Assessing a 2-System Model of Quasi-Hyperbolic Temporal Discounting

Pranav Nanda

People make intertemporal choices, decisions regarding costs and benefits at different points in time, as a matter of routine. Such choices range from mundane decisions, like when to set one's alarm clock, to critical decisions, like whether or not to invest in retirement plans or health insurance. Beyond individuals' decisions, nations must also make intertemporal choices when charting courses of action that will affect both the present population and future generations. The centrality of such choices in determining behavior begs the question: What principles guide peoples intertemporal choices? This question is of deep interest and importance in a number of fields, including economics and cognitive psychology. As a rule, people tend to value immediate outcomes more than future ones and we may begin to fashion an answer beyond that by describing axiomatically the process of temporal discounting.

1 Axiomatic Derivation

By employing simple and intuitive axioms, we may develop a powerful framework in which to analyze temporal discounting. Using these axioms, we will also be able to derive a fairly specific representation for discounting functions. To begin, we let X be a metric space and define it as an undated outcome space. It represents the outcomes of people's choices independent of the times at which they occur. We discretize the time schedules on which people receive these outcomes and so we define the set of all relevant time periods times as T with:

$$T \in \mathcal{T} = \{\{0, 1, \ldots, t\} : t \in \mathbb{N}^0\} \cup \mathbb{N}^0$$

The preferences of an agent exist in the product space $X \times T$. A discrete metric is applied to T whereas a product metric is applied to $X \times T$. $(x, t) \in X \times T$ may be interpreted as the outcome x being received at time period t. When we refer to a preference relationship on $X \times T$, we mean a binary relationship $\precsim$ consisting of a symmetric ($\sim$) and asymmetric ($\prec$) portion. Given any $t \in T$, $\precsim_t$ refers to the t^{th} time projection of the preference relationship on X. For $x, y \in X, x \precsim_t y \Leftrightarrow (x, t) \precsim (y, t)$.

Definition A preference relationship $\precsim$ is called a time preference if:
i) $\precsim$ is complete and continuous
ii) $\precsim_0$ is complete, continuous, and transitive
iii) $\precsim_i = \precsim_j \; \forall i, j \in \mathbb{N}^0$

Most notably, we do not impose the restriction of transitivity on $\precsim$ in order to account for empirical phenomena such as time preference reversals. For instance, we could reasonably envision someone having the following preference relationship where X refers to dollar values:

$$(100, 0) \succ (250, 2 \text{ weeks}) \succ (200, 1 \text{ week}) \succ (100, 0)$$

Here, the agent believes that the delay of 2 weeks exceeds the value of \$150 but believes \$50 to be worth the price of a week delay. However, we only want such intransitivities to arise from variations in time, and so $\precsim_0$ is indeed taken to be transitive. Moreover, condition iii) is taken in order to ensure that agents' outcome preferences are unchanging over time. We will deal in this paper specifically with one-dimensional time preferences, in which $\precsim_i$ is a linear order. With the foundation of these definitions, we may present the axiomatic basis for time preference analysis. We will use six axioms in order to obtain a general representation for time preferences. The first three axioms (A1-A3) can be considered fairly intuitive properties of temporal discounting while the second three (B1-B3) are more technical requirements.

(A1): For any $x \in X$ and $s, t \in T$, $\exists y, z \in X$ such that $(z, s) \succsim (x, t) \succsim (y, s)$

(A2): For any $x, y, z \in X$ and $r, s, t \in T$, if $t \leq r$ and $x \leq z$:

$$(x, r) \succsim (y, s) \Rightarrow (z, t) \succsim (y, s)$$

(A3): For any $x, y, z \in X$ and $s, t \in T$:

$$(x, t) \succsim (y, s) \succsim (z, s) \text{ or } (x, t) \succsim (y, t) \succsim (z, s) \Rightarrow (x, t) \succsim (z, s)$$

