THE PARTITION FUNCTION AND MODULAR FORMS

JUSTIN HSU

Contents

1. Intro to partition function and modular forms

The partition function $p(n)$, counting the number of ways of writing n as a sum of positive integers (the number of unrestricted partitions of n), was initially studied from a combinatorial point of view. For example, Euler, working with the generating function

$$\sum_{i=1}^{\infty} p(i)x^i = (1+x+x^2+\dots)(1+x^2+x^4\dots) = \prod_{i=1}^{\infty}\left(\frac{1}{1-x^i}\right)$$

proved that the number of partitions into odd parts is equal to the number of partitions into distinct parts.

However, in the early 20th century, Hardy and Ramanujan presented a novel result that came from purely analytical methods. In 1918, they proved the asymptotic

$$p(n) \sim \frac{\exp\left(\pi\sqrt{2n/3}\right)}{4n\sqrt{3}}$$

At first glance, this result looks very mysterious. Even more remarkably, in 1937, Hans Rademacher proved [8] a convergent series expansion for $p(n)$, which had as it's first term

Hardy and Ramnujan's asymptotic

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}} \right)$$

where

$$A_k(n) = \sum_{\substack{0 \le m \le k \\ (m,n)=1}} \exp(\pi i s(m,k) - 2nm/k) \tag{1.1}$$

and $s(m,k)$ is the Dedekind sum

$$s(m,k) = \sum_{r \mod k} \left(\left(\frac{r}{k}\right)\right) \left(\left(\frac{mr}{k}\right)\right) \tag{1.2}$$

$$((x)) = \begin{cases} x - \lfloor x \rfloor - 1/2 & : x \notin \mathbb{Z} \\ 0 & : x \in \mathbb{Z} \end{cases}$$

Both Hardy, Ramanujan and Rademacher's results all come from considering the partition function as a type of modular function. In the remainder of this paper, we will provide an overview of modular functions at the level needed to understand Rademacher's proof. Then, we will show an asymptotic upper bound similar to the one discovered by Hardy and Ramanujan, using analytic methods, but without modular functions.

As a second example of the use of modular forms to study the partition function, we will also consider the Ramanujan congruences for the partition function, which state that for $l \in 5, 7, 11$, and $\delta_l = \frac{l^2-1}{24}$, we have

$$p(ln - \delta_l) \equiv 0 \pmod{l}$$

These congruences have been much studied, and have been proved in several ways, either via the theory of Hecke operators, or via a pure combinatorics approach, like Dyson's "rank and crank". We sketch a proof of these congruences using the theory of modular forms, due to [7]. Finally, we discuss areas of active research centered around the partition function, including some of Ono's recent results in this area.

2. Partition function leading term, without modular forms

As a first step in understanding the partition function analytically, we will derive an asymptotic bound for $p(n)$, without the use of modular forms. Define the generating function

$$F(z) = \sum_{n=1}^{\infty} p(n) z^n = \prod_{k=1}^{\infty} \frac{1}{1 - z^k} \tag{2.1}$$

This product converges absolutely in the disk $|z| < 1$. Also, if $0 < r < 1$, we have $\left|F\left(re^{i\theta}\right)\right| \le F(r)$ since $\left|1 - z^k\right| \ge 1 - |z|^k = 1 - r^k$. Integrating around the circle of radius r gives an upper bound for $p(n)$, namely

$$p(n) = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z^{n+1}} dz \le \frac{F(r)}{2\pi r^{n+1}} 2\pi r = \frac{F(r)}{r^n} \tag{2.2}$$

Hence, we try to minimize this upper bound. Taking the derivative of $\log F(r)$, we find that the minimum is achieved at

$$\frac{F'}{F}(r_0) = \frac{n}{r_0} \tag{2.3}$$

However, we can calculate $\frac{\partial}{\partial r} \log F(r)$ another way:

$$\log F(r) = \sum_{k=1}^{\infty} -\log(1-r^k) = \sum_{k,l=1}^{\infty} \frac{r^{kl}}{l} = \sum_{m=1}^{\infty} r^m \left(\sum_{kl=m} \frac{1}{l} \right)$$

$$= \sum_{m=1}^{\infty} \frac{\sigma(m)}{m} r^m \tag{2.4}$$

$$r \log F'(r) = r\frac{F'}{F}(r) = \sum_{m=1}^{\infty} \sigma(m) r^m \tag{2.5}$$

Letting $r = e^{-1/x}$ in (2.4), and using the inverse Mellin transform for $\Gamma(s)$, we find that

$$\log F(r) = \sum_{m=1}^{\infty} \frac{\sigma(m)}{m} e^{-m/x}$$

$$= \sum_{m=1}^{\infty} \frac{\sigma(m)}{m} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{m}\right)^s \Gamma(s) ds \tag{2.6}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\sum_{m=1}^{\infty} \frac{\sigma(m)}{m^{1+s}} \right) x^s \Gamma(s) ds \tag{2.7}$$

But the parenthesized sum is simply $\zeta(s)\zeta(1+s)$, so the integrand is $x^s\Gamma(s)\zeta(s)\zeta(1+s)$, which has poles at $s = 0, \pm 1$. Calculating residues, we find

$$\mathrm{Res}_{s=-1} = -x^{-1}\zeta(0)\zeta(-1) = -\frac{1}{24x} \tag{2.8}$$

$$\mathrm{Res}_{s=1} = x\zeta(2) = \frac{\pi^2}{6} x$$

For $s = 0$, we Taylor expand the integrand:

$$x^s\Gamma(s)\zeta(s)\zeta(s+1)ds = (1 + s\log x + \dots)(\zeta(0) + s\zeta'(0) + \dots)\left(\frac{1}{s} + \gamma + \dots\right)\left(\frac{1}{s} + \Gamma'(1) + \dots\right)$$

$$\mathrm{Res}_{s=0} = \zeta(0)\log x + \zeta'(0) + \gamma\zeta(0) + \Gamma'(1)\zeta(0)$$

$$= -\frac{1}{2}\log x + C_1$$

Now, we will calculate what happens to the integrand when we shift the line of integration from $\Re(z) = c$ to $\Re(z) = -c$. By the function equation for the zeta function, we know that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

so the quantity

$$g(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-\frac{1+s}{2}}\Gamma\left(\frac{1+s}{2}\right)\zeta(1+s)$$

is invariant under the transformation $s \to -s$. Also, by the duplication formula $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, we have

$$g(s) = \pi^{-s}\pi^{-1/2}2^{1-s}\sqrt{\pi}\Gamma(s)\zeta(s)\zeta(s+1)$$

So our integral (2.7) becomes:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^s\Gamma(s)\zeta(s)\zeta(s+1)ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(2\pi x)^s((2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+1))ds \\
&= \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}(2\pi x)^s\left(\frac{g(s)}{2}\right)ds + R \\
&= \frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty}(-1)(2\pi x)^{-s}(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+1)ds + R \qquad (2.9)
\end{aligned}$$

where R is the sum of the residues at $s = 0, \pm 1$, by first shifting the integral to the line $\Re(z) = -c$, then transforming $s \to -s$, since $g(s) = g(-s)$. This final integral is very similar to our original integral on the left hand side. Indeed, writing out the residues, we have

$$\begin{aligned}
\log F\left(e^{-\frac{2\pi}{x}}\right) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(2\pi x)^s\left(\frac{g(s)}{2}\right)ds \\
&= \frac{x}{2\pi}\frac{\pi^2}{6} - \frac{2\pi}{x}\frac{1}{24} - \frac{1}{2}\log\frac{x}{2\pi} + C_1 + \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} x^s(2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+1)ds \\
&= \frac{\pi x}{12} - \frac{\pi}{12x} - \frac{1}{2}\log\frac{x}{2\pi} + C_1 + \log F\left(e^{-2\pi x}\right)
\end{aligned}$$

