An Extensive Survey of Graceful Trees

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Abstract. A tree with n vertices is called *graceful* if there exists a labeling of its vertices with the numbers from 1 to n such that the set of absolute values of the differences of the numbers assigned to the ends of each edge is the set $\{1, 2, .., n-1\}$. The problem of whether or not all trees are graceful is still open. In this paper, we give an extensive survey of most of the results related to this problem published so far. The paper also contains some original proofs. A good deal of effort was spent to give the reader friendly descriptions of technical topics.

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1. Introduction

1.1. Definitions.

Let $G=(V,E)$ be a graph with set of vertices V and set of edges E. Let $n=|E|+1$. A labeling of the vertices of G with the numbers from 1 to n is an injective map $f\colon V\rightarrow \{1,2,\ldots,n\}$. A graph $G=(V,E)$ is *graceful* if there exists a labeling of its vertices f such that the map $g\colon E\rightarrow\{1,2,\ldots,n-1\}$ given by $g(u\,v)=|f(u)-f(v)|$, where $u,v\in V$, $u\,v\in E$, is a bijection. Throughout the literature some authors define the vertex labelings to start from 0.

For the case when G is a tree, we have that $|V|=|E|+1=n$, so, a tree is graceful if there exists a bijection $f\colon V\rightarrow\{1,2,\ldots,n\}$ such that the map $g\colon E\rightarrow\{1,2,\ldots,n-1\}$ given by $g(u\,v)=|f(u)-f(v)|$ is a bijection.

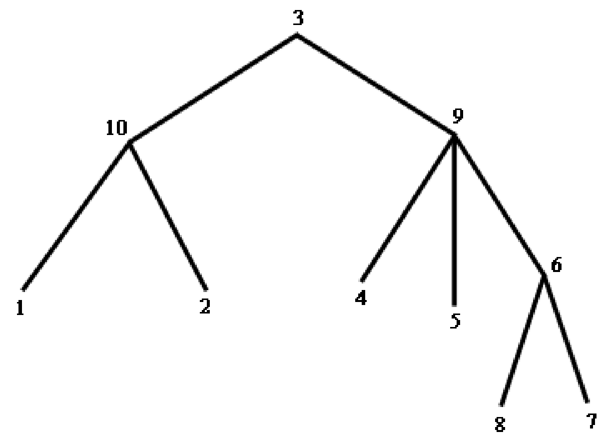

Figure 1. An example of a graceful labeling.

Conjecture. *All trees are graceful.*

1.2. Approaches.

A lot of work has been done by many toward proving the above conjecture. The problem still remains open. In this paper we describe the majority of the significant results related to the conjecture. In the existing literature there are two main approaches to proving the graceful tree conjecture. The more mathematical approach is by showing that all trees having a particular structure are graceful. The other approach is more computational and consists of showing that all trees with up to a certain number of vertices are graceful. The most recent result using the second approach is that all trees with up to 35 vertices are graceful.

2. ORIGINS

In this section we describe how the graceful tree problem first came up. We follow the description of [EH06].

Definition 1. *A* decomposition *of a graph G is a collection $\{H_i\}$ of nonempty subgraphs such that $E(H_i) = E_i$ for some nonempty subset E_i of $E(G)$, where $\{E_i\}$ is a partition of $E(G)$. If $\{H_i\}$ is a decomposition of a graph G such that $H_i = H$ for each i for some graph H, then, G is said to be H-*decomposable*. A* cyclic decomposition *is a decomposition of a graph G into k copies of a subgraph H that can be obtained in the following manner:*

1. Draw G appropriately

2. Select a subgraph H_1 of G isomorphic to H

3. Rotate the vertices and edges of H_1 through an appropriate angle $k-1$ times to produce k copies of H in the decomposition.

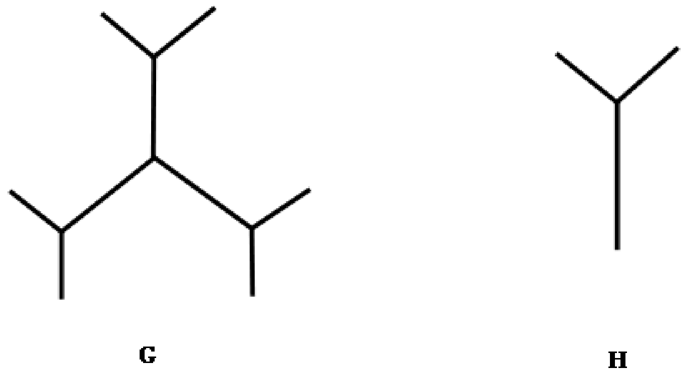

Figure 2. In this example, G decomposes cyclically into 3 H's.

R.M. Wilson proved in [Wil75] that for every graph H without isolated vertices, there exist infinitely many positive integers n such that K_n is H-decomposable. The following is Wilson's theorem as it appears in [EH06]:

Theorem 2. *For every graph H of size m without isolated vertices, there exists a positive integer N such that if* (1). $n \geqslant N$, (2). $m \mid \binom{n}{2}$ *and* (3). $d \mid m-1$, *where* $d = \gcd\{\deg v \colon v \in V(H)\}$, *then K_n is H-decomposable.*

In 1963, Ringel posed the following problem, which has since been known as **Ringel's Conjecture** [Ros67]:

Conjecture 3. *Every tree with $m+1$ vertices decomposes K_{2m+1}.*

This problem is to date still unsolved. According to Rosa [Ros67], Kotzig conjectured a stronger statement than Ringel's, **Kotzig's Conjecture**:

Conjecture 4. *Every tree with $m+1$ vertices cyclically decomposes K_{2m+1}.*

In 1967 Rosa published the paper [Ros67] with the intention of providing insight into attacking Ringel's Conjecture. His idea was to use a labeling of the vertices of a graph H of order m to show that it can cyclically decompose K_{2m+1}. He referred to a labeling as a *valuation* of the graph. Let O_G be a labeling of the vertices of G. Let V_{O_G} denote the set of numbers assigned to the vertices of G and let E_{O_G} denote the set of numbers assigned to the edges of G in the edge labeling induced by O_G, in which, as usual, an edge is labeled with the absolute value of the difference between the labels of its end vertices. Consider the following conditions

(1). $V_{O_G} \subseteq \{1, 2, ..., n\}$,
(2). $V_{O_G} \subseteq \{1, 2, ..., 2n\}$,
(3). $E_{O_G} = \{1, 2, ..., n-1\}$,
(4). $E_{O_G} = \{x_1, x_2, ..., x_n\}$, where $x_i = i$ or $x_i = 2n+1-i$,
(5). There exists $x \in \{1, 2, ..., n\}$, such that for an arbitrary edge $v_i v_j$ of the graph either $a_i \leqslant x < a_j$ or $a_j \leqslant x < a_i$, where a_k is the label assigned to v_k for each k.

From these conditions Rosa defines four types of labelings:
A ρ-*valuation* satisfies conditions (2) and (4).
A σ-*valuation* satisfies conditions (2) and (3).
A β-valuation satisfies conditions (1) and (3).
An α-*valuation* satisfies conditions (1), (3), and (5).

Rosa's β-valuation is what we now call a graceful labeling (the term was introduced by Golomb in [Gol72]). A graph with α-labeling is also know as a graph which has a bipartite labeling.

The above definitions also suggest a hierarchy of labelings: from strongest to weakest it is α-, β-, σ-, and ρ-valuation and each labeling is a special case of one of its successors in the hierarchy. Thus, if it could be shown that each tree has a bipartite labeling, it would also follow that each tree has a graceful labeling. However, Rosa showed that not every tree has a bipartite labeling and was able to classify a family of trees that do not admit such a labelling.

Example 5. An example of a tree with no bipartite labeling is the tree T in the figure below, which was proved by Bloom [Blo79] in 1979 to not admit such a labeling.

T

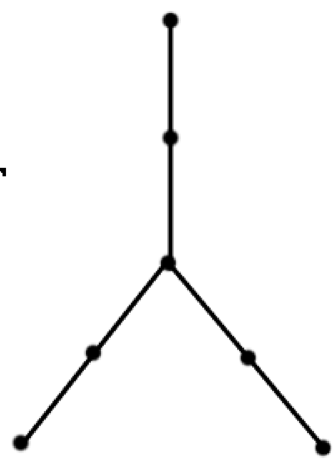

Figure 3. A tree with no bipartite labeling.

In his paper [Ros67], Rosa also showed the following, which turned concentration toward showing that all trees are graceful.

Theorem 6. *If H is a graceful graph with m edges, then $K_{2m+1}$is$H$-decomposable. In fact, K_{2m+1} can be cyclically decomposed into copies of H.*

Proof. (as appears in [EH06])

Since H is graceful, there is a graceful labeling of H, i.e. the vertices of H can be labeled from a subset of $\{1,2,...,m+1\}$ so that the induced edge lebels are $1,2,...,m$. Let $V(K_{2m+1})=\{v_1,v_2,...,v_{2m+1}\}$ where the vertices of K_{2m+1} are arranged cyclically in a regular $(2m+1)$-gon, denoting the resulting $(2m+1)$-cycle by C. Each vertex labeled i $(1\leqslant i\leqslant m+1)$ in H is placed at v_i in K_{2m+1}. Ever edge of H is drawn as a straight line segment in K_{2m+1}, denoting the resulting copy of H as H_1. Thus, $V(H_1)\subseteq\{v_1,v_2,...,v_{m+1}\}$.

Each edge v_sv_t of K_{2m+1} $(1 \leqslant s, t \leqslant 2m+1)$ is labeled $d_C(v_s, v_t)$, the number of edges in the shorter path from v_s to v_t in C, so, $1 \leqslant d_C(v_s, v_t) \leqslant m$. Consequently, K_{2m+1} contains exactly $2m+1$ edges labeled i for each $i \in \{1, 2, ..., m\}$ and H_1 contains exactly one edge labeled i for each $i \in \{1, 2, ..., m\}$. Note that whenever an edge of H_1 is rotated through an angle (say, clockwise) of $2\pi k/(2m+1)$ radians, where $1 \leqslant k \leqslant m$, an edge of the same label is obtained. Denote the subgraph obrained by rotating H_1 through a clockwise angle of $2\pi k/(2m+1)$ radians by H_{k+1} for each $1 \leqslant k \leqslant 2m+1$. Then, H_{k+1} is isomorphic to H and H_k and H_l have different sets of edges if $k \neq l$. Thus, H_1, H_2, ..., H_{2m+1} give a cyclic decompositino of K_{2m+1} into $(2m+1)$ copies of H. □

In fact, a graph H with $m+1$ edges does not need to be graceful in order to cyclically decompose K_{2m+1}. In his paper Rosa also characterizes exactly when H cyclically decomposes K_{2m+1}.

Theorem 7. *A cyclic decomposition of K_{2m+1} into subgraphs isomorphic to a graph H with m edges exists if and only if there exists a ρ-valuation of the graph H.*

Rosa's work turned concentration toward showing that all trees are graceful in oder to prove Ringel's Conjecture.

3. Classes of Graceful Trees

In the attempt to prove that all trees are graceful, many classes of trees have been proven graceful. A great source for finding a list of graceful classes of trees is [Gal10]. However, knowing all of them is still not enough to conlcude that all trees are graceful.

In this section, we are going to exhibit a lot of those classes. Where there are no references, the proofs have been given independently by the author of the current paper, even though they have been proved graceful by others in the past.

3.1. A Preliminary Lemma.

We start with a useful lemma which will be applied in a few of the proofs below.

Lemma 8. *Let T be a tree on n vertices and let f: $V(T) \rightarrow \{1, 2, ..., n\}$ give a graceful labeling of T. Then, the function f': $V(T) \rightarrow \{1, 2, ..., n\}$ given by $f'(v) = n+1-f(v)$ also gives a graeful labeling of T.*

Remark 9. The function f' is sometimes called the *inverse transformation* of the given labeling.

Proof. Note that f' is also a bijection (since f) is and for each $uv \in E(T)$, $g'(uv) = |f'(u) - f'(v)| = |n+1-f(u)-n-1+f(v)| = |f(u)-f(v)| = g(uv)$, so, the set of labels assigned to the edges is still in bijection with the set $\{1, 2, ..., n-1\}$. So, f' also gives us a graceful labeling of T. □

In the rest of this section we are going to describe the majority of the classes of trees that have been proved to be graceful along with some of the proofs. We start with the simplest types of trees.

3.2. Paths and Caterpillars.

Definition 10. *A* path *is a tree with only two leaves, or equivalently a tree in which all vertices have degree* 0 *or* 1*. A* caterpillar *is a tree such that if one removes all of its leaves, the remaining graph is a path.*

Rosa proved in his paper [Ros67] that all caterpillars are graceful. We are going to exhibit a proof of that fact, developed independently by the author. Since paths are also caterpillars, it will follow that paths are also graceful.

Theorem 11. *All caterpillars are graeful.*

Proof. Let C be a caterpillar on n vertices. Let's denote the vertices of the path obtained by removing all leaves by v_1, v_2, ..., v_k. For each $i = 1$, 2, ..., k, let the leaves coming out from v_i be called $u_{i,1}, u_{i,2}, ..., u_{i,n_i}$.

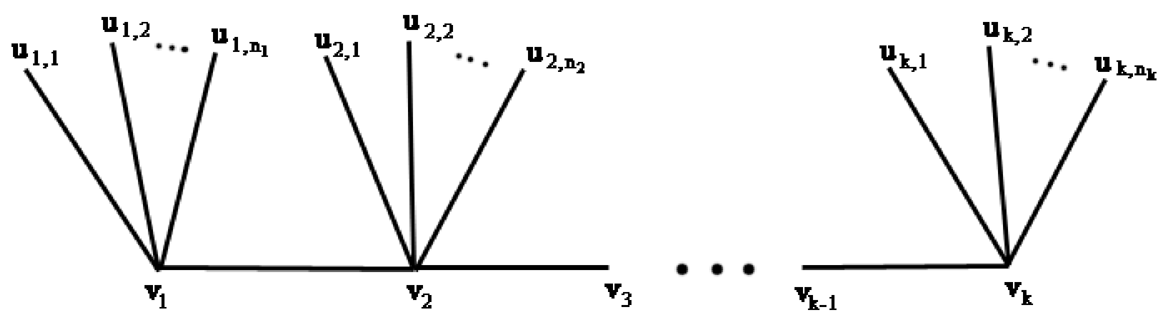

Figure 4.