We may interpret A1 as effectively saying that any difference in time can be compensated by a difference in reward. A2 suggests that the time preference $\succsim$ is increasing in X but decreasing in T. A3 provides a restricted condition of transitivity. In both A2 and A3, the $\succsim$ preferences may be replaced by strict $\succ$ preferences. Before introducing the technically motivated axioms, we must introduce some new terminology. First, given a binary relation $\approx$ on $X \times T$, we call $x_2 \in X$ $\approx$-reachable from x_1 through $(s_i, t_i)_{i=1}^n$ with n 1 if $\exists y_1, \ldots, y_{n-1} \in X$ so that:

$$(x_1, t_1) \approx (y_1, s_1), (y_1, t_2) \approx (y_2, s_2), \ldots, (y_{n-1}, t_n) \approx (x_2, s_n)$$

x_2 is $\approx$-reachable from x_1 if it is $\approx$-reachable through some $(s_i, t_i)_{i=1}^n \in (T \times T)^n$. The set of all $y \in X$ that is $\approx$-reachable from x is referred to as $O^{\approx}(x)$, the $\approx$-orbit of x. With that definition, we may present the remaining three axioms:

(B1): For any $x_1, x_2, y_1, y_2 \in X$ and $s_1, s_2, t_1, t_2 \in T$:

$$(x_1, s_1) \sim (y_1, t_1), (x_2, s_1) \sim (y_2, t_1), and (y_1, s_2) \sim (y_2, t_2) \Rightarrow (x_1, s_2) \sim (x_2, t_2)$$

(B2): For n 1, if $x \in X$ is $\sim$-reachable from x through $(s_i, t_i)_{i=1}^n$, then any $z \in X$ is $\sim$-reachable from z through $(s_i, t_i)_{i=1}^n$.

(B3): There exists some $x \in X$ where $O^{\sim}(x)$ is either somewhere dense or has at least one isolated point.

B1 is a separability condition ensuring the consistency of through-time rankings. Although B1 does not define a condition that is intuitively seen in decision making and although it does not define an unexceptionable property, it allows for a rich class of models of time preferences that includes the vast majority of ones routinely employed. B2, which may also be interpreted as a weak separability condition, similarly is satisfied by most practiced discounting models. B3 helps us avoid time preference systems that engender odd and pathologic mathematical features. It instead abets the maintenance of regularity in the discounting system.

As demonstrated by Masatlioglu and Ok (2003), these axioms can be used to characterize the structure of time preferences.

Proposition 1.1 *Given X and T as above with $\precsim$ being a binary relationship on $X \times T$, $\precsim$ is a one-dimensinoal time preference satisfying A1-A3 and B1-B3 if and only if there exist $U : X \to \mathbb{R}$ and $\varphi : T^2 \to \mathbb{R}$ so that:*

$$(x,t)\{\succ \ or \sim\}(y,s) \Leftrightarrow U(x)\{> \ or =\}U(y)+\varphi(s,t)$$

$\forall (x,t),(y,s) \in X \times T$ and:
i) U is a homeomorphism
ii) $\varphi(\cdot,t)$ is decreasing and $\varphi(s,\cdot)$ is increasing.
iii) $\varphi(s,t)+\varphi(t,s)=0 \ \forall s,t \in T$

Classical treatments of temporal discounting often involve stationary time preferences, in which time only affects the comparison of two options (x,s) and (y,t) through the difference $t-s$ between the receival times.

Definition A time preference $\precsim$ on $X \times T$ is considered stationary if

$$(x,s) \prec (y,t) \Leftrightarrow (x,s+d) \prec (y,t+d)$$

By assuming stationarity, we may develop a more precise understanding of how temporal discounting out to be modelled.

Corollary 1.2 *If $\precsim$ is a binary relationship on $X \times T$, then $\precsim$ is a stationary one-dimensional time preference satisfying A1-A3 and B1-B3 if and only if there exists a homeomorphism $U : X \to \mathbb{R}$ and a decreasing odd function $\eta : T-T \to \mathbb{R}$ where:*

$$(x,t)\{\succ \ or \sim\}(y,s) \Leftrightarrow U(x)\{> \ or =\}U(y)+\eta(s-t)$$

Proof The "if" direction of the claim follows directly from Proposition 1.1 as we may simply take $\varphi(s,t)=\eta(s-t)$. By this definition, $\varphi(s,t)$ is clearly decreasing in s and increasing in t as $\eta(s-t)$ is decreasing. We also know that $\varphi(s,t)=-\varphi(t,s)$ because $\eta(s-t)$ is odd so that $\eta(s-t)=-\eta(t-s)$. Clearly, $(x,s)\{\succ \text{ or } \sim\}(y,t) \Leftrightarrow U(x)\{> \text{ or } =\}U(y)+\eta(s-t)=U(y)+\varphi(s,t)$

