_n} \rho_\mu) &= \sum_{S_\lambda \sigma S_\mu} \dim \mathrm{Hom}(\rho_\lambda|_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}, \rho_\mu|_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}) \\
&= \sum_{S_\lambda \sigma S_\mu} \dim \mathrm{Hom}(\rho_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}, \rho_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}) \\
&= \sum_{S_\lambda \sigma S_\mu} 1.
\end{aligned}$$

Similarly, we can compute

$$\dim \operatorname{Hom}(Ind_{S_\lambda}^{S_n}\rho_\lambda, Ind_{S_\mu}^{S_n}\pi_\mu) = \sum_{S_\lambda \sigma S_\mu} \dim \operatorname{Hom}(\rho_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}, \pi_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}).$$

Now both $\rho_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}$ and $\pi_{S_\lambda \cap \sigma S_\mu \sigma^{-1}}$ are irreducible. Further, it is clear that these two representations are equal if and only if the intersection $S_\lambda \cap \sigma S_\mu \sigma^{-1}$ is contained in the alternating group, i.e., $S_\lambda \cap \sigma S_\mu \sigma^{-1} \leq A_n$. But now we note that since $S_\lambda \cap \sigma S_\mu \sigma^{-1}$ is isomorphic to a product of symmetric groups $\prod_{i=1}^m S_{x_i}$, we will have $S_\lambda \cap \sigma S_\mu \sigma^{-1} \leq A_n$ if and only if $x_i = 1$ for $i = 1, \dots, m$, i.e., the intersection is trivial $S_\lambda \cap \sigma S_\mu \sigma^{-1} = \{1\}$. Thus, with this extra condition we deduce:

$$\dim \operatorname{Hom}(Ind_{S_\lambda}^{S_n}\rho_\lambda, Ind_{S_\mu}^{S_n}\pi_\mu) = \sum_{\substack{S_\lambda \sigma S_\mu \\ S_\lambda \cap \sigma S_\mu \sigma^{-1} = \{1\}}} 1.$$

We see then that our investigation about the symmetric group reduces to the combinatorial problem of counting double cosets of the form $S_\lambda \sigma S_\mu$.

To this end, we first introduce some terminology and prove a useful result. Given a positive integer n and $\lambda = (\lambda_1, \dots, \lambda_k)$ a partition of n, we let n_i^λ for $1 \leq i \leq k$ denote subsets of $\{1, \dots, n\}$ that satisfy $|n_i^\lambda| = \lambda_i$ for all $1 \leq i \leq k$, $n_i \cap n_j = \varnothing$ for $i \neq j$, and $\bigcup_{i=1}^k n_i^\lambda = \{1, \dots, n\}$. We call such a collection of n_i^λ for $i = 1, \dots, k$ a dissection of $\{1, \dots, n\}$. Now, we prove the following useful fact:

Proposition 15. *Let n be a positive integer. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1 \dots, \mu_l)$ be partitions of n. Let S_λ, S_μ be defined as above. Then $\tau \in S_\lambda \sigma S_\mu$ if and only if for every $1 \leq i \leq k$ and $1 \leq j \leq l$ it holds: $|n_i^\lambda \cap \sigma(n_j^\mu)| = |n_i^\lambda \cap \tau(n_j^\mu)|$.*

Proof. Assume $\tau \in S_\lambda \sigma S_\mu$. Say $\tau = \psi\sigma\phi$ where $\psi \in S_\lambda$ and $\phi \in S_\mu$. Then for each $j \in \{1, \dots, l\}$ it will hold:

$$\tau(n_j^\mu) = \psi\sigma\phi(n_j^\mu) = \psi\sigma(n_j^\mu).$$

Thus, if $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$, it holds:

$$n_i^\lambda \cap \tau(n_j^\mu) = n_i^\lambda \cap \psi\sigma(n_j^\mu) = \psi(n_i^\lambda \cap \sigma(n_j^\mu))$$

since $\psi \in S_\lambda$. Thus, since σ is a bijection, this yields $|n_i^\lambda \cap \sigma(n_j^\mu)| = |n_i^\lambda \cap \tau(n_j^\mu)|$, as desired.

Now conversely, suppose that $|n_i^\lambda \cap \sigma(n_j^\mu)| = |n_i^\lambda \cap \tau(n_j^\mu)|$. Then for each fixed i, the subsets $n_i^\lambda \cap \sigma(n_j^\mu)$ and the subsets $n_i^\lambda \cap \tau(n_j^\mu)$ form dissections of n_i^λ which can be put into ordered pairs of the form $(n_i^\lambda \cap \sigma(n_j^\mu), n_i^\lambda \cap \tau(n_j^\mu))$ of subsets of equal order. To see that these are indeed dissections, we note it is clear that they are pairwise disjoint for distinct j_1, j_2 and that indeed:

$$\begin{aligned} \bigcup_{j=1}^{l}(n_i^\lambda \cap \tau(n_j^\mu)) &= n_i^\lambda \cap \bigcup_{j=1}^{l}\tau(n_j^\mu) \\ &= n_i^\lambda \cap \{1, \ldots, n\} \\ &= n_i^\lambda \end{aligned}$$

and similarly for σ. Now for each i then we pick $\psi_i \in S_\lambda$ such that ψ_i fixes $\{1, \ldots, n\} - n_i^\lambda$ and such that

$$\psi_i(n_i^\lambda \cap \sigma(n_j^\mu)) = n_i^\lambda \cap \tau(n_j^\mu)$$

for each $j = 1, \ldots, l$. Now define $\psi = \psi_1 \cdots \psi_k$. Then we see that for every $j = 1, \ldots, k$:

$$\psi\sigma(n_j^\mu) = \tau(n_j^\mu)$$

so that indeed there exists $\phi \in S_\mu$ that satisfies $\tau = \psi\sigma\phi$, i.e., $\tau \in S_\lambda \sigma S_\mu$ and the proof is complete. □

This shows that for two fixed partitions $\lambda \vdash n$ and $\mu \vdash n$ the double cosets $S_\lambda \sigma S_\mu$ are characterized by the numbers

$$x_{ij} = |n_i^\lambda \cap \sigma(n_j^\mu)|, 1 \le i \le k, 1 \le j \le l.$$

Now let $m = max\{k, l\}$. Then by letting

$$x_{ij} = \begin{cases} |n_i^\lambda \cap \sigma(n_j^\mu)| & \text{if } 1 \le i \le k, 1 \le j \le l \\ 0 & \text{if } i > k \text{ or } j > l \end{cases}$$

we attain an injective map $g : S_\lambda \backslash S_n / S_\mu \to M_m(\mathbb{N})$ defined by $S_\lambda \sigma S_\mu \mapsto (x_{ij})$ where $M_m(\mathbb{N})$ is the set of $m \times m$ matrices over the natural numbers. It is clear that the image of g is

$$im(g) = \{(x_{ij}) \in M_m(\mathbb{N}) | \sum_{i=1}^{m} x_{ij} = \mu_j \text{ and } \sum_{j=1}^{m} x_{ij} = \lambda_i \text{ and } x_{ij} = 0 \text{ if } i > k \text{ or } j > l\}.$$