We proceed by induction on i from 1 to k. First start by labeling v_1 with $f(v_1) = 1$ and $u_{1,1}, u_{1,2}, ..., u_{1,n_1}$ by $f(u_{1,j}) = n + 1 - j$. So, this means we have used up the labels, 1 and $n, n-1, ..., n-j+1$. Moreover, note that $g(v_1\, u_{1,j}) = |f(v_1) - f(u_{1,j})| = |1 - n - 1 + j| = n - j$. So, we also have the labels $n-1, n-2, ..., n-n_1$.

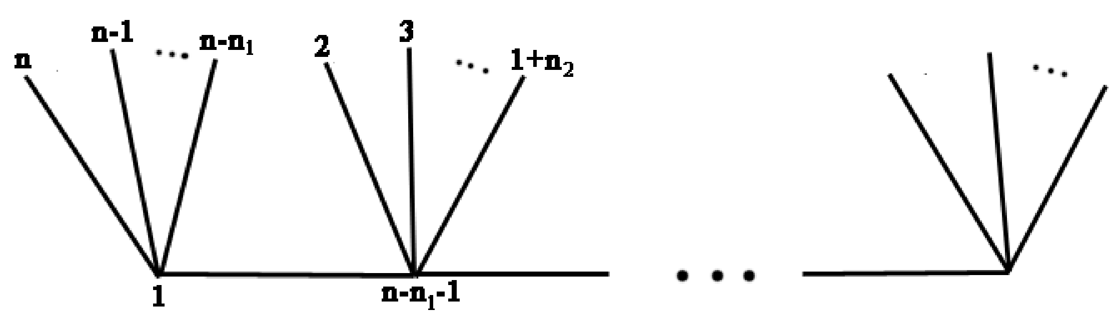

Figure 5.

For the inductive step suppose that we have used up all the vertex labels $1, 2, \ldots, l$ and $n-m, n-m+2, \ldots, n$ and we have obtained all the edge labels $n-1, n-2, \ldots, n-m-l$ and we have already labeled $v_1, v_2, \ldots, v_{i-1}$ and $u_{1,1}, \ldots, u_{1,n_1}, u_{2,1}, \ldots, u_{2,n_2}, \ldots, u_{i-1,1}, \ldots, u_{i-1,n_{i-1}}$ and if $i-1$ is even, $f(v_{i-1}) = n-m$ and if $i-1$ is odd, $f(v_{i-1}) = l$.

Now, we are going to number v_i and $u_{i,1}, \ldots, u_{i,n_i}$. If i is even, then, $i-1$ is odd, so, $f(v_{i-1}) = l$. We set $f(v_i) = n-m-1$, so that $g(v_{i-1}v_i) = |n-m-1-l| = n-m-l-1$ and then we set $f(u_{i,j}) = l+j$ for $j = 1, 2, \ldots, n_i$ so that $g(v_i u_{i,j}) = |f(v_i) - f(u_{i,j})| = |n-m-1-l-j| = n-m-l-j-1$. So, after this labeling, we have used up the labeles 1, 2, ..., $l+n_i$ and $n-m-1, n-m, \ldots, n$ and we have labeled $v_1, v_2, \ldots, v_i$ and $u_{1,1}, \ldots, u_{1,n_1}, u_{2,1}, \ldots, u_{2,n_2}, \ldots, u_{i,1}, \ldots, u_{i,n_i}$ and since i is even we have $f(v_i) = n-m-1$ and we have obtained the edge labels $n-1, n-2, \ldots, n-m-l-n_i-1$. So, we can set $l' = l+n_i$ and $m' = m+1$.

Similarly, if i is odd, then, $i-1$ is even, so, $f(v_{i-1}) = n-m$, so, we label $f(v_i) = l+1$ and $f(u_{i,j}) = n-m-j$ for $j = 1, 2, \ldots, n_i$, so we obtain edge labels $g(v_{i-1}v_i) = n-m-l-1$ and $g(v_i u_{i,j}) = n-m-l-j-1$. So, we can set $l' = l+1$ and $m' = m+n_i$.

So, in both cases, we have used up labeles $1, 2, \ldots, l'$ and $n-m', n-m'+1, \ldots, n$ and we have obtained all the edge labels $n-1, n-2, \ldots, n-m'-l'$ and if i is even, we have $f(v_i) = l'$ and if i is odd, we have $f(v_i) = n-m'$. This completes the induction.

In the end, when we do this process until $i = k$, we are going to have used up all labels from 1 to n since l will be equal to $n-m-1$ and we will have obtained all edge labels $n-1, n-2, \ldots, n-l-m = n-n+m-1-m = 1$.

□

Note that if we apply this construction to a path P with vertices $v_1, v_2, \ldots, v_n$ and edges $v_i v_{i+1}$ for each $i = 1, 2, \ldots, n-1$, we get the following labeling: $f(v_1) = 1$, $f(v_2) = n$, $f(v_3) = 2$, $f(v_4) = n-1$, etc., which is the most natural graceful labeling of a path.

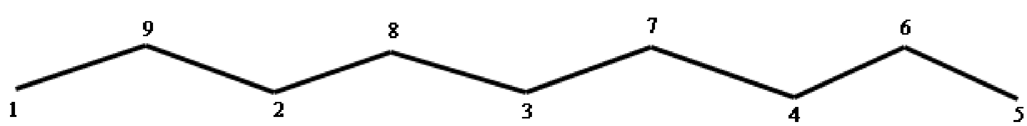

Figure 6.

In a later section, we are going to discuss results showing the growth of the number of different graceful labelings for paths.

3.3. Symmetrical Trees.

Definition 12. *A* symmetrical tree *is a rooted tree in which every level contains vertices of the same degree.*

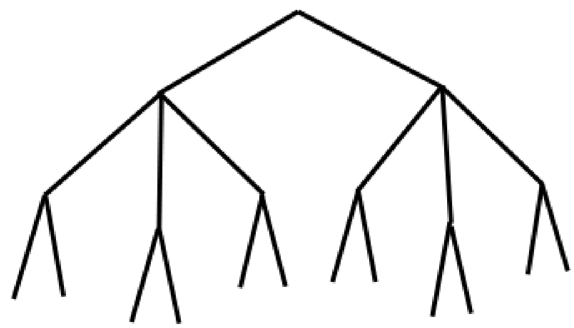

Figure 7. An example of a Smmetrycal Tree.

It has been shown in [BS76] as well as by the present author that all symmetrical trees are graceful. We are going to exhibit our own proof. This result is much harder and more technical than the ones that we have seen so far.

Lemma 13. *Let T be a symmetrical tree with root v, let $v_1, v_2, \ldots, v_k$ be the vertices on the first level, and let $T_1, T_2, \ldots, T_k$ be the symmetrical rooted subtrees of T with roots $v_1, v_2, \ldots$, v_k respectively. Then, $T_1, T_2, \ldots, T_k$ are isomorphic as rooted trees.*

Proof. Clearly the trees $T_1, T_2, \ldots, T_k$ are symmetrical with roots $v_1, v_2, \ldots, v_k$. First, note that if $k = 1$, there is nothing to prove. We are going to show that T_1 and T_2 are isomorphic as rooted trees and it will follow that all of the T_i's are isomorphich as rooted trees. We exhibit an explicit isomorphism φ: $T_1 \to T_2$. We start by level 0, i.e. the roots v_1 and v_2. Let $\varphi(v_1) = \varphi(v_2)$. Then, we look at all of the direct children of v_1 and v_2. Since T_1 and T_2 are symmetrical, then v_1 and v_2 have the same number k_1 of children, so let's denote them by $u_1, u_2, \ldots, u_{k_1}$ for T_1 and $w_1, w_2, \ldots, w_{k_1}$ for T_2. So, let $\varphi(u_j) = w_j$ for each $j = 1, 2, \ldots, k_1$. We proceed the evaluation of φ on $V(T_1)$ following a Breadth First Search algorithm. Suppose we have eavluated all vertices until the lth layer from T_1 with all vertices until the lth layer from T_2. Then, for each vertex u in the lth layer of T_1 we know that it has k_l children and since $\varphi(u) \in V(T_2)$ is also in the lth layer of T_2, it also has k_l children. So, as above, we just let φ be a bijection of these children.

Let $e = uu' \in E(T_1)$ be an edge and let u be in the lth layer and u' be n the $l + 1$st layer. Then, because of the way we evaluated φ, we have that $\varphi(u)\varphi(u')$ also corresponds to an edge in T_2.

So, note that by induction on the layers φ is an isomorphism of T_1 restricted to the first l layers and T_2 restricted to the first l layers since it clearly holds for $l = 0$ and the BFS algorithm in our construction ensures that the inductive step for getting from layer l to layer $l + 1$ also works. Thus, T_1 and T_2 are isomorphic (and the isomorphism induces a bijection between the vertices on each level). □

Now we proceed to the proof of the main theorem of this subsection.

Theorem 14. *All symmetrical trees are graceful.*

Proof. We are going to show by induction on the number of layers that all symmetrical trees are graceful and there exists a graceful labeling which assigns the number 1 to the root.

If T is a symmetrical tree with 0 layers, then, it consists of 0 edges and just one vertex, and clearly there is a graceful labeling which assigns 1 to that vertex. Suppose we have proved that for some $l > 0$ all symmetrical trees with $\leqslant l - 1$ layers are graceful and each of them has a graceful labeling which assigns the number 1 to the root.

Idea of Induction Step. The idea of the induction step is to consider a rooted symmetrical tree for which we know that its k children $T_1, T_2, \ldots, T_k$ are graceful (and isomorphic to each other). We label the children with their (identical) graceful labelinga and then add certain numbers to each of the vertices. The way we do this is the following. We order the childern from left to right. Then, if n is the number of vertices in each child, we start from the 0th layer of the children and add $(k-1)n$ to the root of T_1, $(k-2)n$ to the root of T_2, ..., and 0 to the root of the k-th one. Then, for the first layer, we start from right to left this time and add $(k-1)n$ to each of the vertices in the 1st layer of T_k, then, we add $(k-2)n$ to each of the vertices in the 1st layer of T_{k-1}, ..., and 0 to each of the vertices in the first layer of T_1. So, then we go on with the second layer and we start from left to right, and so on until we finish with the last layer. Then, we write $nk + 1$ on the root of the new tree. Then, we the transformation $x \mapsto nk + 2 - x$ to each of the vertices, so that we can have 1 at the root and the resulting labeling, as we show in the sequel, is graceful.

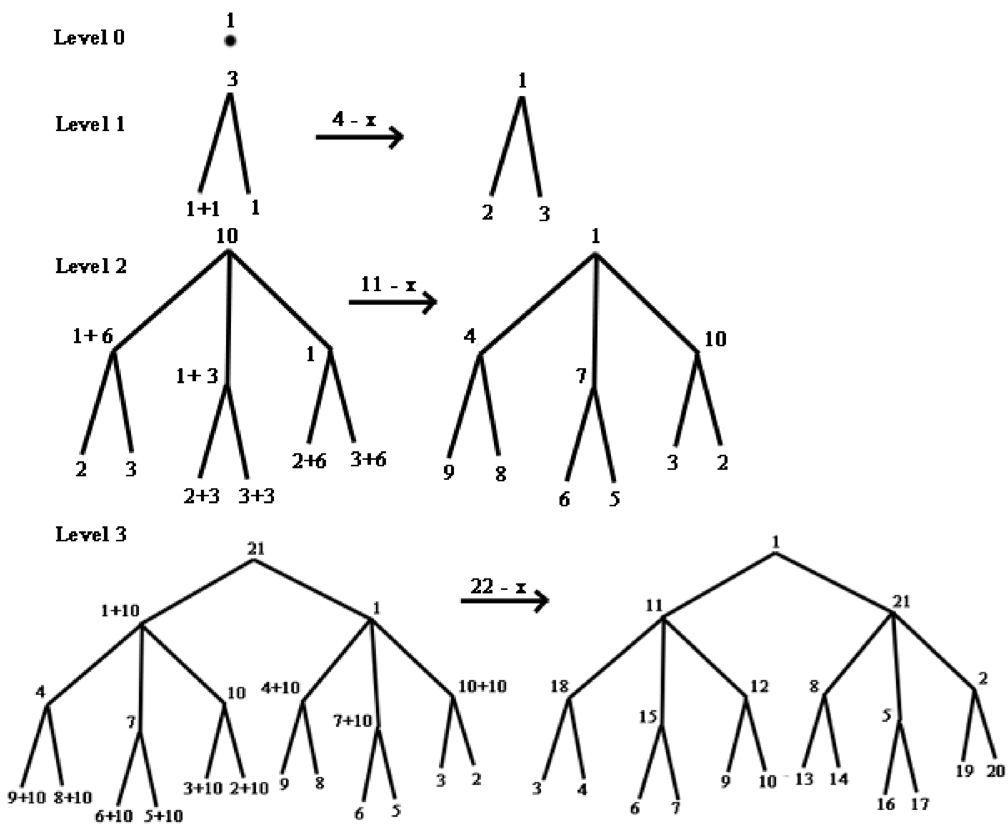

Figure 8. All the steps in a graceful labeling of a symmetrical tree.