For the "only if" part, we may let $\eta(k)=\varphi(k,0)=-\varphi(0,k)$ for $k \geq 0$ and $\eta(k)=\varphi(0,-k)=-\varphi(-k,0)$ otherwise. Because of part ii) of Proposition 1.1 we know that $\eta(k)$ is decreasing, and part iii) informs us that that $\eta(k)=-\eta(-k)$. Finally, we also know that $(x,s)\{\succ \text{ or } \sim\}(y,t) \Leftrightarrow (x,s-t)\{\succ \text{ or } \}(y,0) \Leftrightarrow U(x)\{> \text{ or } =\}U(y)+\varphi(s-t,0)=U(y)+\eta(s-t)$ □

Time preferences are transitive if $\eta(r-t)=\eta(r-s)+\eta(s-t)\forall r,s,t \in T$.

Corollary 1.3 *If* $\precsim$ *is a binary relationship on* $X \times \mathbb{Z}_+$*, then precsim is a stationary and transitive one-dimensional time preferences satisfying A1-A3 and B1-B3 if and only if there exists a homeomorphism* $u : X \to \mathbb{R}^+ \cup \{0\}$ *with* $\delta \in (0,1)$ *so that,* $\forall(x,s),(y,t) \in X \times \mathbb{Z}_+$*:*

$$(x,s) \precsim (y,t) \Leftrightarrow \delta^s u(x) \geq \delta^t(y)$$

Proof For the "if" part of the proof, let $U = ln(u)$ and let $\eta(k) = ln(\delta^k)$. Then, we see that $(x,s) \precsim (y,s) \Leftrightarrow \delta^s u(x) \geq \delta^s(y) \Leftrightarrow ln(u(x)) \geq ln(u(y)) + ln(\delta^{s-t}) \Leftrightarrow U(x) \geq U(y) + \eta(s-t)$

For the "only if" part of the proof, we know that $\eta(t) = \alpha t$ where $\alpha = \eta(1) \leq 0$. As a result, we know that $(x,s) \precsim (y,t) \Leftrightarrow U(x) \geq U(y) + \alpha(s-t)$. If we then let $u = e^U$ and let $\delta = e^\alpha$, we find that $(x,s) \precsim (y,t) \Leftrightarrow \delta^s u(x) \geq \delta^t(y)$. □

Axiomatic derivations of this form have often been taken to bolster the legitimacy of Samuelson's exponential discounted utility (DU) model which contends that the value of sequences of rewards $\{c_i\}_{i=t}^T$ can be represented and compared using an intertemporal utility function:

$$U^t(c_t, \ldots, c_T) = \Sigma_{k=0}^{T-t} D(k) u(c_{k+t})$$

Here, $D(k) = \frac{1}{(1+\rho)^k} = \delta^k$ where ρ is dubbed the decision maker's pure rate of time preference and $u(c_i)$ is interpreted as cardinal instantaneous utility. However, rather than promote this DU model, these derivations instead serve to formulate a tractable mathematical framework wherein to analyze a number of experimental findings. In fact, a simple and seemingly natural exponential DU model, in which u behaves regularly with $u(c_i) = c_i$, runs counter to a number of experimental findings, most notably declining discount rates and dynamic inconsistency. Other empirical anomalies exist as well, including the divergent treatment of gains and losses and the shallower discounting of higher values.

2 Empirical Anomalies

To elucidate some experimental findings, we will introduce some new terminology. First, we will let $\psi_s(x,t)$ be the reward at time s that an agent would be require to be indifferent to (x,t). In other words:

$$(\psi_s(x,t), s) \sim (x,t)$$

This function ψ_s gives us an easy way to compare different choices within the frame of a single time. Given a fixed s, by convention taken to be 0 unless denoted otherwise, we can then determine a discount function for any reward x:

$$\phi^x(t) = \frac{\psi_s(x,t)}{x}$$

For the simple exponential DU model described above, we may extend any sequence of rewards $\{c_i\}_{i=t}^T$ to a sequence $\{c_i\}_{i=0}^T$ simply by setting $c_i = 0$ for $i = 0, \dots, t-1$. Then, by setting $U^0(c_t) = U^0(c'_0) = c'_0 = \psi_0(c_t, t)$, we find that $\psi_0(c_t, t) = \delta^t c_t$ so that $\phi^x(t) = \delta^t$. Notably, $\phi^x(t)$ in this case is independent of x. With this terminology in hand, we may formally introduce some anomalies found empirically in temporal discounting.