Differentiating in x, and setting $r = e^{-\frac{2\pi}{x}}$ gives

$$\begin{aligned}
\frac{F'}{F}\left(e^{-\frac{2\pi}{x}}\right)e^{-\frac{2\pi}{x}}\left(\frac{2\pi}{x^2}\right) &= \frac{\pi}{12}(1+x^{-2}) - \frac{1}{2x} + \frac{F'}{F}\left(e^{-2\pi x}\right)e^{-2\pi x}(-2\pi) \\
r\frac{F'}{F}(r) &= \frac{x^2+1}{24} - \frac{x}{4\pi} - x^2e^{-2\pi x}\frac{F'}{F}\left(e^{-2\pi x}\right) \\
&= \frac{x^2+1}{24} - \frac{x}{4\pi} - x^2e^{-2\pi x}\log F'\left(e^{-2\pi x}\right)
\end{aligned}$$

So the last term goes to 0 exponentially fast as $x \to \infty$. Also, to minimize the upper bound on $p(n)$, we want the left hand side to be n in (2.3). So, we choose x to satisfy

$$\frac{x^2}{24} - \frac{x}{4\pi} + \frac{1}{24} = n \Rightarrow x = \frac{3}{\pi} + \sqrt{\left(\frac{6}{\pi}\right)^2 + 4(24n-1)}$$

Thus by the upper bound (2.2),

$$\begin{aligned}
p(n) &\leq \exp\left(\frac{\pi x}{12} - \frac{\pi}{12x} - \frac{1}{2}\log x + \frac{2\pi n}{x}\right) \cdot \exp\left(O\left(e^{-\sqrt{n}}\right)\right) \\
&\leq \exp\left(\frac{\pi x}{12} - \frac{\pi}{12x} - \frac{1}{2}\log x + 2\pi\left(\frac{x}{24} - \frac{1}{4\pi} + \frac{1}{24x}\right)\right) \\
&= \exp\left(\frac{\pi x}{6} - \frac{1}{2} - \frac{1}{2}\log x\right) \\
&= \exp\left(\frac{\pi}{12}\sqrt{4(24n-1) + \left(\frac{6}{\pi}\right)^2} - \frac{1}{2}\log x\right) \\
&\leq \frac{\exp\left(\pi\sqrt{\frac{2n}{3}} + O\left(n^{-1/2}\right)\right)}{\sqrt{x}}
\end{aligned}$$

Since $x \approx \sqrt{24n}$ for large n, we have the final bound

$$p(n) \ll \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{n^{1/4}}$$

This is a bit off from Hardy and Ramanujan's result, but the method of analysis is similar to the "circle method" used by Hardy and Ramanujan. More careful analysis would given their asymptotic.

3. Modular form basics

Here, we will briefly go over the basic theory of modular forms. Proofs can be found in a standard textbook, such as [9]. First, we must define what a modular function is.

Definition 3.1. Let k be an integer. We say a complex valued function f is weakly modular of weight $2k$ if it is meromorphic on the upper half plane H, and for all elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the full modular group $SL_2(\mathbb{Z})$, f satisfies the modular relation

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right)$$

By plugging in matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ we find that a weakly modular function f satisfies $f(z+1) = f(z)$, and $f(-1/z) = z^{2k} f(z)$. In particular, as f is invariant under translation by 1, we can express f as a function of $q = e^{2\pi i z}$.

Let $f(z)$ be a weakly modular function, and $\tilde{f}(q)$ it's expression in terms of q. As $f(z)$ is meromorphic on the upper half plane, $\tilde{f}(q)$ is automatically meromorphic on the punctured disk $0 < |q| < 1$. If $\tilde{f}$ extends to a holomorphic function at 0, we say that f is holomorphic at infinity. Additionally, we say that the value of f at infinity (denoted $f(\infty)$) is $\tilde{f}(0)$.

Definition 3.2. If a weakly modular function f is holomorphic at infinity, then we say that f is a modular form. If $f(\infty) = 0$, then we say that f is a cusp form. We denote the $\mathbb{C}$-vector space of modular forms of weight k by M_k, and the $\mathbb{C}$-vector space of cusp forms of weight k by S_k.

Example 3.3. Let $k > 1$ be an integer. The Eisenstein series of weight $2k$ is defined as

$$G_k(z) = \sum_{\substack{m=0,n=0 \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(mz+n)^{2k}}$$

It can be shown that G_k is a weight $2k$ modular form, with $G_k(\infty) = 2\zeta(2k)$. For notational convenience, it is useful to define these multiples of the Eisenstein series:

$$g_2 = 60G_2; g_3 = 140G_3$$
$$E_k = \frac{G_k}{2\zeta(2k)}$$

With this notation, we can define the modular discriminant Δ as

$$\Delta = g_2^3 - 27g_3^2$$

Evidently, Δ is a modular form of weight 12. Furthermore, as $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$, we can compute that $\Delta(\infty) = 0$, hence Δ is actually a cusp form of weight 12.

It turns out that Δ has a remarkably simple expression, from the following theorem due to Jacobi.

Theorem 3.4. *Let $q = e^{2\pi iz}$ as before. Then we have*

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Definition 3.5. We define the following 24th root of the modular discriminant to be the Dedekind eta function on the upper half space:

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi inz}\right)$$

4. First application: Rademacher's formula

We will generally follow the account of Rademacher's proof given in [4].

4.1. **A Transformation Formula for the η Function.** Before we give Rademacher's proof, we need a couple of results about how the η-function transforms under the modular group. We first prove a functional equation that will be useful later.

Lemma 4.1 (Functional Equation for Φ)**.** *Let $\zeta(s, \alpha)$ denote the Hurwitz zeta function, and $F(x, s)$ denote the periodic zeta function, defined as*

$$\zeta(s, \alpha) = \sum_{r=0}^{\infty} (r + \alpha)^{-s}$$
$$F(x, s) = \sum_{m=1}^{\infty} \frac{e^{2\pi imx}}{m^s}$$

Also, define a function $\Phi(\alpha,\beta,s)$ *by*

$$\Phi(\alpha,\beta,s)=\frac{\Gamma(s)}{(2\pi)^s}(\zeta(s,\alpha)F(\beta,1+s)+\zeta(s,1-\alpha)F(1-\beta,1+s))$$

Then Φ *satisfies the functional equation*

$$\Phi(\alpha,\beta,s)=\Phi(1-\beta,\alpha,-s)$$

for $\alpha,\beta\in(0,1)$*, and* $\Re(s)>0$*.*

Proof. This functional equation is a direct result of Hurwitz's formula, detailed in [3], which states that for $\alpha\in(0,1),\Re(s)>1$,

$$\zeta(1-s,a)=\frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2}F(a,s)+e^{\pi i(s-1)/2}F(-a,s)\right)$$