Let T be a symmetrical tree with l layers. Let v be the root, v_1, v_2, ..., v_k its children, and T_1, T_2, ..., T_k the corresponding rooted subtrees. Then, by the previous lemma, we know that $T_1, T_2, \ldots, T_k$ are symmetrical trees and are isomorphic as rooted trees. Let φ_i: $T_1 \rightarrow T_i$ be the isomorphisms as constructed in the lemma. So, in particular, φ_i sends the lth level of T_1 to the lth level of T_i and for each vertex $u \in T_1$, φ_i sends the children of u to the children of $\varphi_i(u)$. By the induction hypothesis, there exists a graceful labeling f_1: $T_1 \rightarrow \{1, 2, \ldots, n\}$, where $n = |V(T_i)|$ such that $f_1(v_1) = 1$. So, f_1 induces graceful labelings $f_i = f_1 \circ \varphi_i^{-1}: T_i \rightarrow \{1, 2 \ldots, n\}$ such that $f_i(v_i) = 1$ for each $i = 2, 3, \ldots, k$. For convenience of notation we also define $\varphi_1 = \mathrm{Id}_{T_1}$, so, $f_1 = f_1 \circ \varphi_1$.

Now, we consider the following labeling $f: T \rightarrow \{1, 2, \ldots, nk + 1\}$. (Note that $|V(T)| = nk + 1$). Let $f(v) = nk + 1$ and $f(v_i) = 1 + (i - 1)n$. Let the vertices on level l of T_1 be $u_{1,1}, u_{1,2}, \ldots, u_{1,m_l}$. So, the vertices on level l of T_i are $\varphi_i(u_{1,1}), \varphi_i(u_{1,2}), \ldots, \varphi_i(u_{1,m})$. If $l \geqslant 1$ is odd, let $f(\varphi_i(u_{1,j})) = f_i(\varphi_i(u_{1,j})) + (i - 1)n = f_1(u_{1,j}) + (i - 1)n$ and if $l \geqslant 2$ is even, let $f(\varphi_i(u_{1,j})) = f_i(\varphi_i(u_{1,j})) + (k - i)n = f_1(u_{1,j}) + (k - i)n$.

Then, first of all, the edges from the root to its children are assigned the labels: $|f(v) - f(v_i)| = |nk + 1 - 1 - (i - 1)n| = n(k - i + 1)$ for $i = 1, 2, \ldots, k$, so, these are eactly the numbers $n, 2n, \ldots, kn$. Consider two adjacent vertices $u, u' \in T_1$ and the corresponding $\varphi_i(u), \varphi_i(u') \in T_i$. Let u be in level l and u' in level $l + 1$. Then, if l is odd, we have that the set $\{g(\varphi_i(u)\varphi_i(u')): i = 1, 2, \ldots, k\} = \{|f(\varphi_i(u)) - f(\varphi_i(u')): i = 1, 2, \ldots, k\} = \{|f_1(u) + (k - i)n - f_1(u') - (i - 1)n|: i = 1, 2, \ldots, k\} = \{|f_1(u) - f_1(u') + (k - 2i + 1)n|: i = 1, 2, \ldots, k\}$. If l is even, then, $\{g(\varphi_i(u)\varphi_i(u')): i = 1, 2, \ldots, k\} = \{|f(\varphi_i(u)) - f(\varphi_i(u')): i = 1, 2, \ldots, k\} = \{|f_1(u) + (i - 1)n - f_1(u') - (k - i)n|: i = 1, 2, \ldots, k\} = \{|f_1(u) - f_1(u') - (k - 2i + 1)n|: i = 1, 2, \ldots, k\} = \{|f_1(u') - f_1(u) + (k - 2i + 1)n|: i = 1, 2, \ldots, k\}$.

Note that for $1 \leqslant |m| \leqslant n - 1$, then,

$$\{|m + (k - 2i + 1)n|: i = 1, 2, \ldots, k\} =$$

$$= \{m + (k - 2i + 1)n: 1 \leqslant i < \frac{k + 1}{2}\} \cup$$

$$\{-m - (k - 2i + 1)n: \frac{k + 1}{2} < i \leqslant k\} \cup \{|m|: 2i = k + 1, i \in \mathbb{N}\} =$$

$$= \{m + (k - 2i + 1)n: 1 \leqslant i < \frac{k + 1}{2}\} \cup$$

$$\{-m - (k - 2(k + 1 - i) + 1)n: 1 \leqslant i < \frac{k + 1}{2}\} \cup \{|m|: 2i = k + 1, i \in \mathbb{N}\} =$$

$$= \{m + (k - 2i + 1)n: 1 \leqslant i < \frac{k + 1}{2}\} \cup$$

$$\{-m + (k - 2i + 1)n: 1 \leqslant i < \frac{k + 1}{2}\} \cup \{|m|: 2i = k + 1, i \in \mathbb{N}\} =$$

$$= \{|m| + (k - 2i + 1): 1 \leqslant i < \frac{k + 1}{2}\} \cup \{n - |m| + (k - 2i): 1 \leqslant i < \frac{k + 1}{2}\} \cup$$

$$\{|m|: 2i = k + 1, i \in \mathbb{N}\}.$$

Now, let $1 \leqslant m \leqslant n-1$ be such that $m \neq n-m$. So, there are two edges $uu' \in E(T_1)$ and $ww' \in E(T_1)$ such that $|f_1(u) - f_1(u')| = m$ and $|f_1(w) - f_1(w')| = n-m$. Let u nad w be in odd layers of T, so that u' and w' will be in even layers of T. And so, the following two sets are among the labels of T induced by f:

$$\{|f_1(u) - f_1(u') + (k-2i+1)n| : i = 1, 2, \ldots, k\} \cup$$
$$\{|f_1(w) - f_1(w') + (k-2i+1)n| : i = 1, 2, \ldots, k\} =$$
$$= \{|f_1(u) - f_1(u')| + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\{n - |f_1(u) - f_1(u')| + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{|f_1(u) - f_1(u')| : 2i = k+1\} \cup$$
$$\{|f_1(w) - f_1(w')| + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\{n - |f_1(w) - f_1(w')| + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{|f_1(w) - f_1(w')| : 2i = k+1\}$$
$$= \{m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{n - m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\{n - m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\{m : 2i = k+1\} \cup \{n - m : 2i = k+1\}.$$

Now, if k is odd, then, $k+1$ is even and $\{m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m : 2i = k+1\} = \{m + in : 0 \leqslant i < k\}$ and if k is even, then, $k+1$ is odd, so, $\{m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m : 2i = k+1\} = \{m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} = \{m + in : 0 \leqslant i < k\}$. Similarly, in both cases $\{n - m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{n - m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{n - m : 2i = k+1\} = \{n - m + in : 1 \leqslant i < k\}$.

So, for each $1 \leqslant m \leqslant n-1$ such that $m \neq n-m$, the set $\{m + in : 1 \leqslant i < k\}$ is among the induced labels for the edges of T.

If $m = n-m$, then and if u, u' are such that $|f_1(u) - f_1(u')| = m$ and u is in an odd layer and u' is in an even layer of T, then the following set of labels is among the labels of edges induced by f.

$$\{|f_1(u) - f_1(u') + (k-2i+1)n| : i = 1, 2, \ldots, k\} =$$
$$= \{|f_1(u) - f_1(u')| + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\{n - |f_1(u) - f_1(u')| + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{|f_1(u) - f_1(u')| : 2i = k+1\}$$
$$= \{m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{n - m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\cup \{m : 2i = k+1\} =$$
$$= \{m + (k-2i+1)n : 1 \leqslant i < \frac{k+1}{2}\} \cup \{m + (k-2i)n : 1 \leqslant i < \frac{k+1}{2}\} \cup$$
$$\cup \{m : 2i = k+1\}.$$

So, again, if k is odd, so that $k+1$ is even, this is exactly $\{m+in: 0\leqslant i<k\}$ and if k is even, so that $k+1$ is odd, agan this set is exactly $\{m+in: 0\leqslant i<k\}$.

So, from the edges in $T_1, T_2, \ldots, T_k$, we get all labels $\{m+in: 0\leqslant i<k\}$ for $1\leqslant m\leqslant n-1$ and from the edges between the root v and $v_1, v_2, \ldots, v_k$, we get all labels $\{n+in: 0\leqslant i\leqslant k\}$. So, this is a graceful labeling of T. Moreover, $f(v)=nk+1=|V(T)|$, so, if we apply the preliminary lemma and switch the labels $f(v)\mapsto nk+2-f(v)$, then, we are going to get $f(v)=1$ and the labeling is still graceful. This completes the induction.

Thus, all symmetrical trees are graceful.

Example 15. Complete balanced binary trees are an instance of symmetrical trees. Here are the labelings that we would get by using the construction above for the first three complete balanced binary trees.

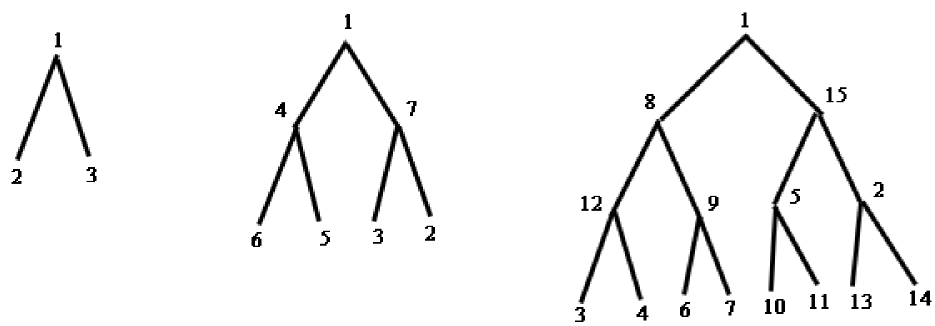

Figure 9.

□

3.4. Spider Trees.

Definition 16. *A* spider tree *is a tree with at most one vertex of degree greater than* 2. *If such a vertex exists, it is called the* branch point *of the tree. A* leg *of a spider tree is any one of the paths from the branch points to a leaf of the tree.*

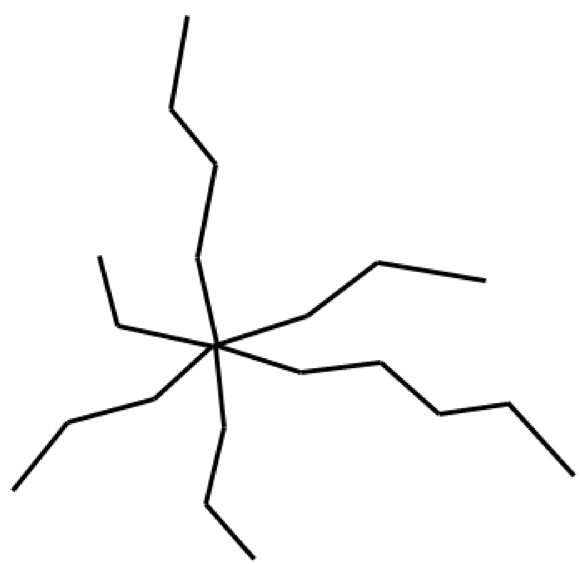

Figure 10. An Example of a Spider Tree.

Bahls, Lake and Wertheim proved the following theorem [BLW]:

Theorem 17. *Let T be a spider tree with l leg, each of which has length in $\{m, m+1\}$ for some $m \geqslant 1$. Then, T is gracefil.*

Proof. (as in [BLW]) We assume that $l \geqslant 3$ since otherwise T is a path and we already showed that paths are graceful. We look at two cases.

Case 1. l is odd. Let $l = l_0 + l_1$, where l_i is the number of legs of length $m+i$ for $i \in \{0, 1\}$. Then, T has $n = lm + l_1 + 1$ vertices. We call the legs $L_1, L_2, \ldots, L_l$ where $L_1, L_2, \ldots, L_{l_1}$ are the legs of length $m+1$ and $L_{l_1+1}, L_{l_2+1}, \ldots, L_l$ are the legs of length m. Let v^* be the branch point of T and let $v_{i,j}$ be the vertex in L_i of distance j from v^*.

We exhibit the following labeling f:

(1). $f(v^*) = 1$,

(2). If i and j are both odd, then $f(v_{i,j}) = n - \frac{i-1}{2} - \frac{(j-1)l}{2}$,

(3). If i and j are both even, then, $f(v_{i,j}) = n - \frac{l-1}{2} - \frac{(j-2)l}{2}$,

(4). If i is even and j is odd, then, $f(v_{i,j}) = \frac{i}{2} + \frac{(j-1)l}{2} + 1$, and,

(5). If i is odd and j is even, then, $f(v_{i,j}) = \frac{l-1}{2} + \frac{i+1}{2} + \frac{(j-2)l}{2} + 1$.

This labeling assignes all numbers from 1 to n to the vertices of T since it starts by assigning 1 to v^* and then traverses the longer legs first, alternating between highest and lowest remaining unused labels, spirally away from the center.

[Insert picture].

Then, we have that for $i \equiv j \pmod 2$, $f(v_{i,j}) - f(v_{i,j+1}) = n - 1 - \frac{l-1}{2} - i + (1-j)l > 0$ and $f(v_{i,j}) - f(v_{i,j-1}) = n - 1 - \frac{l-1}{2} - i + (2-j)l > 0$. Suppose that there exist $(i,j) \neq (i', j')$ and $i \equiv j \pmod 2$, $i' \equiv j' \pmod 2$ and $f(v_{i,j}) - f(v_{i,j+1}) = f(v_{i',j'}) - f(v_{i',j'+1})$. So, we get that $i - i' + (j - j')l = 0$, so, $l = \frac{i-i'}{j-j'}$ (note that if $j = j'$, then also $i = i'$, so, $(i,j) = (i', j')$, so, $j \neq j'$). Thus, $|i - i'| < l$ and $|j - j'| \geqslant 1$, and $l = \left|\frac{i-i'}{j-j'}\right| < \frac{l}{1} = l$, which gives us a contradiction. Thus, $f(v_{i,j}) - f(v_{i,j+1}) \neq f(v_{i',j'}) - f(v_{i',j'+1})$. Similarly, we get that $f(v_{i,j}) - f(v_{i,j+1}) \neq f(v_{i',j'}) - f(v_{i',j'-1})$ and that $f(v_{i,j}) - f(v_{i,j-1}) \neq f(v_{i',j'}) - f(v_{i',j'-1})$.