2.1 Declining discount rates

People tend to discount less over longer time horizons. In other words, they exhibit decreasing impatience so that they will discount rewards over intervals of time more heavily when those intervals are presented as being more immediate (Figure 1). More precisely, a discounting model demonstrates declining discount rates if:

$$\frac{\phi^x(t+1)}{\phi^x(t)} \text{ is non-constant and monotone increasing in } t$$

For the simple exponential DU model, $\frac{\phi^x(t+1)}{\phi^x(t)} = \delta$, and so it does not predict this phenomenon.

2.2 Dynamic Inconsistency

A reversal in choice preferences can be induced by simply introducing a constant shift in time to both choices. In other words, it is possible for $(X,t) \succ (X+k, t+c)$ and $(X, t+d) \prec (X+k, t+c+d)$ simultaneously. This phenomenon is canonically exemplified by the case where an individual prefers \$100 now to \$110 tomorrow, but prefers \$110 in 366 days to \$100 in 365 days. The simple exponential DU model cannot account for this preference reversal as $(X,t) \succ (X+k, t+c) \Leftrightarrow (X, t+d) \succ$

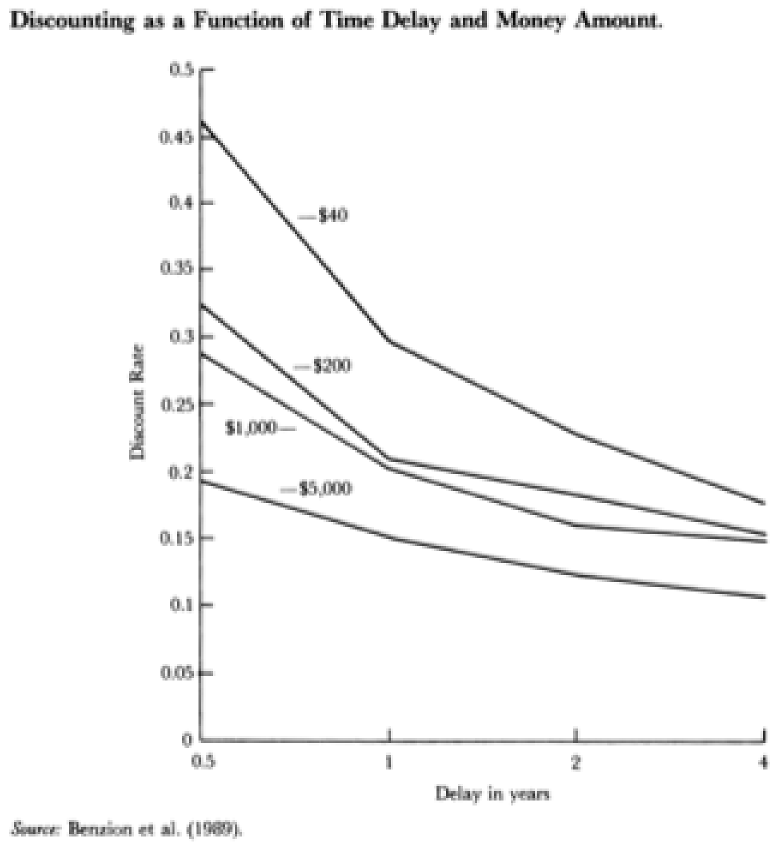

Figure 1

$(X+k, t+c+d)$ under this model because $X\delta^t > (X+k)\delta^{t+c} \Leftrightarrow X\delta^{t+d} > (X+k)\delta^{t+c+d}$ simply by multiplying through by a factor of δ^d. Instead, the phenomenon of declining discount rates may be invoked to explain this reversal in preferences as it causes the interval $[t, t+c]$ to be discounted more heavily upon than the interval $[t+d, t+c+d]$.