We claim that

$$F(a,s)=\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{-\pi(1-s)/2}\zeta(1-s,a)+e^{\pi i(s-1)/2}\zeta(1-s,1-a)\right) \tag{4.1}$$

Denote the right hand side by R, and plug in Hurwitz's formula, gives

$$\begin{aligned}R&=\frac{\Gamma(1-s)\Gamma(s)}{(2\pi)}\left(e^{\pi i(1/2-s)}F(a,s)+e^{\pi i(s-1/2)}F(a-1,s)+iF(-a,s)-iF(1-a,s)\right)\\&=\frac{\Gamma(1-s)\Gamma(s)}{(2\pi)}\left(2\cos((s-1/2)\pi)F(a,s)\right)\\&=\Gamma(1-s)\Gamma(s)\frac{\sin(s\pi)}{\pi}F(a,s)=F(a,s)\end{aligned}$$

where we use the reflection formula for Γ, and also the fact that F has period 1 in the first variable. Plugging (4.1) into the definition of Φ gives

$$\begin{aligned}\frac{\Phi(\alpha,\beta,s)}{\Gamma(s)\Gamma(-s)}=&e^{\pi is/2}\left(\zeta(s,\alpha)\zeta(-s,1-\beta)+\zeta(s,1-\alpha)\zeta(-s,\beta)\right)\\&+e^{-\pi is/2}\left(\zeta(-s,1-\beta)\zeta(s,1-\alpha)+\zeta(-s,\beta)\zeta(s,\alpha)\right)\end{aligned} \tag{4.2}$$

Now both sides are invariant under the change of variables $\alpha\to1-\beta,\beta\to\alpha,s\to-s$, proving the functional equation. □

We next prove some bounds on the growth of Φ

Lemma 4.2 (Asymptotic Bound on Φ)**.** *For* $s=\sigma+it$*,* $\sigma\geq-3/2,|t|\geq1$*, we have*

$$\left|z^{-s}\Phi(\alpha,\beta,s)\right|=O\left(e^{-|t|\delta}\right)$$

For some $\delta>0$*, for fixed* z *with* $\Re(z)>0$*.*

Proof. Using the identity $\Gamma(s)\Gamma(-s)=-\Gamma(s)\Gamma(1-s)s^{-1}=-\pi^{-1}s\sin(\pi s)$, (4.2) gives

$$\begin{aligned}\frac{\Phi(\alpha,\beta,s)}{z^s}=&\frac{-\pi}{z^s s\sin(\pi s)}\{e^{\pi is/2}\left(\zeta(s,\alpha)\zeta(-s,1-\beta)+\zeta(s,1-\alpha)\zeta(-s,\beta)\right)\\&+e^{-\pi is/2}\left(\zeta(-s,1-\beta)\zeta(s,1-\alpha)+\zeta(-s,\beta)\zeta(s,\alpha)\right)\}\end{aligned} \tag{4.3}$$

For z with $\Re(z)$, we have $\arg(z) \leq \pi/2$, and picking $\delta > 0$ with $\delta + \arg(z) \leq \pi/2$, we can bound z^{-s} by

$$\left|z^{-s}\right| = \left||z|^{-s} e^{-si\arg(z)}\right| = O\left(e^{t\arg z}\right) = O\left(e^{|t|(\pi/2-\delta)}\right)$$

since $|z|^{-s}$ has norm bounded above by a constant for $\sigma \geq -3/2$. Likewise, we can bound

$$\frac{1}{|s\sin(\pi s)|} = \frac{2}{|s(e^{\pi is} - e^{-\pi is})|} \leq \frac{2}{|te^{\pm\pi is}|} = O\left(\frac{e^{-\pi|t|}}{|t|}\right)$$

since $|s|$ is bounded below by $|t| \geq 1$, and we can choose the sign so that $|e^{\pm\pi is}| = |e^{\pi|t|}|$ For s in the given range, we can estimate $|\zeta(s,a)| = O(|t|^c)$ for some $c > 0$ (see, for example, [3]), and using the bounds $|e^{\pm\pi is/2}| = O(e^{\pi|t|/2})$, we can estimate (4.3)

$$\left|z^{-s}\Phi(\alpha,\beta,s)\right| = O\left(e^{|t|\left(\frac{\pi}{2}-\delta\right)}\cdot\frac{e^{-\pi|t|}}{|t|}\cdot e^{\frac{\pi|t|}{2}}|t|^{2c}\right) = O\left(e^{-|t|\tilde{\delta}}\right)$$

for $\tilde{\delta} > 0$, as desired.

□

Theorem 4.3 (Iseki's Formula)**.** *For $\Re(z) > 0, \alpha, \beta \in [0,1]$, we define functions*

$$\lambda(z) = -\log(1 - e^{-2\pi iz}) = \sum_{m=1}^{\infty}\frac{e^{-2\pi imz}}{m}$$

$$\Lambda(\alpha,\beta,z) = \sum_{r=0}^{\infty}\lambda((r+\alpha)z - i\beta+)\lambda((r+1-\alpha)z + i\beta)$$

Then if either α or β are in $(0,1)$, we have

(4.4) $$\Lambda(\alpha,\beta,z) = \Lambda(1-\beta,\alpha,z^{-1}) - \pi z(B_2(\alpha)B_0(\beta) - 2iz^{-1}B_1(\alpha)B_1(\beta) - z^{-2}B_0(\alpha)B_2(\beta))$$

The sum of the last few terms on the right, involving the Bernoulli polynomials B_k, is equal to

$$-\pi z\left(\alpha^2 - \alpha + \frac{1}{6}\right) + \frac{\pi}{z}\left(\beta^2 - \beta + \frac{1}{6}\right) + 2\pi i\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)$$

Proof. Using inverse Mellin transform for $\Gamma(s)$ as in (2.6), we get

$$\begin{aligned}
\sum_{r=0}^{\infty}\lambda((r+\alpha)z - i\beta) &= \sum_{r=0}^{\infty}\sum_{m=1}^{\infty}\frac{e^{2\pi im\beta}}{2\pi im}\int_{c-i\infty}^{c+i\infty}\Gamma(s)(2\pi m(r+\alpha)z)^{-s}ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{(2\pi z)^s}\sum_{r=0}^{\infty}(r+\alpha)^{-s}\sum_{m=1}^{\infty}\frac{e^{2\pi im\beta}}{m^{1+s}}ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{(2\pi z)^s}\zeta(s,\alpha)F(\beta,1+s)ds
\end{aligned}$$

Similarly, we find that

$$\sum_{r=0}^{\infty}\lambda((r+1-\alpha)z + i\beta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{(2\pi z)^s}\zeta(s,1-\alpha)F(1-\beta,1+s)ds$$

This gives

$$\Lambda(\alpha,\beta,s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} z^{-s}\Phi(\alpha,\beta,s)ds$$

We let $c=3/2$, and consider the integral around the loop visiting $3/2-iT, 3/2+iT, -3/2+iT, -3/2-iT$, and let $T\to\infty$. The horizontal integrals go to 0 by the estimate in Lemma 4.2, and if R is the sum of the residues in this loop, we are left with

$$\int_{3/2-i\infty}^{3/2+i\infty} z^{-s}\Phi(\alpha,\beta,s)ds=\int_{-3/2-i\infty}^{-3/2+i\infty} z^{-s}\Phi(\alpha,\beta,s)+R$$