Case 2. l is even. WLOG L_l is a leg of length m (otherwise the tree is symmetric which we already proved is graceful). Remove the leg L_l to get a tree T_0 with an odd number of legs $l-1$. From above we get a graceful labeling f_0 of T_0 with $f_0(v^*) = 1$. Let $V(T_0) = n'$. Define a new graceful labeling f_0' of T_0 by $f_0'(v) = n' + 1 - f_0(v)$ for each $v \in V(T_0)$.

Now, construct a new tree T_1 by appending a new vertex, w_1, to T_0's center. Extend f_1 on $V(T_1)$ by $f_1(w_1) = 1$ and $f_1(v) = f_0'(v) + 1$ for all $v \in V(T_0)$. Define f_1' on T_1 by $f_1(v) = n' + 2 - f_1(v)$ for all v. Note that $f_1'(w_1) = n' + 1$. We can consecutively append vertices $w_2, w_3, \ldots, w_m$ to our l-th leg to obtain a graceful labeling of T. Note that we can append as many vertices as we want, not just m. □

3.5. Trees of diameter at most five.

Trees with diameter 2 are star trees and they are instances of caterpillars, hence they are graceful. Rosa proved that trees of diameter at most three are graceful. In 1989 Zhao [Zha89] showed that all trees of diameter 4 are graceful.

In 2001, Hrnciar and Haviar [HH01] showed that all trees of diameter 5 are graceful.

Lemma 18. *Let a tree T with n edges have a graceful labeling f and let $u \in V(T)$ be such that $f(u) = 0$ or $f(u) = n$. Let H be a caterpillar, $V(T) \cap V(H) = \emptyset$ and let $v \in V(H)$ be a vertex which either has a maximal eccentricity or is adjacent to a vertex of maximal eccentricity. If T' is the tree obtained by gluing the trees T and H in such a way that the vertices u and v are identified, then, T' is a graceful tree as well.*

Definition 19. *Let T be a tree and let $uv \in E(T)$. Then, $T_{u,v}$ is the subtree of T induced by the set $V(T_{u,v}) = \{w \in V(T): w = u \,\mathrm{or}\, v \,\mathrm{is\,in}\, a\, u - w\, \mathrm{path}\}$.*

The transformations that Hrnciar and Haviar use in their paper are based on the following lemma.

Lemma 20. *Let T be a tree witha graceful labeling f and let u be a vertex adjacent to the vertices u_1 and u_2. Let T' be the subtree of T induced by the set $V(T') = (V(T) - (V(T_{u,u_1}) \cup V(T_{u,u_2}))) \cup \{u\}$ and let $v \in V(T')$ with $v \neq u$. Then,*

(a). If $u_1 \neq u_2$, $f(u_1) + f(u_2) = f(u) + f(v)$ and the tree T'' is obtained by gluing the trees T_{u,u_1}, T_{u,u_2} and T' in such a way that the vertex v of the tree T' is identified with the vertex u of the trees T_{u,u_1}, T_{u,u_2}, then f is a graceful labeling. (see the figure below)

(b). If $u_1 = u_2$, $2f(u_1) = f(u) + f(v)$ and T'' is a tree obtained by gluing the trees T' and T_{u,u_1} in such a way that the vertex v of the tree T' is identified with the vertex u of the tree T_{u,u_1}, then f is a gracdeful labeling of the tree T'' as well.

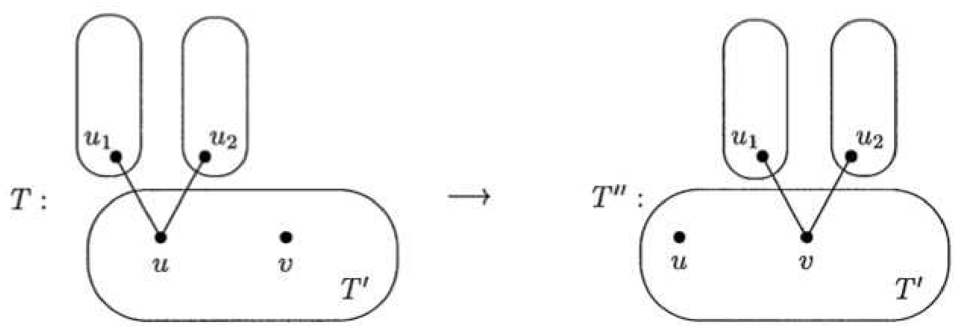

Figure 11. Picture taken from [HH01].

Proof. The proof is based ona few simple equalities:

For part (a)., $|f(u_1) - f(u)| = |f(u) + f(v) - f(u_2) - f(u)| = |f(v) - f(u_2)|$ and $|f(u_2) - f(u)| = |f(u) + f(v) - f(u_1) - f(u)| = |f(v) - f(u_1)|$, so, the labeling of the new tree is also graceful.

For part (b)., $|f(u_1) - f(u)| = |\frac{f(u) + f(v)}{2} - f(u)| = |\frac{f(v) - f(u)}{2}|$ and $|f(u_1) - f(v)| = |\frac{f(u) + f(v)}{2} - f(v)| = |\frac{f(u) - f(v)}{2}|$ and so the labeling of the new tree is also graceful. □

Using the notation of the Lemma, we are going to say that the tree T'' is obtained from T by a transfer of teh trees T_{u,u_1} and T_{u,u_2} from the vertex u to the vertex v. For example, from the star on the left figure below, we can get the tree on the right.

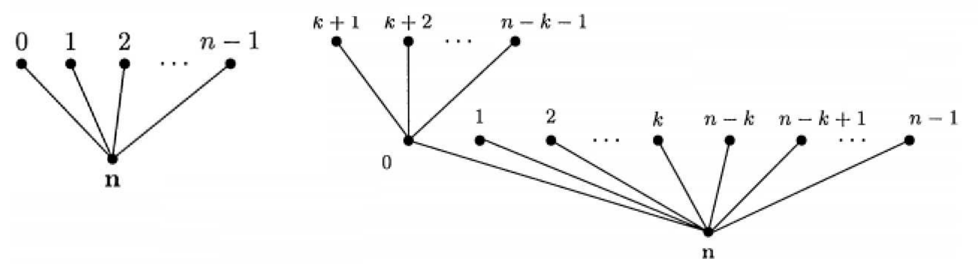

Figure 12. Picture taken from [HH01].

Definition 21. *Hrnciar and Haviar use two types of transfers of end-edges.*

A $u \to v$ transfer is a transfer of the first type *if the end-vertices of the transfered end-edges have labels $k, k+1, \ldots, k+m$ for some k and m.*

A $u \to v$ transfer is a transfer of the second type if the labels of the end-vertices of the transfered end-edges form two sections $k, k+1, \ldots, k+m$ and $l, l+1, \ldots, l+m$ for some k, l, m.

Transfers of the first type work according to the lemma above if $f(u) + f(v) = k + (k + m)(= k + 1 + (k + m - 1) = k + 2 + (k + m - 2) = \cdots)$. Transfers of the second type work if $f(u) + f(v) = k + l + m(= k + 1 + (l + m - 1) = k + 2 + (l + m - 2) = \cdots)$.

Theorem 22. *Every tree of diameter 4 is graceful.*

Proof. (as appears in [HH01]) It is sufficient, using the above lemma, to prove that every tree T of diameter 4 having the central vertex of an odd degree has a graceful labeling such that the label of the central vertex is maximal.

Let x be the number of vertices of an even degree that are adjacent to the central vertex of T. Let y be the number of vertices of odd degree greater than 1 that are adjacent to the central vertex of T. Let the degree of the central vertex of T be $2k + 1$ and let T have n edges. We can obtain a graceful labeling of T starting with the tree in the figure on the right in the figure above by carrying out the following transfers:

$$0 \to n - 1 \to 1 \to n - 2 \to 2 \to n - 3 \to \cdots,$$

where the first x transfers are of the first type and the next $y - 1$ (if $y \geqslant 1$) transfers are of the second type (to get the desirable sets of end-edges of even cardinality). ☐

We are now going to look at trees of diameter 5. The proof given by Hrnciar and Haviar in [HH01] is a bit too technical and involve a large number of cases, so, we are only going to give a sketch here. The authors first show using the above methods that every tree with diameter 5 is “nearly” graceful and then they prove the main result.

So, let T be a tree of diameter 5. Then it has two central vertices which we denote by a and b. Let x be a vertex adjacent to the central vertex a such that $x \neq b$. The subtree $T_{a,x}$ is a *branch* (at the vertex a) if $T_{a,x}$ is a subtree of diameter 2. A branch $T_{a,x}$ is an *odd branch* if the degree of the vertex x is even, otherwise, $T_{a,x}$ is an *even branch*. Similarly, we define even and odd branches $T_{b,y}$ adjacent to b.

Now, let $p = \#$odd branches at a, $r = \#$even branches at a, and $i = \#$endedges at a. Similarly, let $q = \#$odd branches at b, $s = \#$even branches at b, and $j = \#$endedges at b.

The graceful labelings defined in the sequel depend on those cardinalities, mostly on their parties. In fact, the authors introduced the following notation: for example $(p, ri; q, s, j) \equiv (e, o, o; e, e, e)$ if p, q, s, j are even and r, i are odd.

Theorem 23. *Every tree T of diameter 5 is graceful or nearly graceful, i.e. if the cardinality of its edge set is n, then, there exists a vertex labeling with the numbers from 1 to n such that such that the cardinality of the induced edge labeling is either $n-1$ or $n-2$, i.e. at most 2 edges have the same label.*

Proof. The proof of this theorem looks at a number of cases and exhibits specific transfers which give graceful or nearly graceful labelings of the given tree. One can find the proof of this theorem in [HH01]. We omit it here since it is very long and technical. □

Now, the main theorem of this section.

Theorem 24. *Every tree of diameter 5 is graceful.*

Proof. Similarly as in the proof of the previous theorem, the authors distinguish a number of different cases, and using the already found graceful or nearly graceful labelings of the trees, they produce graceful labelings via the transfers defined in the beginning of the section. □

3.6. Lobster Trees.

Definition 25. *A* lobster tree *is a tree such that if you remove all of its leaves, it becomes a caterpillar.*

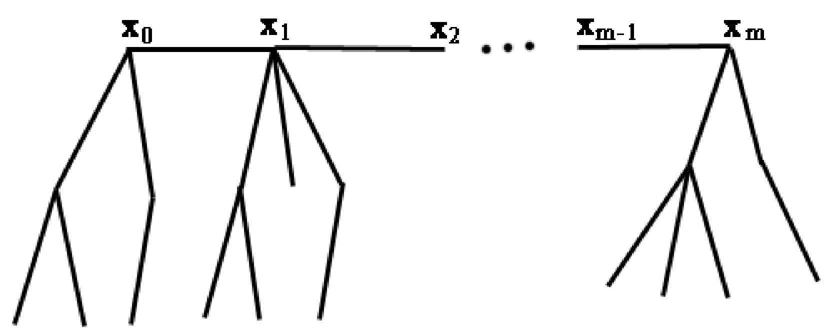

Figure 13. An example of a lobster tree.

In 1979 Bermond [Ber79] conjectured that all lobsters are graceful. Many have attempted to resolve this conjecture, although no one has been able to do it yet.

In 2002 Morgan [Mor02] published a paper showing that all lobster trees with perfect matchings are graceful. However, the current author is failing to see why the result is correct and believes there is a flaw in it, which would mean that even that class is stil not known to be graceful.

In 2008 Mishra and Panigrahi [PM08] and [MP05] found classes of graceful lobsters of diameter at least five. They also showed another class of lobsters is graceful in [MP08]. They observed that a lobster having diameter at least five has a unique path $H = x_0x_1...x_m$ satisfying the property that, besides the adjacencies in H, both x_0 and x_m are adjacent to the centers of at least one $K_{1,s}$ (which is a star with s leaves), where $s > 0$, and each x_i, for $1 \leqslant i \leqslant m-1$, is at most adjacent to the centers of some $K_{1,s}$, where $s \geqslant 0$. This unique path H is called the *central path* of the lobster. There are three types of branches that the vertices of H can be adjacent to: even, odd, and pendant. If x_i is adjacent to the center of a $K_{1,s}$, where $s \geqslant 2$ is even, then $K_{1,s}$ is an *even branch*. If x_i is adjacent to the center of a $K_{1,s}$, where s is odd, then $K_{1,s}$ is an *odd branch*. If x_i is adjacent to the center of a $K_{1,s}$, where $s = 0$, i.e. x_i is adjacent to a leaf, then $K_{1,s}$ is called a *pendant branch*. In their paper [MP08], they give graceful labelings to the lbsters having one of the following features. Let l_1, l_2, and l_3 be integers such that $1 \leqslant l_1 < l_2 \leqslant l_3 \leqslant m$.

(1). The vertex x_0 is attached to an even number of odd branches and an odd number of pendant branches. Each vetex x_i, for $1 \leqslant i \leqslant l_1$, is attached to some combination consisting of odd pendant branches, whereas each vertex x_i, for $l_1 + 1 \leqslant i \leqslant l_2$, is attached to some combination consisting of all three types of branches. If $l_2 < m$, then they have one of the following properties.

(i) Each vertex x_i, for $l_2 + 1 \leqslant i \leqslant l_3$, is attached to some combination consisting of odd and evenbranches and each of the rest of the vertices x_i, if any, is attached to only odd (or even) branches.

(ii) Each vertex x_i, for $l_2 + 1 \leqslant i \leqslant l_3$, is attached to some combination consisting of odd (respecitvely even) and pendant branches. Each of the rest of the vertices x_i, if any, is attached to only odd (respectively even) branches.

(2). The vertex x_0 is attached to an even number of odd branches and an odd number of pendant branches and combinations of branches incident on the vertices x_i, for $1 \leqslant i \leqslant m$, pendant branches and combinations of branches incident on the vertices x_i, for $1 \leqslant i \leqslant m$, are the subcases of those mentioned in (1), where one or more combinations are absent.

More special cases of classes of graceful lobsters have been found by Sethuraman and Jesintha in [SJ08], by Ng in [Ng86] , by Wang, Jin, Lu, and Zhang in [WJLZ94] , by Abhyanker in [Abh02].