This proves the following result:

Proposition 16. *Suppose $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ are two partitions of a positive integer n. Then there exists a bijection:*

$$g : S_\lambda \backslash S_n / S_\mu \to X$$

where

$$X = \{(x_{ij}) \in M_m(\mathbb{N}) | \sum_{i=1}^{m} x_{ij} = \mu_j \text{ and } \sum_{j=1}^{m} x_{ij} = \lambda_i \text{ and } x_{ij} = 0 \text{ if } i > k \text{ or } j > l\}$$

and g is given by $S_\lambda \sigma S_\mu \mapsto (x_{ij})$ where

$$x_{ij} = \begin{cases} |n_i^\lambda \cap \sigma(n_j^\mu)| & \text{if } 1 \le i \le k, 1 \le j \le l \\ 0 & \text{if } i > k \text{ or } j > l. \end{cases}$$

With this we now have the information to compute the formulas given by Mackey's intertwining number formula.

Now if we restrict our view to double cosets $S_\lambda \sigma S_\mu$ with the trivial intersection property:

$$S_\lambda \cap \sigma S_\mu \sigma^{-1} = \{1\}$$

and restrict g to this subset, then we obtain the following corollary:

Proposition 17. *The number of double cosets $S_\lambda \sigma S_\mu$ with the property $S_\lambda \cap \sigma S_\mu \sigma^{-1} = \{1\}$ is equal to the number of $n \times n$ $0-1$ matrices with row sums λ_i and column sums μ_j.*

Thus, we have that dim $\mathrm{Hom}(Ind_{S_\lambda}^{S_n} \rho_\lambda, Ind_{S_\mu}^{S_n} \pi_\mu)$ is equal to the number of $n \times n$ $0-1$ matrices with row sums λ_i and column sums μ_j.

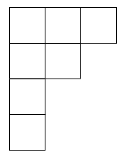
To deal with the combinatorial problem of counting such matrices, we define the following relation on partitions of n:

Definition 3. *Let n be a positive integer and suppose that $\lambda, \mu \vdash n$ are partitions of n. Then we say that $\lambda \ge \mu$ if $\sum_{i=1}^{k} \lambda_i \ge \sum_{i=1}^{k} \mu_i$ for all k.*

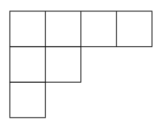
Proposition 18. *Let n be a positive integer. Then $\ge$ defines a partial order on the set of partitions of n.*

Proof. Reflexivity is obvious. To prove antisymmetry, suppose $\lambda \ge \mu$ and $\mu \ge \lambda$. Then for any j, we have $\sum_{i=1}^{j} \lambda_i \ge \sum_{i=1}^{j} \mu_i$ and $\sum_{i=1}^{j-1} \lambda_i \le \sum_{i=1}^{j-1} \mu_i$ so that $\lambda_j = \sum_{i=1}^{j} \lambda_i - \sum_{i=1}^{j-1} \lambda_i \ge \sum_{i=1}^{j} \mu_i - \sum_{i=1}^{j-1} \mu_i = \mu_j$. Similarly $\mu_j \ge \lambda_j$ and thus $\lambda_j = \mu_j$ for each j, i.e., $\lambda = \mu$. Transitivity also follows directly from the definition of $\ge$. ∎

Now for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of a positive integer n, define $D(\lambda)$ to be the diagram consisting of rows of blocks where the ith row consists of λ_i blocks. For example, if $n = 7$ and $\lambda = (3, 2, 1, 1)$ then $D(\lambda)$ is given by



Now noting that $\lambda_i \geq \lambda_{i+1}$ for each i, we see that the lengths of the columns of $D(\lambda)$ form a partition λ' of n. This partition is called the conjugate partition of λ. $D(\lambda')$ can be clearly obtained by simply rotating $D(\lambda)$ around it's main diagonal or by interchanging rows and columns. For example with $\lambda = (3, 2, 1, 1)$ as above, we have $\lambda' = (4, 2, 1)$ and $D(\lambda')$ is given by



.

Now to denote the sum of the column vectors of a matrix A we write $c(A)$ and to denote the sum of the row vectors of the same matrix we write $r(A)$

Proposition 19. *Let $\lambda = (\lambda_1, \ldots, \lambda_k), \mu = (\mu_1, \ldots, \mu_l)$ be partitions of a positive integer n. Then there exists a 0,1 matrix with $c(A) = \lambda$ and $r(A) = \mu$ if and only if $\lambda' \geq \mu$.*

Proof. To prove one direction, assume that $A = (a_{ij})$ is a $k \times l$ matrix with $c(A) = \lambda$ and row sums $r(A) = \mu$. Now if there is no $i \leq k$, $j < h \leq l$ such that $a_{ij} = 0$ and $a_{ih} = 1$ then it is easy to see that $\lambda' = \mu$.

Now if, on the contrary, there exists (i, j) satisfying the conditions above so that $a_{ij} = 0$ is a gap in the matrix, then let $h > j$ be maximal such that $a_{ih} = 1$. Then swapping a_{ij} and a_{ih} we attain a matrix A' with $c(A') = \lambda$ and that satisfies $r(A') \geq r(A)$. Thus, it follows by induction on the number of such gaps that $\lambda' \geq \mu$.

Now to prove the converse, suppose that $\lambda' \geq \mu$. Then we note that there exists a 0-1 $k \times l$ matrix A with $c(A) = \lambda$ and $r(A) \geq \mu$ namely the matrix whose ith row is 1 for $1 \leq j \leq \lambda_i$ and 0 for $j \geq \lambda_i$, i.e., this is the matrix constructed by putting 1's on the diagram $D(\lambda)$ and 0 elsewhere.

Now I claim that given a $k \times l$ matrix A such that $c(A) = \lambda$ and $r(A) \geq \mu$ and such that $r(A) \neq \mu$, we can find a $k \times l$ 0-1 matrix A' such that $c(A') = \lambda$ and $r(A') \geq \mu$ that satisfies $\|r(A') - \mu\| \geq \|r(A) - \mu\|$ where $\|\cdot\|$ is the usual Euclidean norm. Note that since $\|r(A) - \mu\|^2$ is an integer, this process must eventually terminate after a finite number of steps, i.e., we will be able to find a matrix $\bar{A}$ such that $c(\bar{A}) = \lambda$ and $r(\bar{A}) = \mu$. Thus, to complete our proof all we need to do is verify this claim.