2.3 Magnitude Effect

Beyond just observing declining discount rates, Benzion et al. (1989) also noted that discount rates declined as the magnitude of money offered increased (Figure 1). This finding is fairly universal, and larger gains are generally seen to be discounted less than are smaller ones. This result can be formalized by letting $\phi^x(t_0)$ be non-constant and monotone increasing for $x\,0$ with any fixed t_0. However, under the simple exponential DU model, the discount rate is solely t dependent so that $phi^x(t) = \delta^t = \phi^y(t)$, it cannot account for the magnitude effect.

2.4 Sign Effect

Gains have consistently been found to be discounted at a steeper rate than are losses. Formally, this effect can be described by $\phi^x(t) < \phi^{-x}(t)$ for all $x, t \in \mathbb{N}$. Again, because the simple exponential DU model suggests that $\phi^x(t) = \delta^t = \phi^y(t)$ as above, it does not justify the sign effect.

Given these experimental problems for the simple exponential DU model and because there is no normative rationale behind its special discount function, alternative models accounting for these phenomena have been developed and proposed, including the generalized hyperbolic model. As proposed by Loewenstein and Prelec (1992), the generalized hyperbolic model has a discount function:

$$\phi^x(t) = (1 + gt)^{-h/g} \text{ with } h, g \in \mathbb{R}^+$$

This model is quite flexible as it encompasses various previously proposed and expounded hyperbolic models. For instance, if we take $g = 1$, we have $\phi^x(t) = (1+t)^{-h}$ as in Harvey (1986). For $h = g$, the model becomes $\phi^x(t) = (1+gt)^{-1}$ as proposed by Mazur (1987). This model is also closely tied to the exponential form as if we take $g \to 0$, we have $\phi^x(t) \to e^{-h}$. This generalized hyperbolic model provides a very close fit to experimental data and this goodness of fit has led to common application.

Another proposed alternative discounting function is the quasi-hyperbolic discount function, first leveraged in the study of intergenerational altruism. This model is defined for $t > 0$ by the discount function:

$$\phi^x(t) = \beta_x \gamma^t \text{ where } \beta_x, \gamma \in (0, 1) \text{ and } \phi^x(0) = 1$$

This model can account for many of the experimental anomalies challenging the simple exponential DU model, and as such, it too has been found to be quite tractable in describing experimental evidence.

Proposition 2.1 *The quasi-hyperbolic discount function may exhibit the phenomena of:*
a.) Declining discounting
b.) Dynamic inconsistency
c.) The magnitude effect
d.) The sign effect

Proof We can easily determine that this discount function will enable these four phenomena.

a.) $\phi^x(1)/\phi^x(0) = \beta_x\gamma$ while $\dfrac{\phi^x(t+1)}{\phi^x(t)} = \gamma$ for $t > 0$. Because $\beta_x < 1$, $\phi^x(t)$ is non-constant and monotone increasing for $t \geq 0$.

b.) Because this model demonstrates declining discounting, we may induce preference reversals. Since, given $y \in X, d, t \in \mathbb{N}, \beta_x < 1$ strictly, we know that the open interval $(y\beta_y\gamma^t, \dfrac{y\beta_y\gamma^t}{\beta_x})$ is non-empty. Taking x from this interval, we find that $\psi_0(x, 0) > \psi_0(y, t)$ as $x > y\beta_y\gamma^t$ but $\psi_0(x, d) < \psi_0(y, d+t)$ as $x < \dfrac{y\beta_y\gamma^t}{\beta_x}$ and we may multiply through by $\beta_x\gamma^d$. Thus, we have $(x, 0) > (y, t)$ but $(x, d) < (y, d+t)$.

c.) For $x \geq 0$, by setting $beta_x$ as non-constant and monotone increasing in x, we find that $\phi^x(t) = \beta_x\gamma^t$ is also non-constant and monotone increasing in x. The special case of $t = 0$ does not exhibit the magnitude effect, and so it need not be addressed.

d.) Again, we may rig β_x so that this effect is accounted for. We may do so simply by setting $\beta_{-x} > \beta_x$. □