3.7. Firecrackers.

Definition 26. *A* firecracker F *is a tree consisting of a path* $P(F)$ *and a collection of stars, where each vertex on* $P(F)$ *is joined to the central vertex of exactly one star.*

Thus, firecrackers are also a class of graceful trees. They were proved graceful by Chen, Lu and Yeh in [CLY97] (as referenced in [EH06]), however the current author wasn't able to find this paper and came up with a proof independently.

Theorem 27. *All firecracker trees are graceful.*

Proof. Let F be a firecracker tree, let $P(F)$ have k vertices and let each of the stars attached to them have $m-1$ vertices (excluding the vertex in $P(F)$ each of them is attached to, so, m vertices with it). So, the total number of vertices of F is km.

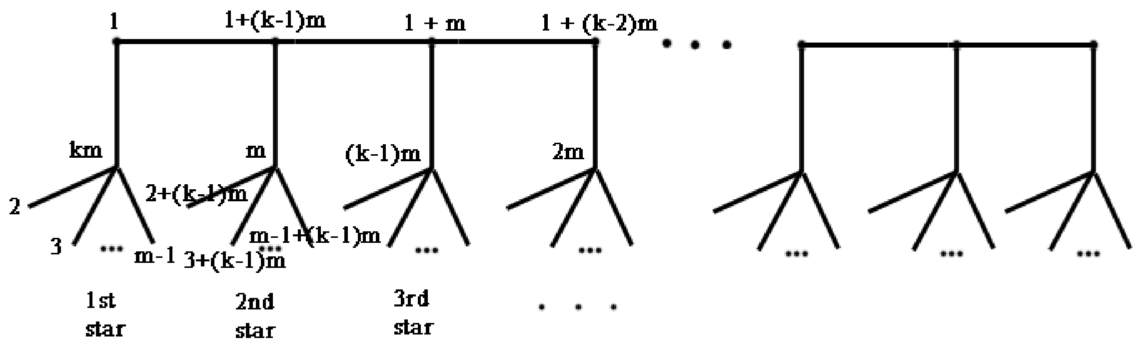

Figure 14. A graceful labeling of a firecracker.

We number the vertices on the central path with the numbers $1, 1+(k-1)m, 1+m, 1+(k-2)m, 1+2m, \ldots$ from left to right. Then, we number the centers of the stars, from left to right with the numbers km, m, $(k-1)m$, $2m$, $(k-2)m$, ... Then, the remaining $m-2$ vertices of each star we number with the following: We start from left to right and

(1). if the star we are looking at is the $2i+1$ - st from left to right, then we have nubered its center vertex with $(k-i)m$ and the top vertex with $1+im$. Then, we number the remaining $m-2$ vertices of this star with $2+im, 3+im, \ldots, m-1+im$.

Thus, we get induced edge valuations (since $2i < k$): $(k-2i)m-1$, $(k-2i)m-2$, ..., $(k-2i)m-(m-1)$.

(2). if the star we are looking at is the $2i$ - th from left to right, then we have numbered its center vertex with im and its top vertex with $1+(k-i)m$. Then, we number the remaining $m-2$ vertices of this star with $2+(k-i)m, 3+(k-i)m, \ldots, m-1+(k-i)m$.

Thus, we get induced edge valuations (since $2i < k$): $(k-2i)m+1$, $(k-2i)m+2$, ..., $(k-2i)m+m-1$.

Thus, all the edge valuations that we get are different, hence we have exhibited a graceful labeling.

□

This proof extends to proving that every tree which contains a central path and each of the vertices of this path is the root of a given symmetrical tree is graceful. The reason that the same proof will work is that as we saw before, symmetrical trees have an interlaced graceful labeling, so, we can keep alternating between the different layers from top to bottom in order to get a graceful labeling for the whole tree.

A generalized firecracker tree is one in which the stars can have different numbers of vertices. According to [EH06], Chen, Lu and Yeh proved in [CLY97] that all generalized firecracker trees are also graceful.

3.8. Banana Trees.

Definition 28. *A* banana tree *consists of a vertex v joined to one leaf of any number of stars.*

Let $(2K_{1,1}, \dots, 2K_{1,n})$ be the tree obtained by adding a vertex to the union of two copeis of each of $K_{1,1}, \dots, K_{1,n}$ and joining it to a leaf of each star. The banana tree obtained in this way is interlaced and therefore graceful. Chen, Lu, and Yeh conjectured in [CLY97] that all banana trees are graceful. More generally, the banana tree $(a_1K_{1,1}, \dots, a_{t-1}K_{1,t-1}, a_tK_{1,t}, a_{t+1}K_{1,t+1}, \dots, a_nK_{1,n})$ denotes the tree obtained by adding a vertex (the *apex*) of the union of a_i copies of each of the stars $K_{1,i}$ and joining the vertex to a leaf of each star.

Bhat-Nayak and Deshmukh [BD96] have constructed three new families of graceful banana trees using an algorithmic labeling proof. Extending the results of Chen, Lu and Yeh [CLY97], they have shown that the following are graceful:

(1). $(K_{1,1}, \dots, K_{1,t-1}, (\alpha+1)K_{1,t}, K_{1,t+1}, \dots, K_{1,n})$, where $0 \leqslant \alpha < t$;

(2). $(2K_{1,1}, \dots, 2K_{1,t-1}, (\alpha+2)K_{1,t}, 2K_{1,t+1}, \dots, 2K_{1,n})$, where $0 \leqslant \alpha < t$;

(3). $(3K_{1,1}, 3K_{1,2}, \dots, 3K_{1,n})$.

Additionally, Murugan and Arumugam [MA01] showed that any banana tree where all the stars have the same size is graceful by constructing a graceful labeling of these banana trees. Note that a banana tree, in which all the stars have the same size is also a symmetrical tree, so, it is also graceful by what we have shown before.

3.9. Other Classes of Graceful Trees.

3.9.1. Regular Bamboo Trees.

Definition 29. *A* regular bamboo tree *is a rooted tree consisting of one cetral vertex, and several legs of equal length attached to it, the leaves of which are identified with leaves of stars of equal size.*

Regular bamboo trees were shown to be graceful by C. Sekar in 2002 (as referenced in [EH06]).

3.9.2. Olive Trees.

Definition 30. *An* olive tree T_k *is a spider tree with k legs with lengths $1, 2, \dots, k$ respectively.*

As referenced in [EH06], Abhyankar and Bhat-Nayak gave direct graceful labeling methods for T_{2n+1} and T_{2n}. Both of these methods involve assigning labels $q = (n+1)(2n+1)$ or n to the roots of the trees T_{2n+1} and T_{2n} respectively and then assigning labels to the vertices on the k paths adjacent to the root depending on the parity of the path label and the tree in question. Finally, the labels are assigned to the remaining vertices of the tree so that the sum of any two adjacent vertices is either $q-1$ or q in the case of T_{2n+1}, or q or $q+1$ in the case of T_{2n}.

3.9.3. Spraying pipes.

Definition 31. *A* spraying pipe tree *is a path $v_1, v_2, \dots, v_n$ such that each vertex v_i is joined to m_i paths at a leaf of each path, and all paths have the same length.*

Cheng, Lu and Yeh [CLY97] (as referenced in [EH06]) proved that a spraying pipe tree is interlaced if n is even and $m_{2i-1} = m_{2i}$ for each $1 \leqslant i \leqslant \frac{n}{2}$.

3.10. 0-rotatable Trees and Bipartite Labelings.

Definition 32. *A tree is* 0-rotatable *if for any vertex in the tree there exists a graceful labeling in which this vertex has the label 1.*

Rosa noted the problem of 0 - rotatability in graceful trees in 1966 and announced that paths are 0 - rotatable in a short note in 1977 [Ros77]. Moreover, he proved the following:

Theorem 33. *There exists a bipartite labeling of any path on which any vertex may be labeled* 1 *if and only if the vertex is not the central vertex of* P_5.

Chung and Hwang [CH81] proved that a class of caterpillars is 0-rotatable.

Definition 34. *Let* C *be a caterpillar and let* v_1, v_2, ..., v_n *be a longest path in* C. *The* head *and the* tail *of* C *are the vertices* v_1 *and* v_n *and the* feet *of* C *are the internal vertices* $v_2, v_3, \ldots, v_{n-1}$. *If every foot has the same degree, then* C *is called a* t-toed caterpillar.

Chung and Hwang proved the following

Theorem 35. *All* t*-toed caterpillars are* 0*-rotatable.*

Note that not all caterpillars are 0-rotatable [EH06]: for example, the spider tree with two legs of length 1 and one leg of length 3 is not 0-rotatable since if we place 1 at the vertex v in the figure below, we can't produce a graceful labeling.

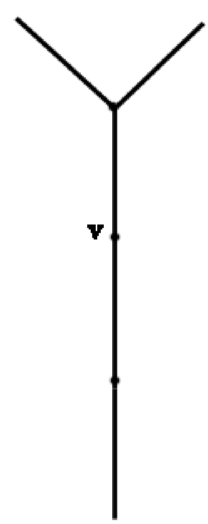

Figure 15.

As we are going to see in the sequel, trees with only two leaves, i.e. paths, have bipartite labelings, in fact they have quite a lot of those. In their paper [HKR82] Huang, Kotzig and Rosa explore trees with three leaves, i.e. spiders with three legs and whether or not they have bipartite labelings. A spider with three legs is characterized by three numbers p, q, r, the lengths of these legs. The authors proved the following:

Theorem 36. *(1). Every tree with three legs has a graceful labeling.*

(2). The spider tree with three legs of lengths p, q, and r has a bipartite labeling if and only if $(p,q,r) \neq (2,2,2)$.

Going on to a tree with four leaves, the tree is either a spider with legs of lengths p, q, r, s and one branch vertex of degree four, or it is a tree with exactly two vertices u and v of degree three. The latter type of tree can be descrbed by p, q, r, s, t, where the numbers give the distances between u, v and the distances between v and the two leaves separated from v by u.

Proposition 37. *(1). If at least two of p,q,r,s do not equal 2, then there exists a bipartite labeling of the spider tree with legs p,q,r,s.*

(2). Every spider tree with legs of lengths p,q,r,s has a graceful labeling.

(3). Every tree with four leaves of the second type described above with parameters p, q, r, s, t has a graceful labeling.

This means that Huang et al. in fact proved that each tree with at most four leaves is graceful.

The authors also proved other results about the existence of bipartite labelings for some spider trees. Let $P_{r,s}$ be a spider tree with r legs of length 2 and s legs of length 4. Then,

Theorem 38. *(1). $P_{r,s}$ has a bipartite labeling if and only if $|r-s| \leqslant 1$.*

(2). $P_{r,s}$ has a graceful labeling for all r, s.

More generally, the authors proved the following

Theorem 39. *Let T be a tree all of whose vertices are of odd degree. Let T^* be obtained from T by replacing every edge of T by a path le length two. If $|V(T)|$ is divisible by 4, then, T^* does not have a bipartite labeling.*

The authors explored the existence of bipartite labelings of a few other classes of trees as well.

4. ALL TREES WITH UP TO 35 VERTICES ARE GRACEFUL

Another approach to unraveling the graceful tree conjecture is by proving that all trees with up to a certain number of vertices are graceful (or finding a counterexample) with the aid of computers. It was first shown by Alfred and McKay [AM98] that all trees with up to 27 vertices are graceful. This result was extended to 29 in 2003 by Horton [Hor03] who used a randomized backtracking search for graceful labelings. Motivated by Horton's work, in 2010 Fang used a deterministic backtracking algorithm to prove that all trees with at most 35 vertices are graceful [Fan01].

The algorithm that Fang uses is a hybrid algorithm consisting of two parts, a back-tracking deterministic search and a hill-climbing tabu search combined with some idea from simulated annealing. Here is a description as it appears in the original paper. Suppose we have a tree T on n vertices.

The first part is the deterministic back-tracking search which tries to construct a graceful labeling f with $f(r)=1$, where r is the root. This is done by assigning values to vertices one by one. At each recursive call, it tries to make sure that a new value k appears in the induced labeling g. The value k decreases as we go deeper in the decision tree, from n to 2. This mechanism assures correctness of this algorithm.

To assure that a new value k appears in the range of g, it finds a not-yet-assigned vertex v connected to another vertex v' that is already assigned a label $f(v')$, then tries to assign to v a not-yet-assigned label $f(v)$ such that $|f(v')-f(v)|=k$. There may be several possibilities, or none. If this attempt fails, it tracks back, restores its status and pursues another possibility if there exists one.

Since the decision tree can grow exponentially in size when n increases, we manually add a treshold on the number of backtrack. This prevents searching for a very ling time. This treshold is tuned with respect to the performance of the probabilistic search described. It is empirically fixed to $11000(n-19)-1000$ in this verification. For a new, improved version of probabilistic earch, it is empirically fixed to $1000(n-18)$. A detailed version of the algorithm can be found in the original paper.

The second part which is a probabilistic search has the goal of minimizing the following evalueation function of a labeling f: $\mathrm{Eval}(f)=\sum_{i\in\{1,\dots,n-1\}\setminus \mathrm{Im}(g)} i$, where $\mathrm{Im}(g)=\{|f(x)-f(y)|\colon \{x,y\}\in E\}$. Then, $\mathrm{Eval}(f)$ is positive if f is not a graceful labeling and 0 if f is a graceful labeling.

By minimizing Eval, we can efficiently explore labellings that are likely to be graceful. The algorithm uses hill-climbing (in this case, hill-descending) to try to minimize Eval. At each iteration, the algorithm trie a number of random modifications and picks the one with the best evaluation. This number is fixed to $2n$.

However, it is known that the hill-climbing method can be trapped in a local minimum. In order to avoid this problem, we use a tabu search. The algorithm keeps track of a number of previous modifications and forbids such modifications unless the result is a graceful labeling. Therefore, the algorithm always goes forth to search for new solutions. The number of forbidden previous modifications is fixed to be $[n/3]$. This value is determinied empirically.

Also in order to solve the local minimum problem, the algorithm accepts with a certain probability determined empirically, modifications that worsen the solution. Thi behavior is intended to emulate simulated annealing, which can escape local minimum with a probability determned by its "temperature".