Now, plugging this into our expression for Λ and changing variables $u=-s$, we find

$$\Lambda(\alpha,\beta,s)=\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} z^{u}\Phi(\alpha,\beta,-u)du+R$$

Using the functional equation $\Phi(\alpha,\beta,-s)=\Phi(1-\beta,\alpha,s)$ from Lemma 4.1, we find that

$$\Lambda(\alpha,\beta,z)=\Lambda(1-\beta,\alpha,z^{-1})+R$$

By the definition of Φ, we see that the poles occur at $s=-1,0,1$. Calculating the pole at $s=1$, we have

$$\begin{aligned}\mathrm{Res}_{s=1}&=\frac{\Gamma(1)}{2\pi z}(F(\beta,2)+F(1-\beta,2))=\frac{1}{2\pi z}\sum_{n=1}^{\infty}\frac{e^{2\pi in\beta}}{n^2}+\frac{e^{-2\pi in\beta}}{n^2}\\&=\frac{1}{2\pi z}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi in\beta}}{n^2}=\frac{\pi}{z}B_2(\beta)\end{aligned}$$

For the $s=0$ residue, we note that $\zeta(0,1-\alpha)=\alpha-1/2$:

$$\begin{aligned}\mathrm{Res}_{s=0}&=\zeta(0,\alpha)F(\beta,1)+\zeta(0,1-\alpha)F(1-\beta,1)=(\frac{1}{2}-\alpha)\sum_{n=1}^{\infty}\frac{e^{2\pi in\beta}-e^{-2\pi in\beta}}{n}\\&=-B_1(\alpha)\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi in\beta}}{n}=2\pi iB_1(\alpha)B_1(\beta)\end{aligned}$$

Finally, for the $s=-1$ residue, we can use the functional equation for Φ:

$$\begin{aligned}\mathrm{Res}_{s=-1}&=\lim_{s\to-1}(s+1)z^{-s}\Phi(\alpha,\beta,s)\\&=\lim_{s\to1}(1-s)z^{s}\Phi(\alpha,\beta,-s)\\&=\lim_{s\to1}(1-s)z^{s}\Phi(1-\beta,\alpha,s)\end{aligned}$$

But this is simply $\mathrm{Res}_{s=1}$, with changes $z\to -z^{-s}, \alpha\to 1-\beta, \beta\to\alpha$. Thus,

$$\mathrm{Res}_{s=-1}=\pi zB_2(\alpha)$$

As $\mathrm{Res}_{s=-1}+\mathrm{Res}_{s=0}+\mathrm{Res}_{s=1}=-\pi z(B_2(\alpha)B_0(\beta)-2iz^{-1}B_1(\alpha)B_1(\beta)-z^{-2}B_0(\alpha)B_2(\beta))$ this proves Iseki's formula in the case $a,b\in(0,1)$. To extend to the case where either a or b

may be in $[0,1]$, we can take limits. For example, suppose $\beta \in (0,1)$, and we want to verify the formula for $\alpha = 0$.

$$\begin{aligned}\sum_{r=0}^{\infty} \lambda((r+\alpha)z - i\beta) &= \sum_{r=0}^{\infty}\sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} e^{-2\pi m(r+\alpha)z} \\ &= \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} e^{-2\pi m\alpha z} \sum_{r=0}^{\infty} e^{-2\pi m r z} \\ &= \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} \frac{e^{-2\pi m\alpha z}}{1 - e^{-2\pi m z}} = \sum_{m=1}^{\infty} \frac{e^{2\pi i m\beta}}{m} f_\alpha(m)\end{aligned}$$

For $\alpha \in [0,1]$, $f_\alpha(m)$ convergs to 0 uniformly, so by the M-test, we can take the limit $\alpha \to 0^+$ term by term. This same trick works for the other boundary values of α, β, by changing the variables:

$$\begin{aligned} &\alpha \to 1-\alpha, && \beta \to 1-\beta && \\ &\alpha \to \beta, && \beta \to 1-\alpha, && z \to z^{-1} \\ &\alpha \to 1-\beta, && \beta \to \alpha, && z \to z^{-1} \end{aligned}$$

□

Lemma 4.4. *For complex z with $\Re(z) > 0$, and integers h, k, H such that $(h,k) = 1$, $k > 0$, and $hH \equiv -1 \pmod{k}$, the following formula holds:*

$$\sum_{n=1}^{\infty} \frac{n}{k}(z - ih) = \sum_{n=1}^{\infty} \frac{n}{k}\left(\frac{1}{z} - iH\right) + \frac{1}{2}\log z - \frac{\pi}{12k}\left(z - \frac{1}{z}\right) + \pi i s(h,k) \tag{4.5}$$

where $s(h,k)$ is the Dedekind sum defined above.

Proof. We first do the case $k = 1$. Since $s(h,1) = 0$, we need

$$\sum_{n=1}^{\infty} \lambda(n(z - ih)) = \sum_{n=1}^{\infty} \lambda\left(n\left(\frac{1}{z} - iH\right)\right) + \frac{1}{2}\log z - \frac{\pi}{12}\left(z - \frac{1}{z}\right)$$

As $\lambda(x) = \lambda(x+i)$, we can rewrite this as

$$\sum_{n=1}^{\infty} \lambda(nz) = \sum_{n=1}^{\infty} \lambda\left(\frac{n}{z}\right) + \frac{1}{2}\log z - \frac{\pi}{12}\left(z - \frac{1}{z}\right)$$

In Iseki's formula (4.4), we set $\beta = 0$ and separate out the first in the series on the left hand side, and the second term in the series on the right hand side. As we take $\alpha \to 0^+$, both these terms tend to infinity. However, their difference limits to a finite value. To see this, we can calculate

$$\lambda(\alpha z) - \lambda(i\alpha) = \log(1 - e^{-2\pi i\alpha}) - \log(1 - e^{-2\pi\alpha z}) = \log\frac{1 - e^{-2\pi i\alpha}}{1 - e^{-2\pi\alpha z}}$$

By L'Hôpital's rule, this difference tends to $\log(i/z) = \frac{\pi i}{2} - \log z$.