So let $r(A) = (r_1, \ldots, r_l)$. Now let i be minimal such that $r_i > \mu_i$, and let j be minimal such that $r_j < \mu_j$. Then $i < j$ since $r(A) \geq \mu$. Now let $r = (r'_1, \ldots, r'_n) = (r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{j-1}, r_j + 1, r_{j+1}, \ldots, r_l$ then it is clear that $\|r' - \mu\| < \|r - \mu\|$. Further, it is clear that $r' \geq \mu$.

Now since $r_i > q_i \geq q_j > r_j$, we can find a number $h \in \{1, \ldots, k\}$ such that $a_{hi} = 1$ and $a_{hj} = 0$. For such an h, define the matrix $A' = (a'_{st})$ by

$$a'_{st} = \begin{cases} 1 & \text{if } (s,t) = (h,j) \\ 0 & \text{if } (s,t) = (h,i) \\ a_{st} & \text{otherwise} \end{cases}$$

By briefly inspecting this definition of A', it is easy to see that indeed $r(A') = r'$ and $c(A) = \lambda$ which verifies our claim and thus completes the proof. □

We note that it is an immediate consequence of this result that $\lambda' \geq \mu$ if and only if $\mu' \geq \lambda$, so that indeed $\lambda = \mu'$ if and only if $\lambda' = \mu$. We will use this result and the following to gain some interesting insight into the irreducible representations of the symmetric group.

Proposition 20. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of a positive integer n. Then there is precisely one 0-1 matrix A with $c(A) = \lambda$ and $r(A) = \lambda'$.*

Proof. One such matrix A is the matrix constructed using $D(\lambda)$, i.e., it is 1 where there is a node in $D(\lambda)$ and 0 elsewhere. More formally, A will be constructed by letting the first λ_i elements of the ith row be 1 and the remaining elements of the row be 0 for $i = 1, \ldots, k$. It is clear that this matrix will satisfy $c(A) = \lambda$ and $r(A) = \lambda'$.

To see that this is the only such matrix, suppose that there is a 0-1 matrix B with $c(B) = \lambda$ and $r(B) = \lambda'$ such that there exists $i \in \{1, \ldots, k\}$ with $b_{ij} = 0$ for some $j < \lambda_i$. Then it is easy to see that $\sum_{l=1}^{k} a_{lj} < \lambda'_j$, a contradiction. So indeed any matrix satisfying our conditions must be of the form of A and we are done. □

Thus, we know that if $\lambda \vdash n$ then dim Hom$(Ind_{S_\lambda}^{S_n}\rho, i_{S_{\lambda'}}^{S_n}\pi) = 1$ so that $Ind_{S_\lambda}^{S_n}\rho$ and $i_{S_{\lambda'}}^{S_n}\pi$ contain precisely one equivalent irreducible representation of multiplicity one in their direct sum decompositions. We will let Π_λ denote this irreducible representation.

Proposition 21. *Let $\lambda \vdash n$ and $\mu \vdash n$ be two partitions of a positive integer n. Then $\Pi_\lambda = \Pi_\mu$ implies $\lambda = \mu$.*

Proof. If $\Pi_\lambda = \Pi_\mu$ then we may deduce that $Ind_{S_\lambda}^{S_n}\rho, i_{S_{\lambda'}}^{S_n}\pi, Ind_{S_\mu}^{S_n}\rho, i_{S_{\mu'}}^{S_n}\pi$ share precisely one irreducible representation in common. Thus, dim Hom$(Ind_{S_\lambda}^{S_n}\rho, i_{S_{\mu'}}^{S_n}\pi) > 0$ so that $\lambda' \geq \mu'$ or equivalently $\mu = \mu'' \geq \lambda'' = \lambda$. Similarly, dim Hom$(Ind_{S_\mu}^{S_n}\rho, i_{S_{\lambda'}}^{S_n}\pi) > 0$ so that it holds that $\mu' \geq \lambda'$ or equivalently that $\lambda \geq \mu$. Thus, since $\geq$ is a partial order, this forces $\lambda = \mu$. □

Now we have proven that for any positive integer n, there exists an injective map from the set of partitions $\lambda \vdash n$ of n into the set of irreducible representations of S_n defined by $\lambda \mapsto \Pi_\lambda$. To see that this is a surjection, we recall the basic group theoretic fact that the number of conjugacy classes of S_n is equal to the number of partitions of n and the basic fact from the representation theory of finite groups that the number of inequivalent irreducible representations of a finite group G is equal to the number of conjugacy classes of G, so that indeed the set of partitions of n and the set of inequivalent irreducible representations of S_n are two finite sets of equal cardinality. We summarize what we have proved in the following proposition:

Proposition 22. *For any positive integer n, there exists a natural bijection between the partitions of n and the set of inequivalent irreducible representations of the symmetric group S_n, given by $\lambda \mapsto \Pi_\lambda$.*

5 Spherical Functions

We begin this section with an important example. Our first goal will be to prove the following proposition.

Proposition 23. *Let G be a finite group, and embed G diagonally into $G \times G$, that is by the map $g \mapsto (g, g)$. Identifying G with the image of this diagonal embedding, G is a Gelfand subgroup of $G \times G$.*

We provide multiple different proofs of this result that are enlightening in different ways.

Proof 1 (Involution Method). Consider the map $\iota : G \times G \to G \times G$ given by $(g_1, g_2) \mapsto (g_2^{-1}, g_1^{-1})$. Observe that given $(g_1, g_2) \in G \times G$ it holds:

$$\iota^2((g_1, g_2)) = \iota((g_2^{-1}, g_1^{-1})) = (g_1, g_2)$$

so that indeed $\iota^2 = id$. Further, if $(g_1, g_2), (g_3, g_4) \in G \times G$, then

$$\begin{aligned}\iota((g_1, g_2)(g_3, g_4)) &= \iota((g_1 g_3, g_2 g_4)) \\ &= (g_4^{-1} g_2^{-1}, g_3^{-1} g_1^{-1}) \\ &= (g_4^{-1}, g_3^{-1})(g_2^{-1}, g_1^{-1}) \\ &= \iota((g_3, g_4))\iota((g_1, g_2))\end{aligned}$$

so that indeed ι reverses multiplication and is thus an involution. To see that ι fixes each double coset $G(g_1, g_2)G$, let $(g_1, g_2) \in G \times G$ be arbitrary. Now let $a = g_1^{-1}$ and $b = g_2^{-1}$ and consider the elements $(a, a), (b, b) \in G$. It is easy to see that

$$\iota((g_1, g_2)) = (g_2^{-1}, g_1^{-1}) = (g_2^{-1} g_1 g_1^{-1}, g_2^{-1} g_2 g_1^{-1}) = (b, b)(g_1, g_2)(a, a) \in G(g_1, g_2)G$$

so that ι acts trivially on double cosets. The result follows. □

Our other proofs require some extra machinery. Let (π_i, V_i) be a representation of the group G_i for $i = 1, 2$. We may define a representation $\pi_1 \otimes \pi_2$ of the direct product $G_1 \times G_2$ by

$$(\pi_1 \otimes \pi_2)(g_1, g_2)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2$$

where $g_1 \in G_1, g_2 \in G_2$ and where $v_1 \otimes v_2 \in V_1 \otimes V_2$ is a simple tensor. Since any element of $V_1 \otimes V_2$ is a linear combination of simple tensors extending this representation by linearity does indeed give a representation of $G_1 \times G_2$. We give conditions on π_1 and π_2 that are necessary and sufficient so that $\pi_1 \otimes \pi_2$ is an irreducible representation of $G \times G$.