A detailed pseudocode of this part of the algorithm can be found in the original paper.

The hybrid algorithm is a combination of the two parts described above. In the first stage the algorithm runs the deterministic back-tracking search. If it fails to find a graceful labeling, it turns to the second stage, where it performs the probabilistic search. The reason for this strategy is that the deterministic back-tracking outperforms the probabilistic search in most cases, but in some cases it takes an enormous amount of time. The probabilistic search is not fast compared to the deterministic one, but its runtime varies much less. Therefore, it is natural to use the deterministic search with a cutoff of runtime, then patch unfinished cases with the probabilistic search.

By applying the hybrid algorithm to every tree with at most 35 vertices, Fang verified that every such tree is graceful. Various statistics for the algorithm are given in detail in the original paper.

5. Constructing Graceful Trees

One of the most natural methods that one explores when looking for a solution to the graceful tree conjecture is the method of induction. It would be easy to apply induction to graceful tree on n vertices if the label 1 could be placed at any vertex in every tree on $n-1$ vertices. A tree with this property is called 0-*rotatable*. Indeed, if this were the case, then we could just add a leaf to a vertex labeled 1 and label the new vertex n to get an induced edge labeling of n on the new edge. However, not all trees are 0-rotatable. For example, the spider tree with two legs of length 1 and one leg of length 3 is not 0-rotatable as we showed in Section 3. This is why other methods were created to construct graceful trees most often from smaller graceful trees.

5.1. Constructing new graceful trees from known graceful trees.

We saw in Lemma 7. that if we have a graceful labeling f of a tree on n vertices, then, the induced labeling $n+1-f$ is also graceful. This transformation is sometimes referred to as the *inverse transformation* of the labeling. By using the elementary transformation, one can generate a family of graceful trees by adding a new vertex with label $n+1$ connected with an edge to the vertex labeled 1 or first applying the inverse transformation and then adding a vertex with label $n+1$ with an edge connected to the vertex labeled 1. One can see that this method of constructing graceful trees gives us exactly the family of all caterpillars.

In 1973, Stanton and Zarnke [SZ73] (as appears in [EH06]) were the first to develop a nontrivial algorithm for constructing graceful trees. The construction is as follows: Let S and T be two trees on n_S and n_T vertices respectively. Copies of T will be attached to either n_S or n_S-1 of the vertices of S. The following steps produce a graceful labeling:

(1). Label the n_S or n_S-1 copies of T by adding some multiple of n_T to the vertex labels used in the graceful labeling of T. This multiple is chosen by the following algorithm:

- Select an arbitrary fixed vertex Z in the original tree T.

- Use $L_i(a)$ to designate the label in T_i of the vertex a of T, and define:

$$L_i(a) = \begin{cases} (r+1-i)n+1-L(a), & \text{if } d(a,z) \text{ is odd} \\ in+1-L(a), & \text{if } d(a,z) \text{ is even,} \end{cases}$$

  where $r = n_S$ or $n_S - 1$ depending on how many copies of T are being attached to S.

(2). Relabel the vertices of S by multiplying all the graceful labels by n_T and adding an arbitrary fixed c, $0 \leqslant c \leqslant n_T$.

(3). Identify vertices in the relabeled S and the copies of T which have identical vertex labels. This operation of identifying vertices is referred to as graphting. It is easy to see that the induced labeling of the reulting tree is also graceful.

The resulting tree is of Type I if it comes from attaching n_S copies of T to S, or Type II if it comes from attaching $n_S - 1$ copies of T to S. Moreover, the author show that the set f all tree derived using a fixed vertex z as a fixed vertex for T is the same as the set of trees derived using a a different vertex y (but the corresponding trees may not be the same).

In [KRT77] Koh, Rogers, and Tan give a variation of Stanton and Zarnke's construction. Let $T(n)$ be a tree on n vertices and f be a graceful labeling of $T(n)$. Then, ($T(n)$, f) is called a graceful system. Let w be the unique vertex in $T(n)$ with $f(w) = n$. For each $p \geqslant 1$, let $T_1(n), T_2(n), \ldots, T_p(n)$ be disjoint isomorphic copies of $T(n)$ and for each $i = 1, 2, \ldots, p$, let w_i be the isomorphic image of w in $T_i(n)$. Adjoin to the graph $\cup_{i=1}^{p} T_i(n)$ a new vertex w_0 and p edges $w_0w_1, w_0w_2, \ldots, w_0w_p$. The new tree is denoted by $T_n^p(w)$. Then, use the exact same labeling given above in the proof given above that all symmetric trees are graceful.

In [KRT79] Koh, Rogers, and Tan created two new constructions based on their original idea in [KRT77]. Let $T_w^p(n)^*$ be the tree obtained by identifying $w_1 = w_2 = \cdots = w_p = w$ on the set $\cup_{i=1}^{p} T_i(n)$.

Theorem 40. *Let* ($T(n)$, f) *be a graceful system and* w *a vertex in* $T(n)$ *with* $f(w) = n$. *Let* $\mathrm{Adj}(w)$ *be the set of vertices in* $T(n)$ *adjacent to* w. *If* $\{f(v) - 1 \mid v \in \mathrm{Adj}(w)\} \subseteq \{0\} \cup \{n - f(v) \mid v \in \mathrm{Adj}(w)\}$, *then there exists a valuation* f^* *on* $T_w^p(n)^*$ *such that the system* ($T_w^p(n)^*$, f^*) *is graceful.*

Another construction that Koh, Rogers, and Tan gave in [KRT79] is the following. Let ($T(m)$, f') and ($T(n)$, f^*) be two given graceful systems where $T(m) = \{w_1, w_2, \ldots, w_m\}$. Let v be an arbitrary fixed vertex in $T(n)$. Based on the tree $T(m)$, adjoin an isomorphic copy $T_i(n)$ of $T(n)$ to each vetex w_i by identifying v^* and w_i. All the m copies $T_i(n)$ of $T(n)$ are pairwise disjoint and no extra edges are added. Such a new tree obtained is denoted by $T(m)\Delta T(n)$. Clearly, $|T(m)\Delta T(n)| = mn$ and $T(m)\Delta T(n) \ncong T(n)\Delta T(m)$ in general.

Theorem 41. *Let $(T(m), f')$ and $(T(n), f^*)$ be two graceful systems. Then there exists a valuation f on $T(m)\Delta T(n)$ such that the system $(T(m)\Delta T(n), f)$ is also graceful.*

The same authors proved another class of trees to be graceful [KRT78]:

Definition 42. *The distance $d(u, v)$ between two vertices in a tree T is the number of edges in the path between u and v. For a vertex v in T, we define the set $\mathfrak{F}(v) = \{u: d(u, v) \text{ is even}\}$. Let f be a labeling of T and b be the vertex for which $f(b) = 1$. Then, f is a* parity valuation *if the set $\{f(u): u \in \mathfrak{F}(b)\} = \{1, 2, \ldots, |\mathfrak{F}(b)|\}$. A labeling is an* interlaced valuation *if it is both graceful and a parity valuation.*

[Insert an example].

Note that a labeling is a parity valuation if we can write the tree T as a bipartite graph so that the labels on the vertices in one of the two groups are all numbers from 1 to the number of elements in that group and the labels on the vertices in the other group are the rest of the numbers.

Theorem 43. *If f is an interlaced valuation of T, then so is the inverse transformation of f as well as the labeling f' such that $f'(v) = p + 1 - f(v)$ if $f(v) \leqslant p$ and $f'(v) = n + p + 1 - f(v)$ if $p < f(v)$, where if b is such that $f(b) = 1$, then, $p = |\mathfrak{F}(b)|$.*

In 1993 the team of Liu, Jin, Liu, Lee, Lu, and Zhang published two papers, [JLLLLZ93_1] and [JLLLLZ93_2] each paper giving a new way of constructing graceful trees from smaller graceful trees.

In the first paper Jin et al. introduced the concept of joint sum operation.

Definition 44. *Given two trees T and R, define a new tree by joining a vertex of T with a vertex of R. This tree is called the* joint sum *of T and R and is denoted by $\langle T + R\rangle$. Similarly, we define $\langle T_1 + T_2 + \cdots + T_n\rangle$ for trees $T_1, T_2, \ldots, T_n$.*

Note that the joint sum of two trees is not unique.

Theorem 45. *Let T and R be two graceful trees. Then, the joint sum $\langle T + R + R\rangle$ by joining the vertices with the graceful label 1 is also graceful.*

This result generalized nicely into the following.

Theorem 46. *If T and R are two graceful trees, then $\langle \lambda T + 2\mu R\rangle$ is also graceful ($\lambda, \mu \in \mathbb{N}$).*

Proof. Use induction and the previous theorem. □

Let R be a graceful tree with an interlaced valuation f. This is equivalent to saying that if (X, Y) is a bipartition of R, then, f is such that $\max_{v\in X} f(v) < \min_{v\in Y} f(v)$. Jin et al. call this type of graceful tree a *glue tree*. In addition, the vertex v^* which satisfies $f(v^*) = \min_{v\in Y} f(v)$ is called a *joint vertex* or simply a *joint*. A vertex v_* which satisfies $f(v_*) = \max_{v\in X} f(v)$ is called a *glue vertex* or simply a *glue*.

Proposition 47. *If R is a graceful tree, then $\langle 2\mu R\rangle$ is a glue tree* ($\mu \in \mathbb{N}$).

Theorem 48. *Let T be a graceful tree and R a glue tree. Then, the joint sum $\langle T + R\rangle$ oned by joining the vertex of T with graceful label 1 witha joint vertex of R is also graceful.*

Corollary 49. *If T is a graceful tree and R is a glue tree, then, $\langle \lambda T + \mu R\rangle$ is a graceful tree* ($\lambda, \mu \in \mathbb{N}$).

In their second paper on graceful trees, Jin et al. [JLLLLZ93_2] used the glue tree definition to give the radical product operation.

Definition 50. *Given two trees T and R, specify for each tree a vertex as the root. By gluing the two roots, we obtain a new tree, called the* radical product *of T and R, denoted by $\langle T \bullet R\rangle$. When T and R are isomorphic, $\langle R \bullet R\rangle$ is denoted as $\langle R^2\rangle$. Similarly, we define $\langle T_1 \bullet T_2 \bullet \cdots \bullet T_n\rangle$ and $\langle T^n\rangle$ for trees $T_1, T_2, \ldots, T_n, T$.*

Theorem 51. *Let T and R be two graceful trees with the same order. If for each tree there is a leaf with graceful label 1, then the radical product $\langle T^m \bullet R^{2n}\rangle$ using these two vertices as roots is a graceful tree, where $m, n \in \mathbb{N}$.*

The above result was also generalized to a list of trees.

Theorem 52. *Let T_i ($i = 1, 2\ldots, k$) be graceful trees with q edges. If for each tree T_i there is a leaf b_i with graceful label 1, then the radical product $\langle T_1^{m_1} \bullet T_2^{2m_2} \bullet \cdots \bullet T_k^{2m_k}\rangle$ obtained by using b_i as the root is also graceful, where $m_i \in \mathbb{N}$ for each i.*

Theorem 53. *If T is a glue tree and R is a graceful tree, then the radical product $\langle T \bullet R\rangle$ is also graceful, where the root of T is the glue vertex and the root of R is the vertex with graceful label 1.*

In 1998, Burzio and Ferrarese [BF98] (as appears in [EH06]) gave a generalization of Koh, Tan and Roger's method to construct $T(m)\Delta T(n)$. They call their new construction the Δ_{+1}-*construction*. Let $(T(m),\ f')$ and $(T(n),\ f^*)$ be two graceful systems as above. Let v^* be an arbitrary fixed vertex in $T(n)$ and w the vertex of $T(m)$ with $f'(w) = m$. Consider $T(m) - w$, which in general is not a connected tree, but just a disjoint union of trees.

We can still perform the generalized Δ-construction as follows.

Let $\overset{<}{v}$ (respectively $\overset{>}{v}$) be the vertex of $T(n)$ with $f'(\overset{<}{v}) = 1$ (respectively $f'(\overset{>}{v}) = n$). Note that $\overset{<}{v}\,\overset{>}{v}$ is an edge of $T(n)$ so $d(v^*, \overset{<}{v})$ and $d(v^*, \overset{>}{v})$ do not have the same parity.

Let $(T(n),\ \tilde{f})$ be either $(T(n),\ f')$ if $d(v^*,\ \overset{<}{v})$ is even and $(T(n),\ f'')$, where f'' is the inverse valuation of f'. Using the Δ-construction, construct the graph $G = (T(m) - w)\Delta T(n)$ with $(m-1)n$ vertices, which in general is a disjoint union of trees, and consider the valuation f: $G \to \{1,\ 2,\ \ldots,\ (m-1)n\}$, defined in the proof of Theorem 17 given by $T(n) = (f^*(w_i) - 1)n + \tilde{f}(v)$ if $d(v^*, v)$ is even and $f(v) = (m - 1 - f^*(w_i))n + \tilde{f}(v)$ if $d(v^*, v)$ is odd for each v in $T_i(n)$, $i = 1, 2, \ldots, m-1$.