The remaining terms on each side double up, and we simply get

$$\frac{\pi i}{2} - \log z + 2\sum_{r=1}^{\infty} \lambda(rz) = 2\sum_{r=1}^{\infty} \lambda\left(\frac{r}{z}\right) - \frac{\pi z}{6} + \frac{\pi}{6z} + \frac{\pi i}{2}$$

This reduces to the desired formula. For $k > 1$, set α, β to be

$$\alpha = \frac{\mu}{k}, 1 \leq \mu \leq k-1$$

with μ an integer. We then have $h\mu = qk + \nu$, with $1 \leq \nu \leq k-1$ an integer. We set $\beta = \frac{\nu}{k}$ Now, $\nu \equiv h\mu \pmod{k}$, so $-H\nu \equiv -Hh\mu \equiv \mu \pmod{k}$, so we have the congruences:

$$\alpha = \frac{\mu}{k} \equiv \frac{-H\nu}{k} \pmod{k}$$
$$\beta = \frac{\nu}{k} \equiv \frac{-h\mu}{k} \pmod{k}$$

Noting that λ is i-periodic, Iseki's formula (4.4) implies

$$\begin{aligned}
&\frac{1}{2}\sum_{r=0}^{\infty} \lambda\left(\left(r+\frac{\mu}{k}\right)z - i\frac{h\mu}{k}\right) + \lambda\left(\left(r+1-\frac{\mu}{k}\right)z + i\frac{h\mu}{k}\right) \\
&= \frac{1}{2}\sum_{r=0}^{\infty} \lambda\left(\left(r+\frac{\nu}{k}\right)\frac{1}{z} - i\frac{H\nu}{k}\right) + \lambda\left(\left(r+1-\frac{\nu}{k}\right)\frac{1}{z} + i\frac{H\nu}{k}\right) \\
&- \frac{\pi z}{2}\left(\left(\frac{\mu}{k}\right)^2 - \left(\frac{\mu}{k}\right) + \frac{1}{6}\right) + \frac{\pi}{2z}\left(\left(\frac{\nu}{k}\right)^2 - \left(\frac{\nu}{k}\right) + \frac{1}{6}\right) + \pi i\left(\frac{\mu}{k} - \frac{1}{2}\right)\left(\frac{\nu}{k} - \frac{1}{2}\right)
\end{aligned}$$

This is equivalent to:

$$\begin{aligned}
&\frac{1}{2}\sum_{r=0}^{\infty} \lambda\left(\frac{(rk+\mu)(z-ih)}{k}\right) + \lambda\left(\frac{(rk+k-\mu)(z-ih)}{k}\right) \\
&= \frac{1}{2}\sum_{r=0}^{\infty} \lambda\left(\frac{(rk+\nu)\left(\frac{1}{z} - iH\right)}{k}\right) + \lambda\left(\frac{(rk+k-\nu)(\frac{1}{z} - iH)}{k}\right) \\
&- \frac{\pi z}{2}\left(\left(\frac{\mu}{k}\right)^2 - \left(\frac{\mu}{k}\right) + \frac{1}{6}\right) + \frac{\pi}{2z}\left(\left(\frac{\nu}{k}\right)^2 - \left(\frac{\nu}{k}\right) + \frac{1}{6}\right) + \pi i\left(\frac{\mu}{k} - \frac{1}{2}\right)\left(\frac{\nu}{k} - \frac{1}{2}\right)
\end{aligned}$$

Summing over $\mu = 1, 2, \ldots k-1$, we note that the terms $rk + \mu$ and $rk + k - \mu$ hit exactly the numbers which are not multiples of k. Also, as $\nu \equiv h\mu \pmod{k}$, and $(h,k) = 1$, ν runs over the same values as μ, but in a different order. Hence:

$$\begin{aligned}
\sum_{\substack{n=1 \\ n\not\equiv 0 \pmod k}}^{\infty} \lambda\left(\frac{n}{k}(z-ih)\right) &= \sum_{\substack{n=1 \\ n\not\equiv 0 \pmod k}}^{\infty} \lambda\left(\frac{n}{k}\left(\frac{1}{z}-iH\right)\right) \\
&\quad + \frac{\pi}{2}\left(\frac{1}{z}-z\right)\sum_{\mu=1}^{k-1}\frac{\mu^2}{k^2} - \frac{\pi}{2}\left(\frac{1}{z}-z\right)\sum_{\mu=1}^{k-1}\frac{\mu}{k} \\
&\quad + \frac{\pi}{12}\left(\frac{1}{z}-z\right)(k-1) + \pi i\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{\nu}{k}-\frac{1}{2}\right) - \frac{\pi i}{2}\sum_{\mu=1}^{k-1}\frac{\nu}{k} + \frac{\pi i}{4}(k-1) \\
&= \sum_{\substack{n=1 \\ n\not\equiv 0 \pmod k}}^{\infty} \lambda\left(\frac{n}{k}\left(\frac{1}{z}-iH\right)\right) + \pi i\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{\nu}{k}-\frac{1}{2}\right) \\
&\quad + \frac{\pi}{12}\left(\frac{1}{z}-z\right)\left(\frac{(k-1)(2k-1)}{k} - 3(k-1) + (k-1)\right) \\
&= \sum_{\substack{n=1 \\ n\not\equiv 0 \pmod k}}^{\infty} \lambda\left(\frac{n}{k}\left(\frac{1}{z}-iH\right)\right) + \frac{\pi}{12}\left(\frac{1}{z}-z\right)\left(1-\frac{1}{k}\right) + \pi i\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{\nu}{k}-\frac{1}{2}\right) \qquad (4.6)
\end{aligned}$$

Since $h\mu = qk + \nu$, we have

$$\frac{\nu}{k} = \frac{h\mu}{k} - \left[\frac{h\mu}{k}\right]$$

So we can simplify the last term:

$$\sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{\nu}{k}-\frac{1}{2}\right) = \sum_{\mu=1}^{k-1}\frac{\mu}{k}\left(\frac{h\mu}{k}-\left[\frac{h\mu}{k}\right]-\frac{1}{2}\right) = s(h,k)$$

If we write $n = mk$, and consider the (4.5) for $k = 1$:

$$\sum_{m=1}^{\infty}\lambda(mz) = \sum_{m=1}^{\infty}\lambda\left(\frac{m}{z}\right) + \frac{1}{2}\log z - \frac{\pi}{12}\left(z-\frac{1}{z}\right)$$

we see that adding this equation to (4.6) gives precisely the missing terms, and

$$\sum_{n=1}^{\infty}\lambda\left(\frac{n}{k}(z-ih)\right) = \sum_{n=1}^{\infty}\lambda\left(\frac{n}{k}\left(\frac{1}{z}-iH\right)\right) - \frac{\pi}{12k}\left(z-\frac{1}{z}\right) + \frac{1}{2}\log z + \pi i s(h,k)$$

as desired. □

We need one final Lemma before we can prove a transformation rule for the η-function.

Lemma 4.5. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *with* $c > 0$*, and* τ *with* $\Im(\tau) > 0$*, we have the following:*

$$\sum_{n=1}^{\infty} \lambda(-in\tau) = \sum_{n=1}^{\infty} \lambda\left(-in\frac{a\tau+b}{c\tau+d}\right) + \frac{\pi i}{12}\left(\tau - \frac{a\tau+b}{c\tau+d}\right) + \pi i\left(\frac{a+d}{12c} + s(-d,c)\right) + \frac{1}{2}\log(-i(c\tau+d)) \tag{4.7}$$

Proof. Pick z, h, k, H via:

$$k = c, h = -d, H = a, z = -i(c\tau + d)$$

Since $\Im(c) > 0$, we have $\Re(z) > 0$. As $ad - bc = 1$, $-hH - bk = 1$, so $(h,k) = 1$ and $hH \equiv -1 \pmod{k}$. So, the assumptions of Lemma 4.4, and we can express (4.7) in terms of z, h, H, k and use (4.5). This gives

$$\begin{aligned}
\tau &= \frac{iz-d}{c} = \frac{iz+h}{k} \\
a\tau + b &= H\frac{iz+h}{k} - \frac{hH+1}{k} = \frac{iz}{k}\left(H + \frac{i}{z}\right) \\
\frac{a\tau+b}{c\tau+d} &= \frac{1}{k}\left(H + \frac{i}{z}\right) \\
\tau - \frac{a\tau+b}{c\tau+d} &= \frac{1}{k}(h-H) + \frac{i}{k}\left(z - \frac{1}{z}\right) = -\frac{a+d}{c} + \frac{i}{k}\left(z - \frac{1}{z}\right)
\end{aligned}$$

and so

$$\frac{\pi i}{12}\left(\tau - \frac{a\tau+b}{c\tau+d}\right) = -\pi i\left(\frac{a+d}{12c}\right) + \frac{\pi}{12k}\left(z - \frac{1}{z}\right)$$

With these simplifications and change of variables, Lemma 4.5 gives the desired result. □

We are finally ready to prove a transformation formula for the η-function.