Proposition 24. *Let* (π_i, V_i) *be a finite dimensional linear representation of the group* G_i *for* $i = 1, 2$. *Then* $\pi_1 \otimes \pi_2$ *is an irreducible representation of* $G_1 \times G_2$ *if and only if* π_1, π_2 *are irreducible representations of* G_1, G_2 *respectively.*

Proof. Suppose without loss of generality that (π_1, V_1) is reducible, say $V_1 \cong U \oplus W$ where $U, W \subseteq V_1$ are nontrivial G_1 invariant subspaces. Then it is easy to see that $V_1 \otimes V_2 \cong (U \oplus W) \otimes V_2 \cong (U \otimes V_2) \oplus (W \otimes V_2)$ so that $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ is reducible.

Now suppose that π_1, π_2 are irreducible. Let $n = dim V_2$. Then $Hom_{G_1}(\pi_1, \pi_1)^n \cong Hom_{G_1}(\pi_1, \pi_1^n)$ via the isomorphism $A_1 \oplus \cdots \oplus A_n \mapsto B$, where B is defined by $B(v) = A_1(v) \oplus \ldots \oplus A_n(v)$.

Because $V_2 \cong \mathbb{C}^n$ and $\mathbb{C} \cong Hom_{G_1}(\pi_1, \pi_1)$, it holds that $V_2 \cong Hom_{G_1}(\pi_1, \pi_1 \otimes 1^n)$, where $\pi_1 \otimes 1^n$ is the representation of G_1 on $V_1 \otimes V_2$ defined by $(\pi_1 \otimes 1^n)(g_1)(v_1 \otimes v_2) = \pi_1(g_1)v_1 \otimes v_2$ for $v_1 \in V_1, v_2 \in V_2$. (Note that this representation can be identified with the restriction of $\pi_1 \otimes \pi_2$ to the subgroup $G_1 \times \{1\}$ of $G_1 \times G_2$. If m is a positive integer, then the map $T : V_1 \otimes Hom_{G_1}(\pi_1, \pi_1^m) \to V_1^m$ defined by $v \otimes A \mapsto A(v)$ is an isomorphism.

These facts show that there is a bijection

$$\{G_1 - \text{invariant subspaces of } V_1 \otimes V_2\} \leftrightarrow \{\mathbb{C} - \text{vector subspaces of } V_2\}$$

given by $V_1 \otimes W \leftarrow W$ and $X \to Hom_{G_1}(\pi_1, X) \subseteq Hom_{G_1}(\pi_1, \pi_1 \otimes 1^n) = V_2$.

Suppose that $X \subseteq V_1 \otimes V_2$ is a nonzero $G_1 \times G_2$-invariant subspace. Then X is also a G_1-invariant subspace, so that by what we have shown it holds that $X = V_1 \otimes W$ for some complex subspace $W \subseteq V_2$. That π_2 is irreducible guarantees

$$\begin{aligned} span\{(\pi_1 \otimes \pi_2)(1, g_2)x : x \in X, g_2 \in G_2\} &= V_1 \otimes span\{\pi_2(g_2)w : w \in W, g_2 \in G_2\} \\ &= V_1 \otimes V_2 \end{aligned}$$

We see then that $G_1 \times G_2$-invariance of X forces that $X = V_1 \otimes V_2$, i.e., the tensor product $\pi_1 \otimes \pi_2$ is irreducible. □

Proposition 25. *Let* (π, V) *be an irreducible representation of the direct product group* $G_1 \times G_2$. *Then there exist irreducible representations* π_1 *and* π_2 *of* G_1 *and* G_2 *respectively such that* $\pi \cong \pi_1 \otimes \pi_2$.

Proof. For $g_1 \in G_1$, $g_2 \in G_2$, and $v \in V$, define $\widetilde{\pi}_1(g_1)v = \pi(g_1, 1)v$ and $\widetilde{\pi}_2(g_2)v = \pi(1, g_2)v$. It is easy to see that these define representations of

G_1, G_2 respectively. It is not hard to see that we can pick a nonzero G_1 invariant subspace V_1 of V such that $\pi_1 = \widetilde{\pi}_1|_{V_1}$ is an irreducible representation of G_1. Now let $v \in V_1$ be nonzero and define $V_2 = span\{\widetilde{\pi}_2(g)v : g \in G_2\}$. V_2 is clearly G_2 invariant and $\pi_2 = \widetilde{\pi}_2|_{V_2}$ is a representation of G_2.

Now for any $v_1 \in V_1$ and $v_2 \in V_2$, there exist numbers $a_i, b_i \in \mathbb{C}$ and elements $g_1^{(i)}, g_2^{(i)} \in G_1 \times G_2$ such that $v_1 = \sum_i a_i \pi_1(g_1^{(i)})v$ and $v_2 = \sum_i b_i \pi_2(g_2^{(i)})v$. Given such elements v_1, v_2, we define a map $T : V_1 \otimes V_2 \to V$ by the rule

$$T(v_1 \otimes v_2) = \sum_i \sum_j a_i b_j \pi(g_1^{(i)}, g_2^{(j)})v$$

on simple tensors and then extend by linearity to define a map on all elements of $V_1 \otimes V_2$.

It is not difficult to verify that T is a well-defined $G_1 \times G_2$-module homomorphism, i.e., $T \in Hom_{G_1 \times G_2}(V_1 \otimes V_2, V)$. Further T is nonzero. For example, $T(v \otimes v) = v$ and thus $image(T) \subseteq V$ is a nontrivial $G_1 \times G_2$-invariant subspace. By irreducibility of V, this forces $image(T) = V$, i.e., T is surjective. It is similarly straightforward to check T is injective. □

Now we have shown that the irreducible representations of the group $G \times G$ are precisely those representations $\pi_i \otimes \pi_j$ where π_i and π_j are irreducible representations of G. In order to utilize this information to prove that G embedded diagonally in $G \times G$ is a Gelfand subgroup, we must utilize some tools from character theory. We introduce some basic tools from character theory below (such as Frobenius reciprocity and Schur orthogonality) in order to prove our desired result.