For each edge vv' in G the values $|f(v) - f(v')|$ are different and the missing values from $\{1, 2, ..., (m-1)n-1\}$ are exactly the multiples np_k, where $p_k = f^*(w, w_k)$ are the weights of the edges incident to w in $T(m)$. We can now recover the missing values in the following way. Add to G a new vertex u and, for each k sucht hat ww_k is an edge in $T(m)$, add to G the edges $u\overset{<}{v}^{(k)}$, if $\tilde{f} = f'$ (respectively $u\overset{>}{v}^{(k)}$, if $\tilde{f} = f''$), where $\overset{<}{v}^{(k)}$ (respectively $\overset{>}{v}^{(k)}$) i the corresponding vertex of $\overset{<}{v}$ (respectively $\overset{>}{v}$) in $T_k(n)$. The graph obtained in a tree and it is denoted by $T(m)\Delta_{+1}T(n)$. Moreover,

Theorem 54. *The mapping f_{+1}: $T(m)\Delta_{+1}T(n) \rightarrow \{1, 2, ..., (m-1)n+1\}$ defined by $f_{+1}(v) = f(v)$ for each v in G and $f_{+1}(u) = (m-1)n+1$ is a labeling on $T(m)\Delta_{+1}T(n)$ and the system $(T(m)\Delta_{+1}T(n), f_{+1})$ is a graceful system.*

Proof. If $\tilde{f} = f'$, it is enough to check that $f_{+1}(u\overset{<}{v}^{(k)}) = np_k$. But $f_{+1}(u\overset{<}{v}^{(k)}) = |f_{+1}(u) - f_{+1}(\overset{<}{v}^{(k)})| = |(m-1)n+1-((f^*(w_k)-1)n+1)| = |(m-1)n+1-((m-p_k-1)n+1)| = |p_kn| = p_kn$. The case $\tilde{f} = f''$ is similar. □

If each edge $e = uv$ of a graph G is replaced by a new vertex w and edges uw and wv, then, the resulting graph is called the *subdivision graph* of G and is denoted $S(G)$. Burzio and Ferrarese used this generalized construction to prove the following:

Theorem 55. *The subdivision graph of a graceful tree is also a graceful tree.*

Proof. Let $(T(m), f^*)$ be a graceful system, w the vertex of $T(m)$ with $f^*(w) = m$, and $(T(2), f')$ be the graceful system with vertices $\overset{<}{v}$ and $\overset{>}{v}$ such that $f'(\overset{<}{v}) = 1$ and $f'(\overset{>}{v}) = 2$. Fix in $T(2)$ the vertex $v^* = \overset{<}{v}$ and perform the Δ_{+1}-contruction obtaining $T(m)\Delta_{+1}T(2)$. Using the generalized Δ-construction, connect the copies $T_i(2)$ and $T_j(2)$ in $T_m - w$ by the edge $\overset{<}{v}^{(i)}\overset{<}{v}^{(j)}$ (respectively $\overset{>}{v}^{(i)}\overset{>}{v}^{(j)}$) if $d(w_j, w) = d(w_i, w) + 1$ is odd (respectively even). Then, $T(m)\Delta_{+1}T(2) = S(T_m)$ and the system $(S(T_m), f_{+1})$ is graceful by the previous Theorem. □

Proposition 56. *Let $S_n(T)$ be the n-th subdivision graph of a tree T, i.e. the tree obtained by inserting n new vertices into each edge of T. Then, if T is graceful, $S_n(T)$ is also a graceful tree.*

Proof. The proof of this proposition is similar to the proof of the previous theorem using as $T(n)$ a path of length n, where $\overset{<}{v}$ is an end vertex. □

In 1973, Kotzig (as referenced in [EH06]) extended the work of Rosa on balanced trees by proving the following theorem.

Theorem 57. *(1). If a leaf of a long-enough path is joined to any leaf of an arbitrary tree, the resulting tree is graceful.*

(2). If a long-enough path replaces an arbitrary edge in an arbitrary tree, the resulting tree is graceful.

Another way of obtaining graceful labelings from known graceful labelings was proved by Poljak and Sura (as referenced in [EH06]) as a tool for showing that equidescendent c-symmetrical trees are graceful.

Theorem 58. *Let T be a gracefully labeled tree. Let G_1 and G_2 be two subgraphs of T each isomorphic to a star $K_{1,rs}$, and labeled with bipartite labelings $\{a\}, \{b+1, ..., b+sr\}$ and $\{b\}, \{a+1, ..., a+sr\}$, where $a+sr < b$. Let G_1' and G_2' be the trees with teh following structure:*

(1). A root with s direct descendants, each of which have $r-1$ additional descendants;

(2). Labels of the three levels above are respectively $\{a\}, \{b+kr\colon 1 \leqslant k \leqslant s\}, \{a+(s-k)r+1, ..., a+(s-k+1)r-1\}$ according to the value of k of the previous level for G_1', and $\{b\}, \{a+kr\colon 1 \leqslant k \leqslant s\}, \{b+(s-k)r+1, ..., b+(s-k+1)r-1\}$ for G_2'.

Then, a tree T' obtained from T by replacing G_1 and G_2 with G_1' and G_2' respectively is graceful.

Applying this theorem to pairs of adjacent vertices of gracefully labeled caterpillars produces new graceful lobster trees.

5.2. Strong Gracefulness. (as appears in [EH06])

Definition 59. *A tree T on n vertices is* strongly graceful *if T contains a perfect matching M and T admits a graceful labeling f such that $f(u)+f(v)=n+1$ for every edge $uv \in M$.*

Conjecture 60. *Every tree containing a perfect matching is strongly graceful.*

Broersma and Hoede proved in [BH99] that the above conjecture is equivalent to the Graceful Tree Conjecture. The way they show this is the following.

Definition 61. *The* spiketree spik(T) *of a tree T on n vertices is obtained by adding n new vertices to T along with n edges. The* contree *of a tree T with a perfect matching M is obtained from T by contracting the edges of M.*

The way Broersma and Hoede prove the equivalence of the two conjectures is by first using a graceful labeling of the contree of a tree with a perfect matchin to show that a tree witha perfect matching is strongly graceful. This is done by finding vertex labels that sum to $n-1$ and have absolute difference as prescribed b double the edge label in the original graceful labeling. The other direction is proved by taking the spiketree Spik(T) of an arbitrary tree T. Spik(T) is trongly graceful since it has a perfect matching, and has $2n$ vertices. Define the two sets matched by the perfect matchin to be those containing odd and even labels. By giving labels to vertices of T according to half the even vertex label of the corresponding edge in the strong graceful labeling of Spik(T), one produces a graceful labeling of T.

Another result that Broersma and Hoede obtained is the following:

Lemma 62. *Let T be a tree containing a perfect matching and let T^c be the contree of T. Then, T is strongly graceful if and only if the spiketree* Spik(T) *with contree T is strongly graceful.*

This result s that in order to prove the Graceful Tree Conjecture, it is sufficient to show that every spiketree is strongly graceful. However, as the authors observed, since a strongly graceful labeling of a spiketree immediately yields a graceful labeling of its contree, this isn't a substantial improvement. Moreover, the label 1 cannot be assigned to an arbitrary vertex of a spiketree, the same obstacle that arises in attempts to prove the original Graceful Tree Conjecture.

Another result that the authors proved in [BH99] is the following:

Theorem 63. *Every tree containing a perfect matching and having a caterpillar as its contree is strongly graceful.*

Thus, trees meeting this hypothesis can be used to generate new strongly graceful trees with the original tree as their contrees, by successively taking the spiketrees of a sequence of trees. This procedure generates what the authors called strongly graceful *long-edged caterpillars*.

6. Number of Graceful Labelings for Particular Types of Trees

6.1. Asymptotic Bounds for Bipartite Labelings for Paths.

Definition 64. *A* bipartite labeling *is an α-valuation as defined in Section 1.1. In other words, it is a graceful labeling f such that tehre exists $x \in \{1, 2, ..., n\}$ ($n = |V(T)|$) such that for any edge $vu \in E(T)$, either $f(v) \leqslant x < f(u)$ or $f(u) \leqslant x < f(v)$.*

Example 65. For example, the labelings of caterpillars and, in particular, paths that we exhibited are bipartite. Consider the following labeling of a path with 9 vertices. Note that for any two adjacent vertices the labeling of one of them is always $\leqslant 5$ and the labeling of the other one is always > 5.

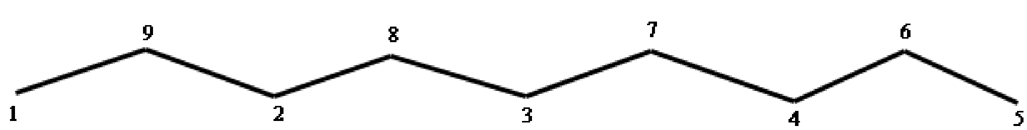

Figure 16. A bipartite labeling of a path with $n = 9$ and $x = 5$.

In fact, paths have many different bipartite labelings. Moreover, paths are 0-rotatable, i.e. there exist graceful labelings with 1 on any of the vertices.

A lot of effort has been put to sutdying the number of bipartite labelings of paths. In 1990 Abraham and Kotzig showed that the number of bipartite labelings of paths of order n grows asymptotically at least as fast as 1.3953^n (as referenced in [EH06]). Alfred et al. [ASS03] improved this to $\left(\frac{5}{3}\right)^n$ by counting bipartite labelings. We are going to exhibit their result.

Definition 66. *A* k-pendant *bipartite labeling is a bipartite graceful labeling of a path* P_n *on* n *vertices, in which the vertex label* k *is assigned to a leaf.*

Theorem 67. *Let* f *be a* k*-pendant bipartite labeling of* P_n *and* g *is a* k*-pendant bipartite labeling of* P_{2t}*. Inserting a new edge joining the vertex labeled* k *in* P_n *to the leaf in* P_{2t}*, which is not labeled* k*, can be made to produce a* k*-pendant labeling of* P_{n+2t}*.*

Proof. The way to do this is by just adding k to each of the vertices in the P_{2t} part of P_{n+2t} which have valuation greater than x, where x is the threshold for the bipartite labeling of P_{2t}. □

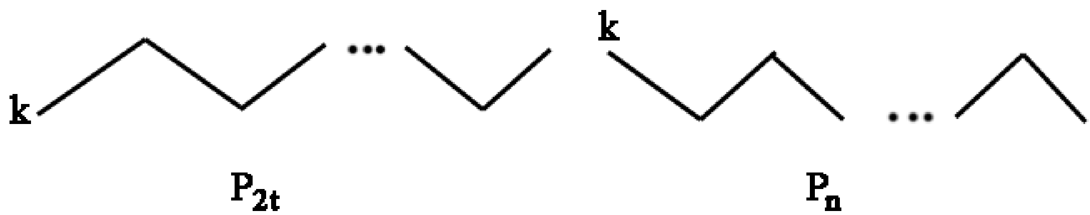

Figure 17.

Let $\tau_k(n)$ denote the number of inequivalent (neither isomorphic nor inverse) k-pendant bipartite labelings of the path P_n. The original result by Abraham and Kotzig used the recurrence $\tau_k(n+2t) \geqslant \tau_k(n)\tau_k(2t)$, with $\tau_1(n) = 1$. The reason that $\tau_1(n) = 1$ for each n is that if we put 1 in the beginning of a path, then we need to put n right next to it in order to have a difference of $n-1$, then we need to put 2 right enxt to n in order to have a difference of $n-2$, etc. So, there is exactly one graceful bipratite labeling of P_n for which 1 is assigned to a leaf.

Alfred et al. defined $b_{k,t} = (\tau_k(n+2t))^{\frac{1}{2t}}$ and $c_{k,t}$ to be the minimum of $\frac{\tau_k(n)}{(b_{k,t})^n}$ taken over all n such that $k+1 \leqslant n \leqslant k+2t$. By showing that $\tau_k(n) \geqslant c_{k,t}(b_{k,t})^n$ for $n, t > k > 1$, and by determining values of $b_{k,t}$ for various pairs k, t, Alfred et al. were able to improve the previous bound. Moreover, as the authors note, their result has an application in topological graph theory. It allows the construction of surface embeddings of large graphs from smaller graphs edge-labeled with certain elements of finite groups. Graceful labelings of paths in this context induce current assignments on certain cubic graphs generating vertex-transitive triangular embeddings of complete graphs. Thus, if the lower bounds on the number of graceful labelings of paths may lead to improved bounds on the number of vertex-transitive triangulations of complete graphs. Moreover, as the authors of [ASS03] also note that the number of vertex-transitive triangulations of surgaces by complete graphs may have super-exponential growth, if $b_{k,t} \to \infty$ as $t \to \infty$.

7. Graceful Graphs.

The definition of gracefulness also extends to general graphs.

Definition 68. *A* graceful labeling *of a graph G with e edges is a labeling of its vertices with distinct integers between* 1 *and $e+1$ such that the set of absolute values of the difer- ence of the labels of the ends of each edge is the set* $\{1,2,\ldots,e\}$.

The classes of graphs that have been studied with respect to gracefulness are even more than the classes of trees that we described already. A very good survey of graph labelings can be found in Gallian's survey of graph labeling [Gal10].

Graham and Sloane showed in [GS80] that almost all graphs are not graceful. They also showed the following assymptotic result. If $g(n)$ is the number of edges in the graceful graph on n vertices with largest size, then, it is known that $\lim_{n\rightarrow\infty}\frac{g(n)}{n^2}$ exists and satisfies $\frac{1}{3}\leqslant\lim_{n\rightarrow\infty}\frac{g(n)}{n^2}\leqslant 0.411$.