Theorem 4.6. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$*, with* $c > 0$*,* τ *with* $\Im(\tau) > 0$*, we have the following:*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\{-i(c\tau+d)\}^{1/2}\eta(\tau) \tag{4.8}$$

where

$$\epsilon(a,b,c,d) = \exp\left\{\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right\} \tag{4.9}$$

and $s(h,k)$ *is the Dedekind sum defined by* (1.2).

Proof. Taking the log of (4.8), we need

$$\log\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \log\eta(\tau) + \pi i\left(\frac{a+d}{12c} + s(-d,c)\right) + \frac{1}{2}\log(-i(c\tau+d))$$

Since $\eta(\tau)$ is defined as an infinite product, we equivalently need

$$\log(\eta) = \frac{\pi i\tau}{12} + \sum_{n=1}^{\infty}\log(1-e^{2\pi in\tau}) = \frac{\pi i\tau}{12} + \sum_{n=1}^{\infty}\lambda(-in\tau)$$

Together, this is

$$\sum_{n=1}^{\infty}\lambda(-in\tau) = \sum_{n=1}^{\infty}\lambda\left(-in\frac{a\tau+b}{c\tau+d}\right) + \frac{\pi i}{12}\left(\tau - \frac{a\tau+b}{c\tau+d}\right) + \pi i\left(\frac{a+d}{12c} + s(-d,c)\right) + \frac{1}{2}\log(-i(c\tau+d))$$

which is exactly Lemma 4.7. □

4.2. **Rademacher's Convergent Series.** We first give an outline for Rademacher's proof of the convergent series for $p(n)$. By Cauchy's integral formula, we can write $p(n)$ in terms of the Euler's generating function $F(x)$:

$$p(n) = \frac{1}{2\pi i}\int_{\gamma}\frac{F(x)}{x^{n+1}}dx$$

where γ is a counterclockwise contour located inside the unit disk. Proceeding using Hardy and Ramanujan's Circle method, we then split the contour into arcs lying near the roots of unity, where the F has a singularity, and approximate the integrand on each piece via a function that is easier to estimate. Though the circle method uses a contour in the x plane, we will follow Rademacher's proof, which uses a contour in the τ plane, where $x = e^{2\pi i\tau}$ to simplify the estimates. First, we note that $F(e^{2\pi i\tau}) = e^{\pi i\tau/12}/\eta(\tau)$, and so we can give a transformation formula for F using our results in the previous section.

Proposition 4.7. *Given* $\Re(z) > 0, k > 0, (h,k) = 1$ *and* $hH \equiv -1 \pmod{k}$*, let*

$$x = \exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right), x' = \exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)$$

Then, we have

$$F(x) = e^{\pi is(h,k)}\left(\frac{z}{k}\right)^{1/2}\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)F(x')$$

Proof. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c > 0$, the transformation formula for the η-function (4.8) gives

$$\frac{1}{\eta(\tau)} = \frac{1}{\eta(\tau')}\left(-i\left(c\tau+d\right)\right)^{1/2}\exp\left(\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right)$$

where $\tau' = (a\tau+b)/(c\tau+d)$. So, we pick

$$a = H, c = k, d = -h, b = -\frac{hH+1}{k}, \tau = \frac{iz+h}{k}$$

when

$$\tau' = \frac{i+h}{zk}$$

and the transformation formula gives

$$\begin{aligned} F(e^{2\pi i\tau}) =& F(e^{2\pi i\tau'})\exp\left(\frac{\pi i(\tau-\tau')}{12}\right)(-i\,(c\tau+d))^{1/2} \\ & \times \exp\left(\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right) \\ F\left(\exp\left(\frac{2\pi i h}{k}-\frac{2\pi z}{k}\right)\right) =& F\left(\exp\left(\frac{2\pi i H}{k}-\frac{2\pi}{kz}\right)\right)z^{1/2} \\ & \times \exp\left(\frac{\pi}{12kz}-\frac{\pi z}{12k}+\pi i s(h,k)\right) \end{aligned}$$

Setting $z = z/k$ gives the desired formula. □

We note that this formula allows us to estimate F close to the singularities: for $|z|$ small, we have that x is approximately a primitive root of unity, and x' is very close to 0, so $F(x') \sim 1$. Thus, near the singularities,

$$F(x) \sim z^{1/2}\exp\left(\frac{\pi}{12z}\right)$$

4.2.1. *Digression: Farey Fractions and Ford Circles.* To construct Rademacher's contour of integration, it will be necessary to use some facts about Farey fractions and Ford circles.

Definition 4.8. The Farey fractions of order n, denoted by F_n, are the set of reduced fractions in $[0, 1]$ with denominator at most n. Within each F_n, we list the fractions in increasing order.

Definition 4.9. Given a reduced rational number h/k, the Ford circle of h/k, denoted $C(h, k)$, is the circle in the complex plane with center $(h/k) + i/(2k^2)$ and radius $1/(2k)^2$.

By definition, the Ford circle of a set of Farey fractions form a line of circles in the upper half plane, all tangent to the real axis. For our purposes, we will need one fact about Ford

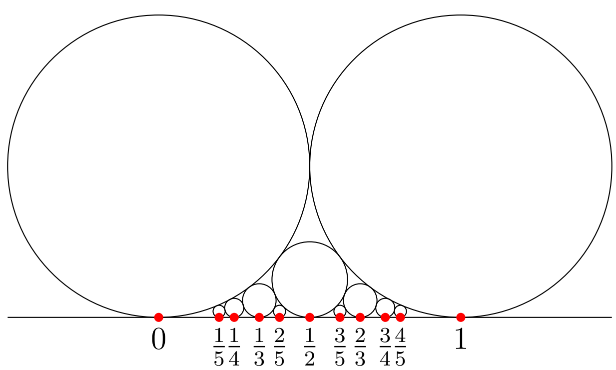

FIGURE 1. Ford circles for Farey fractions F_5

circles, that can be found in many texts on elementary number theory, and in [4].