Proposition 26. *Let G be a finite group and $H \leq G$ a subgroup. Let (π, V) be a representation of H and (ψ, W) a representation of G. Then there is a vector space isomorphism*

$$Hom_G(W, Ind_H^G \pi) \cong Hom_H(W, V)$$

where the vector space isomorphism and its inverse are given thus: for $\sigma \in Hom_G(W, Ind_H^G \pi)$ define $\phi \in Hom_H(W, V)$ by $\phi(w) = \sigma(w)(1)$. For $\phi \in Hom_H(W, V)$ define $\sigma \in Hom_G(W, Ind_H^G \pi)$ by $\sigma(w)(g) = \phi(\psi(g)w)$.

Proof. Suppose that $\sigma : W \to Ind_H^G \pi$ is a G-module homomorphism. Then for $h \in H$, we have

$$\phi(\psi(h)w) = \sigma(\psi(h)w)(1) = (\pi^G(h)\sigma(w))(1) = \sigma(w)(1 \cdot h) = \pi(h)\sigma(w)(1).$$

The right hand side is $\pi(h)\phi(w)$ so that ϕ is an H-module homomorphism. It is a similarly straightforward computation to show that if $\phi : W \to V$ is an H-module homomorphism then $\sigma(w)(g) = \phi(\psi(g)w)$ gives a G-module homomorphism $\sigma : W \to Ind_H^G \pi$ and that indeed the maps $\sigma \mapsto \phi$ and $\phi \mapsto \sigma$ are mutual inverses. □

The following corollary, also referred to as Frobenius reciprocity, is the form of the result as we will use it.

Proposition 27 (Frobenius reciprocity). *Let G be a finite group and $H \leq G$ a subgroup. Let (π, V) be a representation of H and (ψ, W) a representation of G, and let χ_π and χ_ψ be the associated characters. Then*

$$\langle \chi_{Ind_H^G \pi}, \chi_\psi \rangle_G = \langle \chi_\pi, \chi_{\psi|_H} \rangle_H.$$

Proof. This is immediate from the previous proposition. □

The following result is a consequence of Frobenius reciprocity and is actually a statement of the equivalence of the two definitions of Gelfand pairs which were provided in the introduction.

Proposition 28. *Let G be a finite group and $H \leq G$ a subgroup. Then H is a Gelfand subgroup if and only if for every irreducible representation (π, V) of G, the subspace $V^H \subseteq V$ of H-fixed vectors is at most one dimensional, or equivalently, the restriction $\pi|_H$ contains the trivial representation 1_H with multiplicity at most 1.*

Proof. Note that $V^H \cong Hom_H(1_H, V)$. By Frobenius reciprocity, it holds:

$$\dim \mathrm{Hom}_H(1_H, V) = \langle \chi_{1_H}, \chi_\pi|_H \rangle_H = \langle \chi_{Ind_H^G 1_H}, \chi_\pi \rangle_G.$$

The expression on the left is the multiplicity of π in $Ind_H^G 1_H$, so the result follows from the definition of a Gelfand subgroup (Definition 2). □

The following special type of representation will be useful in our example of interest:

Definition 4. *Let G be a a group and (π, V) a linear representation of G. The representation $\hat{\pi}$ of G on V^* the dual space of V defined by*

$$\hat{\pi}(g) = \pi(g^{-1})^*$$

for all $g \in G$ (where $*$ denotes taking the adjoint of a linear transformation) is called the representation of G contragradient to π.

Proposition 29. *Let G be a finite group and (π, V) a representation of G. Then there exists a G-invariant inner product on V.*

Proof. Let $\langle\langle , \rangle\rangle$ be any inner product on V. Then it is easy to see that

$$\langle v, v' \rangle = \frac{1}{|G|} \sum_{g \in G} \langle\langle \pi(g)v, \pi(g)v' \rangle\rangle$$

for $v, v' \in V$ defines a Hermitian inner product on V. It is G-invariant by construction. ☐

Proposition 30. *Suppose that (π, V) is a representation of a finite group G and χ_π is its character. Then $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$ and the character of the contragradient representation $\hat{\pi}$ is the complex conjugate of the character of π, i.e., $\chi_{\hat{\pi}} = \overline{\chi_\pi}$.*

Proof. Let $\langle , \rangle$ be a G-invariant inner product and let $v \in V$ be an eigenvector of $\pi(g)$ with eigenvalue λ (existence of such a v and λ is guaranteed by basic linear algebra). Then

$$|\lambda|^2 \langle v, v \rangle = \langle \lambda v, \lambda v \rangle = \langle \pi(g)v, \pi(g)v \rangle = \langle v, v \rangle.$$

Thus, every eigenvalue is of norm 1. So, if we let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $\pi(g)$. Then

$$tr(\pi(g^{-1})) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\chi(g)}.$$

Now $\hat{\pi}(g)$ is the adjoint of $\pi(g^{-1})$, so its trace is equal to $tr(\pi(g^{-1}))$ which completes the proof. ☐

Proposition 31. *Let (π, V) and (π', V') be irreducible representations of a finite group G. Suppose that $L : V \to \mathbb{C}$ and $L' : V' \to \mathbb{C}$ are linear functionals. Then either $\pi \cong \pi'$ or $L(\pi(g)x)$ and $L'(\pi'(g)x)$ are orthogonal.*

Proof. Since L and L' are linear functionals, they are of the form $L(x) = \langle x, y \rangle$, $L'(x') = \langle x', y' \rangle$ for some $y \in V$ and $y' \in V'$. Define a map $\mu : V \to V'$ by

$$\mu(v) = \sum_{g \in G} \langle \pi(g)v, y \rangle \pi'(g^{-1}) y'.$$

We claim that $\mu(\pi(h)v) = \pi'(h)\mu(v)$ to see this, we make the variable change $g \mapsto gh^{-1}$ and see that

$$\begin{aligned}\mu(\pi(h)v) &= \sum_{gh^{-1}\in G} \langle \pi(gh^{-1})v, y\rangle \pi'(hg^{-1})y' \\ &= \pi'(h)\sum_{g\in G}\langle \pi(g)v, y\rangle \pi(g^{-1})y' \\ &= \pi'(h)\mu(v)\end{aligned}$$

so that μ is a G-module homomorphism. By Schur's Lemma, μ is either identically 0 or an isomorphism, so if $\pi \ncong \pi'$ then it holds:

$$\begin{aligned}0 &= \langle \mu(x), x'\rangle \\ &= \frac{1}{|G|}\sum_{g\in G}\langle \pi(g)x, y\rangle\langle \pi'(g^{-1})y', x'\rangle \\ &= \frac{1}{|G|}\sum_{g\in G}\langle \pi(g)x, y\rangle\langle y', \pi'(g)x'\rangle \\ &= \frac{1}{|G|}\sum_{g\in G}\langle \pi(g)x, y\rangle\overline{\langle \pi'(g)x', y'\rangle} \\ &= \langle L(\pi(g)x), L'(\pi'(g)x')\rangle\end{aligned}$$

where the third equality comes from taking adjoints. Thus, we have shown, $\pi \cong \pi'$ or $L(\pi(g)x)$ and $L'(\pi'(g)x')$ are orthogonal, which is what we had to prove. □

Proposition 32. *Let (π, V) be an irreducible representation of the finite group G, with G-invariant inner product $\langle , \rangle$. Then there exists a positive real constant $d \in \mathbb{R}^+$ such that*

$$\sum_{g\in G}\langle \pi(g)x, y\rangle\overline{\langle \pi(g)x', y'\rangle} = \frac{1}{d}\langle x, x'\rangle\langle y', y\rangle.$$

Indeed we will later show that this formula holds with $d = dim(V)$.