3 Normative Basis for a 2-System Model

However, these models are not supported by any normative rationale. Literature suggests that the act of temporal discounting leverages two separate and distinguishable neural systems. One system, referred to as the β system, favors more immediate rewards and thereby contributes towards impulsivity. Conversely, another system, the δ system, engages in more patient analysis weighing future and even distant rewards more evenly. These two systems arise in disparate regions of the brain, as the β system appears limbic whereas the δ system is cortical. This neural data is suggestive of a two-system multiple-self model, which can naturally be represented by splicing together two quasi-hyperbolic functions so that $\phi^x(t) = (1-\omega)\beta^t + \omega\delta^t$ for $\beta, \delta, \omega \in (0, 1)$ where δ represents of the δ system, β accounts for the β system, and ω varies contextually while dictating the balance between the patient and impulsive systems. A breadth of experimental evidence, most notably in cross-species comparisons and lesion data, implicates the prefrontal cortex in the delta system of temporal discounting. Non-human animals, whose prefrontal cortices differ dramatically in size from those of humans, discount much more severely, preferring immediate rewards far more than delayed ones. When they do choose delayed rewards, there decisions seem to be the product only of stereotyped and instinctive behaviors. Similarly, peo-

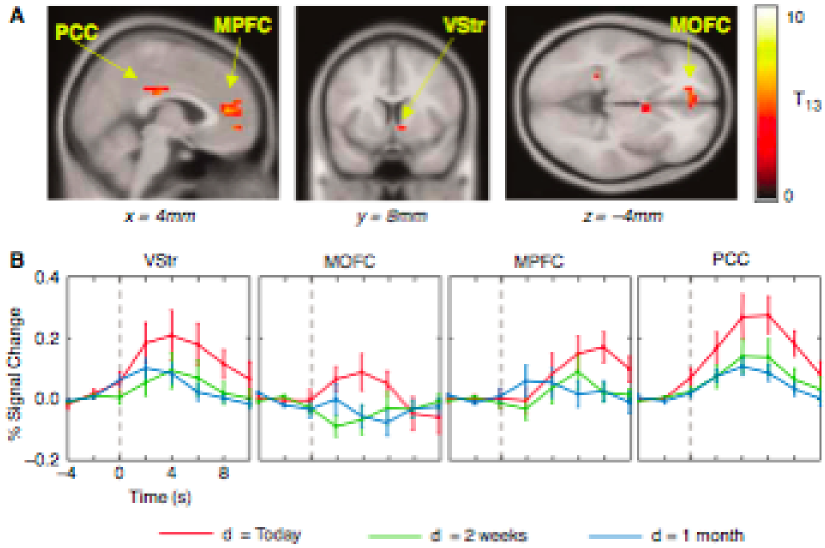

Figure 2: β regions which loaded on immediacy. Source: McClure et al. (2004).