7.1. Complete Graphs.

Golomb showed (as referenced in [EH06]) in 1972 that:

Theorem 69. *If $n>4$, then the complete graph K_n is not graceful.*

The following is also due to Golomb (as referenced in [EH06]):

Theorem 70. *For all $ab\in\mathbb{N}$, the complete bipartite graph $K_{a,b}$ is graceful.*

Proof. If we assign the labels $1,2,\ldots,a$ to the vertices in the set of cardinality a, and the labels $a+1, 2a+1,\ldots, ba+1$, we obtain all edge labels from 1 to ab, so this is a graceful labeling. □

7.2. Cycle - related Graphs.

Rosa showed in [Ros67] the following results on graceful cycles (as described in [EH06]):

Theorem 71. *A bipartite labeling of C_n exists if $n\equiv 0 \pmod 4$.*

Proof. Let the vertices of C_n be $v_1, v_2, \ldots, v_n$, where $4\mid n$. Then, consider the following labeling f: $f(v_i)=\frac{i-1}{2}$ if i is odd, $f(v_i)=n+1-\frac{i}{2}$ if i is even and $i\leqslant\frac{n}{2}$, and $f(v_i)=n-\frac{i}{2}$ if i is even and $i>\frac{n}{2}$. It is easy to check that this labeling is a graceful bipartite labeling. □

Theorem 72. *A graceful labeling of C_n exists if $n\equiv 3 \pmod 4$.*

Proof. Now, let $n\equiv 3 \pmod 4$ and consider the following labeling f of the vertices v_1, $v_2,\ldots,v_n$ of C_n: $f(v_i)=n+1-\frac{i}{2}$ if i is even, $f(v_i)=\frac{i-1}{2}$ if i is odd and $i\leqslant\frac{n-1}{2}$, and $f(v_i)=\frac{i+1}{2}$ if i is odd and $i>\frac{n-1}{2}$. It is easy to check that this labeling is graceful. □

Rosa (as referencedn in [EH06]) completed the theory of the gracefulness of cycles:

Theorem 73. *If $n\equiv 1,2 \pmod 4$, then the graph C_n does not admit a graceful labeling.*

In fact, Rosa proved a more general result:

Lemma 74. *If G is an Eulerian graph with n edges, where $n \equiv 1, 2 \pmod 4$, then, there does not exist a graceful labeling of G.*

Bodendiek conjectured that every graph consisting of a cycle and a cord is graceful (as referenced in [EH06]), and various proofs have established this result later on, including a proof by Chen and Zhi Zeng in 1986. In more recent work, Zhi Zheng (as referenced in [EH06]) has profed a generalizetion of the Bodendiek conjecture:

Theorem 75. *Apart from four exceptional cases, graphs consisting of three independent paths joining two vertices are graceful.*

7.3. Joins of Graphs.

For two graphs G and H, the graph $G+H$ is the graph consisting of the disjoint union of G and H to which are added all edges between vertices of G and vertices of H.

A number of classes of graphs that are joins of graphs have been shown to be graceful.

Acharya (ref. in [Gal10]) showed that if G is a connected graph, then $G + \overline{K_n}$ is graceful. ($\overline{K_n}$ is the complement of K_n, i.e. a graph on n vertices that has no edges).

Definition 76. *A* wheel graph *W_n is a cycle with n vertices and an extra $n+1$-st vertex in the middle, which is connected with an edge to all the n vertices of the cycle. In other words, W_n is the join $C_n + K_1$.*

Hebbare (as referenced in [EH06]) conjectured in 1976 that all wheels W_n are graceful and Hoede and Kuiper (as referenced in [EH06]) proved the result in 1987. Hebbare's work on wheel labelings led him to consider other non-Eulerian graphs, and to formulate the following conjecture (as referenced in [EH06]):

Conjecture 77. *A non-Eulerian graph with at least two blocks, each block being a complete graph on at least three vertices, is not graceful.*

Fun and Wu (as referenced in [EH06]) proved the following result on graceful graphs in 1990:

Theorem 78. *If T is a tree with a graceful labeling and S_n is a star on n vertices, then $T+S_n$ is graceful.*

7.4. Product Graphs.

A second result of Fu and Wu ([EH06]) is the following:

Theorem 79. *Let T be a tree with partite sets of size n_1 and n_2, where $|n_1 - n_2| \leqslant 1$. Then, for $m \geqslant 2$, the Cartesian product $T \times P_m$ has a bipartite labeling.*

As a corollary to this theorem, we have that the net $P_n \times P_m$ has a bipartite labeling for all $n, m \in \mathbb{N}$ ([EH06]).

Prisms are graphs of the form $C_m \times P_n$. Thse can be viewed as grids on cylinders. In 1977 Bodendiek Schumacher, and Wegner (as referenced in [Gal10]) proved that $C_m \times P_2$ is graceful when $m \equiv 0 (\bmod 4)$. According to Gallian's survey, Gangopadhyay and Rao Hebbare did the case that m is even about the same time. In 1979, Frucht (as referenced by [Gal10]) stated without rpoof that he had proved that all $C_m \times P_2$ were graceful. A comoplete proof and some other related results were given in 1988 by Gallian and Frucht.

In 1992 Jungreis and Reid proved that $C_m \times P_n$ are graceful when m and n are even or when $m \equiv 0 (\bmod 4)$ (ref. in [Gal10]). Later on it was shown by Skiena (ref. in [Gal10]) that $C_m \times P_n$ is graceful for all n when m is even and for all n with $3 \leqslant n \leqslant 12$ when m is odd.

Torus grids are graphs of the form $C_m \times C_n$ $(m > 2, n > 2)$. In 1992 (ref. in [Gal10]) Jungreis and Reid showed that $C_m \times C_n$ is graceful when $m \equiv 0 (\bmod 4)$ ad n is even. Gallian suggests that a complete solution to both the graceful and harmonious torus grid problems will most likely involve a large number of cases.

More classes of graceful graphs can be found in Gallian's survey [Gal10] and in Edwards and Lea's survey [EH06]. In fact, apart from more detailed descriptions, Gallian's survey contains a 7-page long table summarizing all results on graceful graphs that have been discovered so far and it is a great reference for the interested reader.

8. Graceful Graphs and Variations on Graceful Labelings

We are now going to describe a few different types of labelings (of both trees and general graphs) and we will note on how they are related to graceful labelings.

8.1. Balanced Labelings.

A balanced labeling, aslo known as an α-labeling or an interlaced labeling, as defined above, is a graceful labeling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leqslant k < f(y)$ or $f(y) \leqslant k < f(x)$. Note that a graph with an α-labeling is bipartite and, thus, cannot contain a cycle of odd length. Wu (ref. in [Gal10]) has shown that a necessary condition ofr a bipartite graph with n edges and degree sequence d_1, d_2, ..., d_p to have an α-labeling is that $\gcd(d_1, d_2, \ldots, d_p, n)$ divides $\frac{n(n-1)}{2}$.

Kotzig (ref in [Gal10]) has shown that almost all trees have α-labelings. However, there do exist trees that do not have α-labelings as we showed before.

An n-cycle has an α-labeling if and only if $n \equiv 0 (\bmod 4)$ whereas P_n always has an α-labeling. Other graphs that have α-labelings include caterpillars and many others [Gal10].

8.2. Harmonious Labelings.

Definition 80. *A connected graph G with n vertices and m edges such that $m \geqslant n$ is* harmonious *if there exists a labelin φ: $V(G) \rightarrow \mathbb{Z}_m$ of the vertices of G with distinct elements from $\mathbb{Z}_m$ such that each edge $uv \in E(G)$ is labeled with $\varphi(u) + \varphi(v)$ and the resulting edge labels are distinct. Such a labeling is a* harmonious labeling.

In the case when G is a tree, φ is a harmonious labeling *if exactly two of the vertices of G are labeled with the same label and everything else in the definition is the same.*

Graham and Sloane [GS80] studied the problem of which graphs are harmonious and which aren't in 1980. Their first result is the following

Theorem 81. *The complete graph K_n is harmonious if and only if $n \leqslant 4$.*

In fact, the authors looked at many classes of graphs and checked whether they were harmoniousor graceful and came up with the following table.

Graph	Harmonious?	Graceful?
Caterpillars	H	G
Trees	Conjectured to be H, true for $\leqq 9$ nodes	Conjectured to be G, true for $\leqq 16$ nodes
Cycle C_{4m}	Not H	G
Cycle C_{4m+1}	H	Not G
Cycle C_{4m+2}	Not H	Not G
Cycle C_{4m+3}	H	G
Ladder L_n	H iff $n \geqq 3$	G
Friendship graph F_n	H iff $n \not\equiv 2$ (mod 4)	G iff $n \equiv 0$ or 1 (mod 4)
Fan f_n	H	G
Wheel W_n	H	G
Complete graph K_n	H iff $n \leqq 4$	G iff $n \leqq 4$
Complete bipartite $K_{m,n}$	H iff m or $n = 1$	G
Small graphs	All with $\leqq 5$ nodes are H except for 5	All with $\leqq 5$ nodes are G except for 3
Petersen	H	G
Cube, octahedron	Not H	G
Icosahedron	H	G
Dodecahedron	H	G
Most graphs	Not H	Not G

Figure 18. Harmonious and graceful graphs table [GS80].

The authors showed that almost all graphs are not harmonious and not graceful. The following is known as the **Graham - Sloane Conjecture**:

Conjecture 82. *Every nontrivial tree is harmonious.*

Alfred and McKay showed (as referenced in [EH06]) that all trees on at most 27 vertices are harmonious.

8.3. *k*-graceful Labelings.

The notion of k-graceful graphs was introduced independently by Slater and Maheo and Thuillier in 1982 (ref. in [Gal10]). A graph G with q edges is k-*graceful* if there is a labeling $f: G \rightarrow \{1, 2, ..., q+k\}$ such that the set of edge labels induced by the absolute value of the difference of the adjacent vertices is $\{k, k+1, ., q+k-1\}$. So, 1-graceful is the same as the concept of graceful. It has been shown that any graph that has an α-labeling is k-graceful for all k. Various special types of graphs have been shown to be k-graceful for various values of k [Gal10].

8.4. Edge-Graceful Labelings. [EH06]

Let G be a graph with p vertices and q edges and let $l: V(G) \to \mathbb{Z}$ and $l^*: E(G) \to \mathbb{Z}$.

Definition 83. *The graph G is* edge-graceful *if there is a function pair (l, l^*) such that l is onto $\{0, 1, ..., p-1\}$ and l^** is onto *{1, 2, .., q}, and for $u \in V(G)$ we have that $l(u) = \sum_{uv \in E(G)} l^*(uv) \pmod p$.*

In other words, for each vertex, its vertex label is equal to the sum of the labels of the edges adjacent to it modulo p. The following is known as **Lee's Conjecture** (as referenced in [EH06]):

Conjecture 84. *All trees with an odd number of edges are edge-graceful.*

Simoson showed (as referenced in [EH06]) that all three-legged spiders and all four-legged spiders with an odd number of edges are edge-graceful. He also showed how to inductively attach new legs (which are long enough) to spiders so that the new spider is edge-graceful.

Now, let $P = \{\pm 1, \pm 2, ..., \pm \lfloor \frac{p}{2} \rfloor\}$ if p is even and $P = \{0, \pm 1, \pm 2, ..., \pm \lfloor \frac{p}{2} \rfloor\}$ if p is odd. Let $Q = \{\pm 1, \pm 2, ..., \pm \lfloor \frac{q}{2} \rfloor\}$ if q is even and $Q = \{0, \pm 1, \pm 2, ..., \pm \lfloor \frac{q}{2} \rfloor\}$ if q is odd.

Definition 85. *The graph G is* super-edge-graceful *if there ecists a function pair (l, l^*) such that l is onto P and l^* is onto Q and $l(u) = \sum_{uv \in E(G)} l^*(uv)$ for each $u \in V(G)$.*

It has been shown (as referenced in [EH06]) that if G is a tree with an odd number of edges and is super-edge-graceful, then G is also edge-graceful. Simson ([EH06]) used this to construct new types of edge-graceful trees. Note that there exist super-edge-graceful graphs that are not edge-graceful. Mitchem and Simoson (as referenced in [EH06]) also used super-edge-graceful trees to construct various new edge-graceful graphs. Their construction also gace a proof that all spiders having legs of the same length are edge-graceful.

8.5. Cordial and Ordered Graceful Labelings.

For a labeling f of a graph G with induced edge labeling $\overline{f}$, denote by $v_f(i)$ (respectively $e_f(i)$) the number of vertices (respectively edges) with the label $i \in \{0, 1, 2, ..., k\}$, $k \leqslant |E(G)|$. The labeling f is called $(k+1)$-equitable if $|v_f(i) - v_f(j)| \leqslant 1$ and $|e_f(i) - e_f(j)| \leqslant 1$ for $i \neq j$, $i, j = 0, 1, ..., k$. For $k = 1$, a 2-equitable labeling is called a *cordial labeling*. For $k = |E(G)|$, the $(|E(G)| + 1)$-equitable labeling is the same as a graceful labeling of G. Cahit proved that (as referenced in [EH06]) that ever tree is cordial. However, all attempts at generalizing this result to equitable labelings have failed.

A graceful labeling of a tree is *ordered graceful* if for every three vertices x, y, and z of the tree T such that xy and $yz \in E(T)$, we have that $(f(x) < f(y)$ and $f(y) > f(z))$ or $(f(x) > f(y)$ and $f(y) < f(z))$. In other words, for each vertex either all its neighbors have labels greater than its label or their labels are smaller than its label. If a graph has a bipartite labeling, then this labeling is also ordered graceful. As referenced in [EH06]

Cahit gives and ordered graceful labeling of spiders with all legs of length 2 and with up to 23 vertices.

8.6. Bigraceful Labelings. (as appears in [EH06])

Let $H = H(A, B)$ be a bipartite graph wtih n vertices and partite sets A and B. A pair of injective mapse $f_A: A \to \{0, 1, ..., n-1\}$ and $f_B: B \to \{0, 1, ..., n-1\}$ is a *bigraceful labeling* of H if the induced labeling on the edges f_e: $E(H) \to \{0, 1, ..., n-1\}$ defined by $f_e(xy) = f_B(y) - f_A(x)$ is also injective. A bigraceful labeling of $H(A, B)$ is *consecutive* if $f(A) = \{0, 1, ..., |A| - 1\}$ and $f(B) = \{|A\lfloor ..., n-1\}$. Then, the labelinf $\overline{f}: V(H) \to \{0, 1, ..., n\}$ defined by $\overline{f}(A) = f(A)$ and $\overline{f}(B) = f(B) + 1$ is a bipartite labeling. As referenced in [EH06] Llado and Lopez used properties of bigraceful labelings to construct bigraceful graphs. These constructions yield graphs that decompose $K_{n,n}$ and, thus, give results towards verifying Haggkvist's Conjecture.

8.7. Odd-Graceful Labelings.

A graph with q edges is *odd-graceful* if there is an injection $f: V(G) \to \{0, 1 ..., 2q-1\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, ..., 2q-1\}$. Gnanajothi (ref. in [Gal10]) proed that the class of odd $-$ graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labeling has an odd-graceful labeling and every graph with an odd cycle is not odd-graceful. She also proved that the following graphs are also odd graceful: P_n; C_n if and only if n is even; $K_{m,n}$; and many others.

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