Proof. μ as defined in the proof of the previous proposition is a G-module homomorphism, so by Schur's lemma it is a scalar multiple of the identity. That is, for fixed y, y' there is a constant $c = c(y, y')$ such that $\mu(x) = c(y, y')x$ for all $x \in X$. Thus, it holds

$$c(y, y')\langle x, x'\rangle = \langle \mu(x), x'\rangle = \sum_{g\in G}\langle \pi(g)x, y\rangle\langle \pi(g^{-1})y', x'\rangle$$

making the variable change $g \mapsto g^{-1}$, conjugating the inner product and taking adjoints gives:

$$c(y,y')\langle x,x'\rangle = \sum_{g\in G}\langle \pi(g)x,y\rangle\overline{\langle \pi(g)x',y'\rangle}$$

Applying the same argument allows us to see that there exists a constant $\hat{c}(x,x')$ such that

$$\hat{c}(x,x')\langle y',y\rangle = \sum_{g\in G}\langle \pi(g)x,y\rangle\overline{\langle \pi(g)x',y'\rangle}$$

Together, these identities give the formula in the proposition. Setting $x = x' = y = y'$, we see that the constant d must be real and positive. □

Proposition 33. *Suppose G is a finite group and (π,V) is an irreducible representation of G of dimension n. Let χ_π denote the character of π. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g) = \begin{cases} 1 & \text{if } \pi \text{ is the trivial representation;} \\ 0 & \text{otherwise.}\end{cases}$$

More generally, if (π,V) is any representation of G and χ_π its character then

$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g) = dim(V^G)$$

where V^G denotes the subspace of V consisting of G-fixed vectors.

Proof. In the case that π is irreducible, we note that if π is trivial, then $\chi_\pi(g) = 1$ for all $g \in G$ so that the conclusion is obvious. If π is not trivial, then we note that $\chi_\pi(g)$ is a sum of functions of the form $L(\pi(g)x)$ where $x \in V$. Indeed if we write pick a basis $v_1,\ldots,v_n$ of V and write

$$\pi(g) = \begin{pmatrix} \pi_{11}(g) & \cdots & \pi_{1n}(g) \\ \vdots & & \vdots \\ \pi_{n1}(g) & \cdots & \pi_{nn}(g)\end{pmatrix}$$

then $\pi_{ij}(g) = L_i(\pi(g)v_j)$ where $L_i(\sum_j c_j v_j) = c_i$. Thus, since $\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)$ is the inner product of χ_π with the trivial character (which is a function of the form $L(1_G v) = 1$) we may extend this inner product by linearity to see that $\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)$ is a sum of inner products of the form $\langle L(\pi(g)v_j), 1_G\rangle$ which are all zero since π and the trivial representation are not isomorphic. This proves the first statement.

More generally, if (π, V) is any representation, we write $\pi = \oplus_i d_i \pi_i$ where d_i are nonnegative integers as the decomposition of π into irreducibles. Then $\chi_\pi = \sum_i d_i \chi_{\pi_i}$ so that

$$\sum_{g \in G} \chi_\pi(g) = \sum_{g \in G} \sum_i d_i \chi_{\pi_i}(g) = \sum_i \sum_{g \in G} d_i \chi_{\pi_i}(g) = \sum_{g \in G} d_{1_G} \chi_{1_G}(g)$$

where d_{1_G} is the multiplicity of the trivial representation. The last equality holds by what we proved in the irreducible case. Recognizing that the multiplicity of the trivial representation in (π, V) is the dimension of V^G yields the result. □

We now have the tools to prove the following important result known as Schur orthogonality:

Proposition 34 (Schur Orthogonality). *Suppose G is a finite group with representations (π_1, V) and (π_2, V_2) and associated characters χ_1, χ_2. Then $\langle \chi_1, \chi_2 \rangle$ = dim $Hom_G(V_1, V_2)$. By extension, if π_1 and π_2 are irreducible, then*

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1 & \text{if } \pi_1 \cong \pi_2; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the vector space of all linear transformations $A = Hom_{\mathbb{C}}(V_1, V_2)$ from V_1 to V_2. We may define a representation Ω of G on A by the rule $\Omega(g)T = \pi_2(g) \circ T \circ \pi_1(g^{-1})$. It is not hard to prove that $\chi_2(g)\overline{\chi_1(g)}$ is the character of $\Omega(g)$. The space of G-fixed vectors Ω^G consists precisely of those linear transformations that commute with the action of G, i.e., the G-module homomorphisms. By proposition two,

$$\langle \chi_1, \chi_2 \rangle = \dim \mathrm{Hom}_G(V_1, V_2).$$

The second statement about irreducible representations is a direct consequence of Schur's lemma. □

Proposition 35. *Let $(\pi_i, V_i), (\pi_j, V_j)$ be irreducible representations of a finite group G. Then the representation $(\pi_i \otimes \pi_j)|_G$ of $G \times G$ restricted to the diagonal subgroup G contains exactly one copy of the trivial representation 1_G if and only if $\pi_i = \hat{\pi}_j$. Otherwise, the multiplicity of the trivial representation in $(\pi_i \otimes \pi_j)|_G$ is 0. Thus, $G \hookrightarrow G \times G$ is a Gelfand subgroup.*

We already gave one proof of this fact. Here are two more which are enlightening in different ways.