Theorem 4.10. *The following two results hold:*

- *The Ford circle of consecutive Farey fractions are tangent to each other.*
- *Let $h_1/k_1, h/k, h_2/k_2$ be three consecutive Farey fractions. Then $C(h,k)$ is tangent to $C(h_1,k_1)$ and $C(h_2,k_2)$ at the points*

$$\alpha_1(h,k) = \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + \frac{i}{k^2+k_1^2}$$
$$\alpha_2(h,k) = \frac{h}{k} + \frac{k_2}{k(k^2+k_2^2)} + \frac{i}{k^2+k_2^2}$$

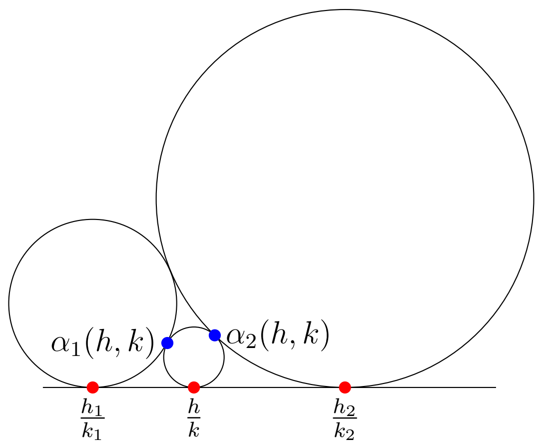

FIGURE 2. Consecutive Ford circles in Theorem 4.10

4.2.2. *Convergent Series for $p(n)$.* With these results in hand, we can define our contour γ, and finish the proof. First, we note that integrating around the circle of radius $e^{-2\pi}$ in the x plane is equivalent to integrating from i to $i+1$ in the τ plane, along a horizontal line segment. So, for a given, fixed integer N, define the path $P(N)$ from i to $i+1$ by taking arcs of the Ford circles of Farey fractions F_N as follows. Given the Ford circles for consecutive Farey fractions $h_1/k_1 < h/k < h_2/k_2$, the points of tangency to $C(h,k)$ define an upper and a lower arc. Take $P(N)$ to be the union of all the upper arcs of $C(h,k), h/k \in F_N$. For the fractions $0/1, 1/1$, take the arcs that lie over $[0,1]$. To traverse these arcs, we integrate over the arcs in increasing order of h/k. This defines a path from i to $i+1$, as in Figure 4.2.2. We note that the arc of a given Ford circle in $P(N)$ corresponds precisely to integrating near a primitive kth root of unity in the x plane, where $k \leq N$.

Instead of considering the integral over this row of Ford circles, it will be useful to transform variables once again, so that the arcs of the different Ford circles will be of similar form.

Lemma 4.11. *If we set*

$$z = -ik^2\left(\tau - \frac{h}{k}\right)$$

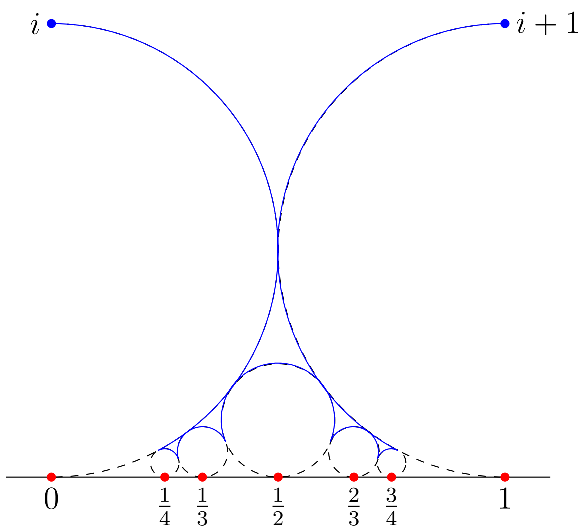

FIGURE 3. The Rademacher path $P(N)$ for $N = 4$ (in blue), τ-plane

the Ford circle $C(h,k)$ in the τ plane is mapped to a circle in the z plane of radius $1/2$ centered at $1/2$. Moreover, the points of tangency $\alpha_1(h,k), \alpha_2(h,k)$ are mapped to points

$$z_1(h,k) = \frac{k^2}{k^2+k_1^2} + i\frac{kk_1}{k^2+k_1^2}$$
$$z_2(h,k) = \frac{k^2}{k^2+k_2^2} - i\frac{kk_2}{k^2+k_2^2}$$

The upper arc of the Ford circle in the τ plane maps onto the arc in the z plane that does not touch the imaginary axis.

Proof. The change of variables is equivalent to translating the Ford circle by $-h/k$, scaling the radius by k^2, and then rotating the circle through $\pi/2$ clockwise. The first translation places the center on the imaginary axis at $i/(2k^2)$. The radius is then scaled up to $1/2$, and the rotation places the center at $1/2$. By repeating these transformations on the points α_1 and α_2 in Theorem 4.10, we get the given expressions for z_1, z_2. □

When integrating over this arc in the z plane, we will want to deform the arc to the chord between $z_1(h,k), z_2(h,k)$. We will use the following estimate.

Lemma 4.12. *For points z_1 and z_2, we have*

$$|z_1(h,k)| = \frac{k}{\sqrt{k^2+k_1^2}}, |z_2(h,k)| = \frac{k}{\sqrt{k^2+k_2^2}}$$

For z on the chord (z_1, z_2), we have the estimate

$$|z| < \frac{\sqrt{2}k}{N}$$

if $h_1/k_1 < h/k < h_2/k_2$ are consecutive in F_N. The length of the chord is at most $2\sqrt{2}k/N$.

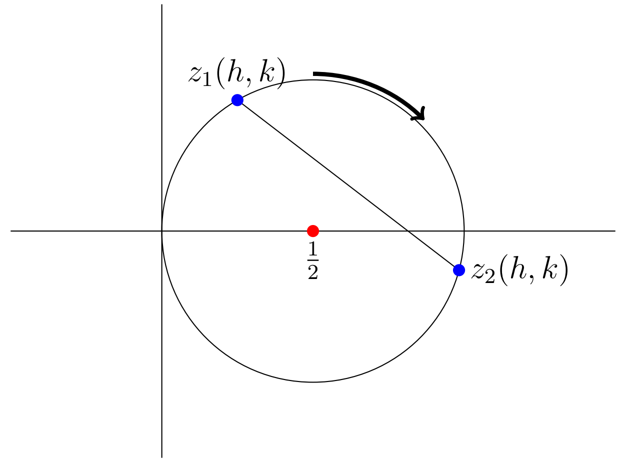

FIGURE 4. Transformed sub arc of Rademacher path, z-plane, Lemma 4.11

Proof. We can directly calculate

$$|z_1|^2 = \frac{k^4 + k^2 k_1^2}{(k^2 + k_1^2)^2} = \frac{k^2}{k^2 + k_1^2}$$

and similarly for z_2. For points on the chord, we note that $|z| \leq \max(|z_1|, |z_2|)$, so it is enough to check the bound for the endpoints z_1, z_2. By the AM-RMS inequality, we have

$$(k^2 + k_1^2)^{1/2} \geq \frac{k + k_1}{\sqrt{2}} \geq \frac{N+1}{\sqrt{2}} \geq \frac{N}{\sqrt{2}}$$

Where the second inequality follows from the fact that $h_1/k_1, h/k$ are consecutive in F_N. Finally, the triangle inequality implies that the length of the chord is at most $|z_1| + |z_2| = 2\sqrt{2}k/N$. □

We are now ready to state and prove Rademacher's convergent series for $p(n)$.