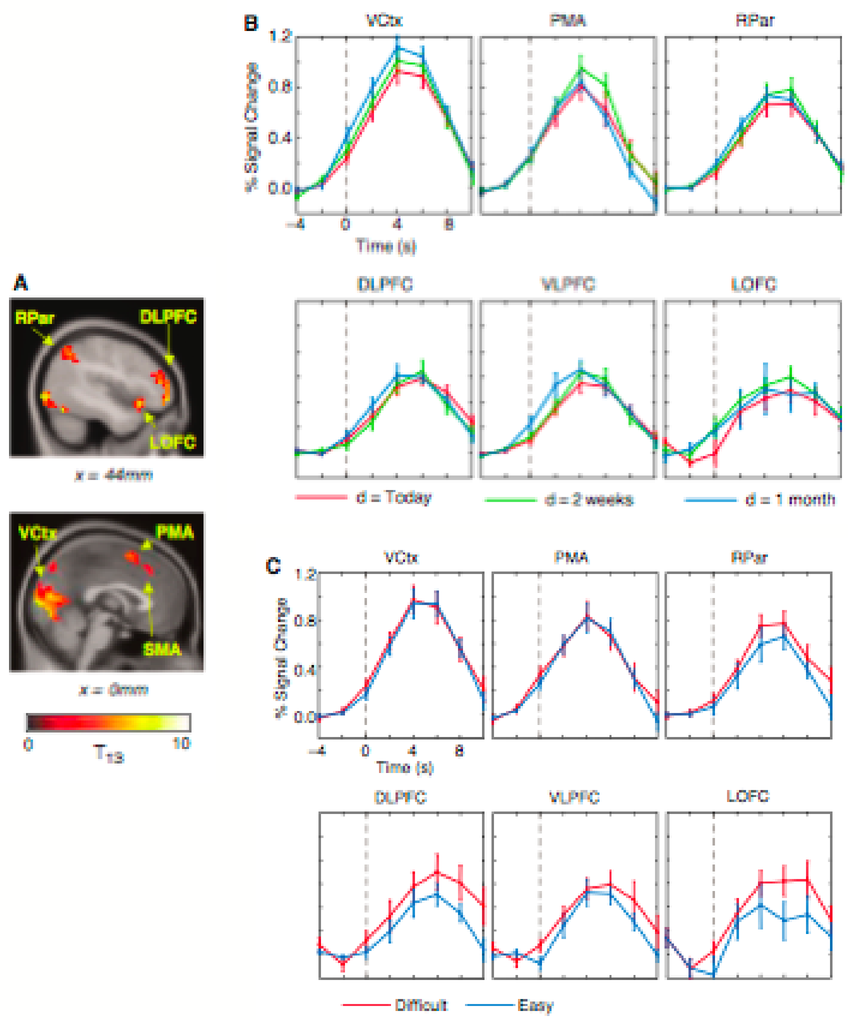

Figure 3: δ candidate regions activating independent of d but more for difficult than easy decisions. Source: McClure et al. (2004).

ple with prefrontal cortex lesions show diminished levels of patience. Most famously, after a tamping iron destroyed the left frontal lobe of his brain, Phineas Gage lost his ability to plan for the future or delay gratification, among various other distortions of his personality. Prefrontal damage has been shown to produce these results in a number of cases, indicating that it plays a critical role in evaluating future rewards.

Research conducted by Professor McClure (2004) firmly establishes the dissociation between this cortical system and a more limbic system. Subjects underwent a battery of choices between smaller-sooner and larger-later monetary rewards. Concurrently, their brains were scanned using functional magnetic resonance imaging technology. The smaller-sooner reward was presented as either immediate ($d = 0$) or after a delay period ($d > 0$) and the regions of the brain differentially activated in the $d = 0$ condition were labeled β regions as they demonstrated a preference for immediate rewards. These β systems were found to be limbic structures associated with paralimbic cortical projections (Figure 2). Areas of the brain activated equally in the $d = 0$ and $d > 0$ conditions were labeled δ system candidate regions. The relationship between activity and task difficulty was further analyzed because some of these δ system candidates were postulated to account for nonspecific task performance, like simply viewing the choice, rather than targeted decision-making. The areas activated by difficult choices were labeled as δ system regions, and were found to be located in the prefrontal and parietal cortices (Figure 3). Difficult choices were defined as those in which the two rewards were within 5% and 25%, a subset which was found to elicit significantly higher response times and in which the sooner and later rewards were selected fairly equally. These cortical δ regions were found to be more activated than the limbic β systems when subjects selected larger-later rewards and, on the other hand, when subjects selected smaller-sooner rewards, the activation of the two systems was similar, with a trend towards greater β system activity (Figure 4). This evidence, especially the divergence in system activity during the selection of different options, points unwaveringly towards a dissociation between an impulsive limbic β system and a patient cortical δ system. Although this division is not reflected in more traditional models for temporal discounting, like the generalized hyperbolic model and standard quasi-hyperbolic model, it naturally emerges in the two-system model with $\phi^x(t) = (1 - \omega)\beta^t + \omega\delta^t$.

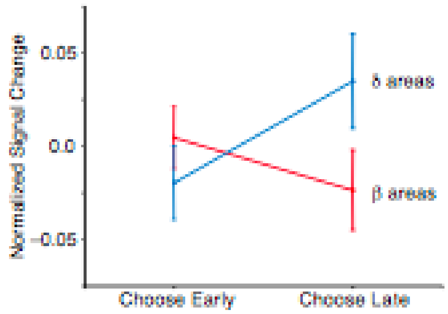

Figure 4: Differences in β and δ region activity based on chosen reward. Source: McClure et al. (2004).