Proof 2. Let χ_i, χ_j denote the characters of π_i, π_j respectively. Then $\chi_i\chi_j$ is the character of $(\pi_i \otimes \pi_j)|_G$. Now let

$$\chi_i\chi_j = d \cdot 1_G + \sum_{\chi_l \neq 1_G} d_l \chi_l$$

be the decomposition of $\chi_i\chi_j$ into irreducible characters. Now note that

$$d = \langle \chi_i\chi_j, 1_G \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_i\chi_j(g) = \langle \chi_i, \overline{\chi_j} \rangle_G$$

Observe that the expression on the left is the multiplicity of 1_G in $(\pi_i \otimes \pi_j)|_G$ and that Schur orthogonality implies that the expression on the right is 0 if $\chi_i \neq \overline{\chi_j}$ and 1 if $\chi_i = \overline{\chi_j}$. We already saw that $\overline{\chi_j}$ is the character of $\hat{\pi}_j$. The conclusion follows. □

Proof 3. Note that $V_i \otimes V_j^* \cong Hom_{\mathbb{C}}(V_j, V_i) = V$ where $(g_i, g_j) \in G \times G$ acts on $T \in Hom_{\mathbb{C}}(V_j, V_i)$ by the rule

$$(g_i, g_j)T = \pi_i(g_i) T \pi_j(g_j^{-1}).$$

Now observe that saying that T is a G-fixed vector is equivalent to saying that T is a G module homomorphism. That is, $(g,g)T = T$ if and only if $T(\pi_2(g^{-1})v) = \pi_1(g^{-1})T(v)$. This shows that $V^G = Hom_G(V_j, V_i)$ where V^G denotes the subspace of V consisting of G-fixed vectors. But Schur's lemma says that

$$\dim Hom_G(V_j, V_i) = \begin{cases} 1 & \text{if } V_j \cong V_i. \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for any irreducible representation V of $G \times G$, V^G is at most one dimensional, i.e., $G \hookrightarrow G \times G$ is indeed a Gelfand subgroup. □

We now introduce the notion that is the purpose of this section:

Definition 5. *Let G be a finite group and $H \leq G$ a Gelfand subgroup. Given an irreducible representation (π, V) of G with a unique H-fixed vector $v \in V$, a spherical function σ associated with this representation, is a function $\sigma : G \to \mathbb{C}$ which satisfies:*

1. *σ is H bi-invariant, i.e., $\sigma(hgh') = \sigma(g)$ for all $g \in G$, $h, h' \in H$,*

2. *σ is of the form $\sigma(g) = f(\pi(g) \cdot v)$ for $g \in G$) where $f : V \to \mathbb{C}$ is a linear functional.*

Proposition 36. *The G be a group and (π, V) a finite dimensional irreducible representation of G with a unique H-fixed vector $v \in V$. Let $\langle , \rangle$ be a G-invariant inner product on V. Then the spherical function σ as defined above is uniquely determined by conditions 1) and 2) and is given by the formula $\sigma(g) = \langle \pi(g)v, v\rangle$.*

Proof. It is easy to see that $\sigma(g) = \langle \pi(g)v, v\rangle$ satisfies condition 2) above. That it satisfies condition 1) follows by taking adjoints on the inner product and noting that v is H-fixed, i.e.,

$$\langle \pi(hgh')v, v\rangle = \langle \pi(g)v, \pi(h)^* v\rangle = \langle \pi(g)v, v\rangle$$

To see that 1) and 2) force σ to take this form observe that 2) implies that we may write $\sigma(g) = \langle \pi(g)v, y\rangle$ for some $y \in V$. But 1) forces that for any $h \in H$ and any $g \in G$ $\langle \pi(g)v, \pi(h)^* y\rangle = \langle \pi(g)v, y\rangle$. But since π is irreducible, this implies that $\langle z, \pi(h)^* y - y\rangle = 0$ for all $z \in V$ so that indeed $\pi(h)^* y = y$ for all $h \in H$. Thus, $\pi(h)y = y$ for all $h \in H$, i.e., y is H-fiixed. By assumption, there is only one H-fixed vector v, so $y = v$ which completes the proof. □

We note then, that it makes sense to talk about the spherical function σ_π associated with an irreducible representation (π, V) of G with a unique H-fixed vector without any ambiguity.

Now we have already shown that G embedded diagonally in $H = G \times G$ is a Gelfand subgroup. So suppose (π, V) is an irreducible representation of G so that $(\pi \otimes \hat{\pi}, V \otimes V^*)$ has a unique G-fixed vector. Identifying $V \otimes V^*$ with $Hom_{\mathbb{C}}(V, V)$ and letting Π denote the representation isomorphic to $\pi \otimes \hat{\pi}$ given by $\Pi((g_1, g_2)) \cdot T = \pi(g_1) T \pi(g_2^{-1})$, we see that $id_V \in Hom_{\mathbb{C}}(V, V)$ is the unique G-fixed vector, since for $(g, g) \in G$ it holds that

$$\Pi((g, g)) \cdot id_V = \pi(g) id_V \pi(g^{-1}) = \pi(gg^{-1}) = id_V.$$

Now let χ_π denote the character of π and define $\sigma((g_1, g_2)) = \chi_\pi(g_1 g_2^{-1})$. It is easy to see that σ is G bi-invariant since for $(g, g), (g', g') \in G$ and

$(g_1, g_2) \in H$, it holds

$$\begin{aligned}\sigma((g,g)(g_1,g_2)(g',g')) &= \chi_\pi(gg_1g'g'^{-1}g_2^{-1}g^{-1})\\ &= \chi_\pi(g_1g_2^{-1})\\ &= \sigma((g_1,g_2))\end{aligned}$$

where the second to last equality holds since χ_π is a class function. Further, for $h = (g_1, g_2) \in H$ we see that

$$\sigma(h) = \chi_\pi(g_1g_2^{-1}) = tr(\pi(g_1)id_V\pi(g_2^{-1})) = tr(\Pi(h)\cdot id_V)$$

so that since we saw that id_V is G-fixed and since the trace operator is a linear functional, we see that σ satisfies condition 2) of being a spherical function. Thus, we have established that for an irreducible representation π of G, $\sigma((g_1, g_2)) = \chi_\pi(g_1g_2^{-1})$ defines the spherical function on $H = G \times G$ associated with $\pi \otimes \hat{\pi}$.

As it turns out, as π_i ranges over the irreducible representations of G, the spherical functions associated with the π_i's form an orthonormal basis for the space of functions $C(G\backslash H/G)$ of all G bi-invariant functions from H into the complex numbers.

To prove this, we require the following proposition:

Proposition 37. *Let G be a group and let $H = G \times G$. Identify G with the diagonal subgroup of H given by $g \mapsto (g, g)$. Then the set of double cosets $G\backslash H/G$ is in bijection with the conjugacy classes of G.*

Proof. First, observe that the left cosets H/G are in bijection with G itself, since for any left coset representative (g, h), multiplying on the right by $(h^{-1}, h^{-1}) \in H$ gives $(gh^{-1}, 1)$. Now suppose that $(g, 1) \in G\backslash(g', 1)/G$ where $g, g' \in G$, i.e., there exist $h, l \in G$ with $(g, 1) = (h, h)(g', 1)(l, l)$. Multiplying on the right by $(l^{-1}h^{-1}, l^{-1}h^{-1})$ keeps us within the same double coset and gives $(g, 1) = (hg'h^{-1}, 1)$, i.e., g and g' are in the same conjugacy class. □

It follows from this result that counting dimensions yields a vector space isomorphism $L^2_{\text{class}}(G) \cong C(G\backslash H/G)$ where $L^2_{\text{class}}(G)$ denotes the set of class functions on G. Now it is a well known fact that the irreducible characters χ_{π_i} of G form an orthonormal basis of $L^2_{\text{class}}(G)$ where the inner product in $L^2_{\text{class}}(G)$ is given by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

for $\phi, \psi \in L^2_{\text{class}}(G)$.

Motivated by this, we note that

$$\langle F, F' \rangle = \frac{1}{|H|} \sum_{h \in H} F(h) \overline{F'(h)}$$

for $F, F' \in C(G \backslash G/H)$ defines a Hermitian inner product on $C(G\backslash H/G)$.

Now given a class function, $f : G \to \mathbb{C}$, define a function $F : H \to \mathbb{C}$ by $F(g_1, g_2) = f(g_1 g_2^{-1})$. To see that $F(g_1, g_2) \in C(G\backslash H/G)$ observe that for any $h, h' \in G$ it holds:

$$F((h,h)(g_1,g_2)(h',h')) = f(hg_1g_2^{-1}h^{-1}) = f(g_1g_2^{-1}) = F(g_1,g_2)$$

since f is a class function. Thus, $f \mapsto F$ gives a well-defined map from $L^2_{\text{class}}(G)$ to $C(G\backslash H/G)$.

This map preserves inner products since given $f, f' \in L^2_{\text{class}}(G)$ and the associated functions $F, F' \in C(G\backslash H/G)$, we observe that

$$\begin{aligned}
\langle F, F' \rangle &= \frac{1}{|H|} \sum_{h \in H} F(h) \overline{F'(h)} \\
&= \frac{1}{|G|^2} \sum_{g_1, g_2 \in G} F(g_1, g_2) \overline{F'(g_1, g_2)} \\
&= \frac{1}{|G|^2} \sum_{g_1, g_2 \in G} f(g_1 g_2^{-1}) \overline{f'(g_1 g_2^{-1})} \\
&= \frac{1}{|G|} \sum_{g \in G} f(g) \overline{f'(g)} \\
&= \langle f, f' \rangle.
\end{aligned}$$

Further, we see that given the character χ_{π_i} of an irreducible character π_i, this map will give us the spherical function $\sigma_i(g_1, g_2) = \chi_{\pi_i}(g_1 g_2^{-1})$. Since the inner product is preserved under this mapping, the spherical functions will thus be be orthonormal in $C(G\backslash H/G)$ and by the vector space isomorphism established, they compose an orthonormal basis for $C(G\backslash H/G)$.

This interesting example motivates us to ask the question, of whether given a Gefland pair (G,H) it holds in general that the associated spherical functions provide an orthonormal basis for $C(H\backslash G/H)$, where the inner product on $C(H\backslash G/H)$ is given by

$$\langle f,f'\rangle = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{f'(g)}.$$

As it turns out, the answer is yes, which gives an interesting characterization of this function space, analogous to the theory of irreducible characters within the space of class functions. The proof of this fact is analogous to that of the example. We begin by showing the following fact.

Proposition 38. *Let $H\le G$ be a Gelfand subgroup and let $Ind_H^G 1_H = \oplus_{i=1}^n \pi_i$ be the decomposition of $Ind_H^G 1_H$ into irreducible representations, which we know to be multiplicity free since H is a Gelfand subgroup. Then the cardinality of the set of double cosets $H\backslash G/H$ is given by $|H\backslash G/H| = n$.*

Proof. This is a simple consequence of Mackey' theorem. That is,

$$\dim \mathrm{Hom}_G(Ind_H^G 1_H, Ind_H^G 1_H) = \sum_{HgH} \dim \mathrm{Hom}(1_H|_{H\cap g^{-1}Hg}, 1_H|_{H\cap g^{-1}Hg}) = \sum_{HgH} 1.$$

Recognizing that the left hand side is equal to n, the result follows. □

Now from Proposition 31 and 38, we can see that for a Gelfand pair (G,H) the spherical functions form an orthogonal basis for the space of functions $C(H\backslash G/H)$. Indeed, we can scale the spherical functions by a constant, so that they form an orthonormal basis.

We verify here that indeed the constant in Proposition 32 is $d = \dim(V)$, which is the last tool we need in constructing an orthonormal basis of $C(H\backslash G/H)$.

Proposition 39. *The constant in Proposition 32 is dim(V).*

Proof. Let $v_1,\ldots,v_n$ be an orthonormal basis of V, where $n=\dim(V)$. Then $\chi_\pi(g) = \sum_{i=1}^n \langle \pi(g)v_i, v_i\rangle$. Combining this with Schur orthogonality gives:

$$1 = \frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_\pi(g)} = \sum_{i,j=1,\ldots,n}\frac{1}{|G|}\sum_{g\in G}\langle \pi(g)v_i, v_i\rangle\overline{\langle \pi(g)v_j, v_j\rangle}$$

Of the n^2 terms on the right hand side, only n are nonzero (those with $i=j$) and each of those is equal to the constant d^{-1} of Proposition 32. Thus, $nd^{-1}=1$. That is, $d=\dim(V)$ which is what we wanted. □

Using this, we attain the following, our desired result.

Proposition 40. *Let G be a finite group and $H \leq G$ a Gelfand subgroup. Let $(\pi_1, V_1), \ldots, (\pi_n, V_n)$ denote the irreducible representations of G with unique H-fixed vectors and $\sigma_1, \ldots, \sigma_n$ denote the associated spherical functions. Then $\sqrt{d_1}\sigma_1, \ldots, \sqrt{d_n}\sigma_n$ (where d_i is the dimension of π_i for each $i = 1, \ldots, n$) forms an orthonormal basis for the vector space $C(H\backslash G/H)$ of all functions $f : H\backslash G/H \to \mathbb{C}$.*

Proof. We have already seen that $d_1\sigma_1, \ldots, d_n\sigma_n$ form an orthogonal basis. Using Proposition 39, we have that for any σ_i, it holds

$$\langle \sqrt{d_i}\sigma_i, \sqrt{d_i}\sigma_i \rangle = d_i \langle \sigma_i, \sigma_i \rangle = d_i \left(\frac{1}{d_i} \langle v, v \rangle \langle v, v \rangle \right) = 1$$

where v denotes the unique H-fixed vector of unit length. Thus, the basis is orthonormal as desired. □

Although we have developed our theory of spherical functions in the setting of finite groups, it should be noted that the finiteness of the groups was not used in any of the proofs. Indeed in the setting that G is a locally compact abelian group, replacing the finite sums in many of these proofs with integrals will give the same results (left to the reader to verify). In the special case that $G = \mathbb{R}$ and $H = \{2\pi k : k \in \mathbb{Z}\} \leq \mathbb{R}$, it is not hard to see that the spherical functions are of the form $\sigma_n(x) = e^{inx}$ for $n \in \mathbb{Z}$. Identifying the spaces $L^2(H\backslash G/H)$ and $L^2([-\pi, \pi])$ (these are different ways of describing the space of 2π periodic functions), we see that indeed the classical theory of Fourier series is recovered as a special case of our more general theory, as was claimed in the introduction.

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