4 Models' Fit to Data

Although normative evidence supports the 2-system quasi-hyperbolic model, in order to gain greater acceptance, recognition, and application, it must also be descriptive of data observed in temporal discounting experiments. In one such study subjects were asked to choose between two options, a smaller reward presented sooner or a larger reward presented later. The thrust of this study was to shed light on the magnitude effect, and so its 15 subjects were asked for rewards ranging from 0-60 or 3000-6000. In order to assess the fit of the 2-system quasi-hyperbolic model in comparison to the generalized hyperbolic model, a hard classification error minimization process was undertaken employing a 15-fold cross-validation protocol. Essentially, the models were trained for the 0-60 or 3000-6000 condition on 14 subjects' choices using an optimization procedure that simply tuned the parameters so as to minimize the number of false classifications of subjects' choices. The resulting parameter values were then tested on the remaining subject's data and a confusion matrix was used to track how the predicted choices matched the actual choices. This procedure was conducted 15 times for both the 0-60 and 3000-6000 conditions so that every subject's data comprise the test set once and are part of the training set 14 times. The results reflect the average proportions of each type of classifications. This cross-validation optimization process has two primary results. First, the 2-system model parameters derived from this procedure match what we would have hypothesized. Across the 0-60 and 3000-6000 conditions, the β and δ values are fairly consistent, only varying by 7% and 3% respectively. However, the ω value increased by 25% in the 3000-6000 condition, as we would anticipate due to the magnitude effect. The fact that this discounting function behaves as normative considerations would indicate lends credence

2 System Model 0-60			
		Actual Choices	
		S-S	L-L
Predicted Choices	S-S	0.500	0.120
	L-L	0.131	0.248
Total Errors		0.252	

Generalized Hyperbolic Model 0-60			
		Actual Choices	
		S-S	L-L
Predicted Choices	S-S	0.497	0.123
	L-L	0.134	0.245
Total Errors		0.258	

2 System Model 3000-6000			
		Actual Choices	
		S-S	L-L
Predicted Choices	S-S	0.304	0.120
	L-L	0.162	0.413
Total Errors		0.283	

Generalized Hyperbolic Model 3000-6000			
		Actual Choices	
		S-S	L-L
Predicted Choices	S-S	0.299	0.124
	L-L	0.167	0.410
Total Errors		0.291	

Figure 5: Average results of cross-validation trials where S-S refers to smaller-sooner rewards and L-L refers to larger-later rewards

		ω	δ	β
2-System Model	0-60	0.791	0.966	0.625
	3000-6000	0.990	0.998	0.668
		g	h	
Generalized Hyperbolic	0-60	0.516	0.148	
	3000-6000	0.619	0.009	

Figure 6: Optimized parameter values

to its application as a natural model.

Second, the 2-system quasi-hyperbolic and the generalized hyperbolic models perform equally well. The proportions of errors using the quasi-hyperbolic and generalized hyperbolic models are within 3% of one another in both the 0-60 and 3000-6000 conditions. Moreover, even the proportions of each cell of the confusion matrices differ for the quasi-hyperbolic and generalized hyperbolic models by less than 4% in both the 0-60 and 3000-6000 conditions, indicating that the two models produce nearly identical results.

The 2-system model's conformity to anticipated behavior and its virtual matching of the generalized hyperbolic model's performance justifies further exploration into its viability as a temporal discounting model.

5 Concluding Thoughts

The 2-system quasi-hyperbolic model has been shown within an axiomatically defined framework of temporal discounting to fit both normative and descriptive conditions while also satisfying a number of experimental anomalies. It therefore warrants

However, in the descriptive analysis, the utilized hard-classification system treats cases in which two choices attributed values are nearly equal and cases in which they diverge vastly as equivalent. But this uniform treatment may not evaluate models appropriately, as people will choose between (x,s) and (y,t) more inconsistently when $\psi_0(x, s)$ is nearer to $\psi_0(y, t)$. As such, future exploration should implement a probability function $P(\psi_0(x, s), \psi_0(y, t))$ describing the probability that an individual selects (x, s) over (y, t), thereby representing this phenomenon of inconsistency. The models could then be assessed using a maximum likelihood estimate technique, in which parameters are selected in order to maximize:

$$\Pi P(\psi_0(x_i, s_i), \psi_0(y_i, t_i))^{\gamma_i}(1 - P(\psi_0(x_i, s_i), \psi_0(y_i, t_i)))^{1-\gamma_i}$$

Here, (x_i, s_i) and (y_i, t_i) represents the battery of binary choices the subjects decide between and γ_i represents the selection they make (1 for (x_i, s_i) and 0 for (y_i, t_i)). Using this technique, the viability of the 2-system quasi-hyperbolic model could be, in the future, more precisely investigated.

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