Theorem 4.13. *For $n \geq 1$, we have*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)}{\sqrt{n - \frac{1}{24}}}\right)$$

where

$$A_k(n) = \sum_{\substack{0 \leq m \leq k \\ (m,n)=1}} \exp(\pi i s(m,k) - 2nm/k)$$

Proof. Proceeding as above, we have

$$p(n) = \frac{1}{2\pi i}\int_\gamma \frac{F(x)}{x^{n+1}}dx$$

with γ a positively oriented curve surrounding $x = 0$ and lying inside the unit disk. We change variables to move the integral to the τ plane, where $x = e^{2\pi i\tau}$. If we integrate around γ a circle of radius $e^{-2\pi}$, τ follows a horizontal line segment from i to $i+1$. We deform the path with the path $P(N)$ described above, for fixed N. We also fix n for now. If we denote the upper arc of $C(h,k)$ in $P(N)$ as $\gamma(h,k)$, we can write

$$p(n) = \int_{P(N)} F(e^{2\pi i\tau})e^{-2\pi i\tau}d\tau = \sum_{k=1}^{N} \sum_{\substack{0\leq h<k \\ (h,k)=1}} \int_{\gamma(h,k)} F(e^{2\pi i\tau})e^{-2\pi i\tau}d\tau$$

Changing variables to $z = -ik^2(\tau - h/k)$, we have that

$$\tau = \frac{h}{k} + i\frac{z}{k^2}$$

By Lemma 4.11, this maps the arc $\gamma(h,k)$ onto the arc between $z_1(h,k)$ and $z_2(h,k)$, where z_1, z_2 lie on a circle of radius $1/2$ centered at $1/2$. Writing $\sum_{h,k}$ as shorthand for the sum, we have

$$\begin{aligned} p(n) &= \sum_{h,k} \int_{z_1(h,k)}^{z_2(h,k)} F\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right)\right) \exp\left(\frac{2\pi inh}{k} - \frac{2\pi nz}{k^2}\right) \frac{i}{k^2}dz \\ &= \sum_{h,k} ik^{-2}e^{-2\pi inh/k} \int_{z_1(h,k)}^{z_2(h,k)} e^{2\pi nz/k^2} F\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right)\right) dz \qquad (4.10) \end{aligned}$$

Since we eventually want to take N large, which corresponds to letting F tend towards the singularities, we want to use the transformation formula for F. We have

$$F\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right)\right) = \omega(h,k)\left(\frac{z}{k}\right)^{1/2} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right)$$

where

$$\omega(h,k) = e^{\pi is(h,k)}, (h,k) = 1, hH \equiv -1 \pmod{k}$$

Define

$$\Psi_k(z) = z^{1/2}\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)$$

and split the integral (4.10) into two parts to get

$$p(n) = \sum_{h,k} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}\left(I_1(h,k) + I_2(h,k)\right)$$

with

$$I_1(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z)e^{2\pi nz/k^2}dz \qquad (4.11)$$

$$I_2(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z)\left\{F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right) - 1\right\}e^{2\pi nz/k^2}dz \qquad (4.12)$$

Let K be the disk in the z plane, centered at $1/2$ with radius $1/2$. By Lemma 4.12, the length of the chord is bounded by $2\sqrt{2}k/N$, and on the chord, z is bounded by $\sqrt{2}k/N$. Furthermore, for points in $z \in K$, let $z = 1/2 + r\cos\theta + r\sin\theta, r \leq 1/2$. Also,

$$\Re\left(\frac{1}{z}\right) = \frac{2 + 4r\cos\theta}{1 + 4r\cos\theta + 4r^2} \geq 1$$

This also implies that $\Re(1/z) > 0$, and on the boundary of the circle $r = 1/2$, $\Re(1/z) = 1$. Now, we proceed to estimate I_2. We can bound the integrand (4.12) by

$$\begin{aligned}
&\left|\Psi_k(z)\left\{F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right) - 1\right\} e^{2\pi nz/k^2}\right| \\
&= |z|^{1/2} \exp\left(\frac{\pi}{12}\Re\left(\frac{1}{z}\right) - \frac{\pi}{12k^2}\Re(z)\right) e^{2\pi n\Re(z)/k^2} \left|\sum_{m=1}^{\infty} p(m) e^{2\pi iHm/k} e^{-2m\pi\Re(1/z)}\right| \\
&\le |z|^{1/2} \exp\left(\frac{\pi}{12}\Re\left(\frac{1}{z}\right)\right) e^{2\pi n/k^2} \sum_{m=1}^{\infty} p(m) e^{-2\pi m\Re(1/z)} \\
&< |z|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} p(m) e^{-2\pi(m-(1/24))\Re(1/z)} \\
&\le |z|^{1/2} e^{2n\pi} \sum_{m=1}^{\infty} p(m) e^{-2\pi(m-(1/24))} \\
&= |z|^{1/2} e^{2n\pi} \sum_{m=1}^{\infty} p(m) e^{-2\pi(24m-1)/24} \\
&< |z|^{1/2} e^{2n\pi} \sum_{m=1}^{\infty} p(24m-1) y^{24m-1} \\
&= c|z|^{1/2}
\end{aligned}$$

where $y = e^{-2\pi/24}$ and

$$c = e^{2n\pi} \sum_{m=1}^{\infty} p(24m-1) y^{24m-1}$$

c is independent of z and N (and we are keeping n fixed). On the chord $z_1(h,k), z_2(h,k)$, $|z| < \sqrt{2}k/N$, and the chord length is bounded by $2\sqrt{2}k/N$, so we have

$$|I_2(h,k)| < Ck^{3/2}N^{-3/2}$$

for some constant C. Thus, we can bound I_2 by

$$\left|\sum_{h,k} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k} I_2(h,k)\right| < \sum_{k=1}^{N} \sum_{\substack{0\le h<k \\ (h,k)=1}} Ck^{-1}N^{-3/2} \le CN^{-3/2} \sum_{k=1}^{N} 1 = CN^{-1/2}$$

so this entire second term is $O(N^{-1/2})$.

For $I_1(h,k)$, we replace the integral along the arc $z_1(h,k), z_2(h,k)$ with an integral around the whole circle K. We have

$$I_1(h,k) = \int_{K_-} - \int_0^{z_1(h,k)} - \int_{z_2(h,k)}^{0} = \int_{K_-} - J_1 - J_2$$

where K_- denotes integrating around K clockwise. For $|J_1|$, we note that we are integrating over an arc with length bounded by $\pi|z_1(h,k)| < \pi\sqrt{2}k/N$. In particular, $|z|$ on this arc is also bounded by $\pi\sqrt{2}k/N$. As $\Re(1/z) = 1$ and $0 < \Re(z) \leq 1$ on this arc, we can bound the integrand of J_1:

$$\left|\Psi_k(z)e^{2n\pi z/k^2}\right| = e^{2n\pi\Re(z)/k^2}\exp\left(\frac{\pi}{12}\Re\left(\frac{1}{z}\right) - \frac{\pi}{12k^2}\Re(z)\right) \leq \frac{e^{2n\pi}\pi^{1/2}2^{1/4}k^{1/2}e^{\pi/12}}{N^{1/2}}$$

So together, we have

$$|J_1| < C_1 k^{3/2} N^{-3/2}$$

which means that the J_1 term in I_1 contributes a $O(N^{-1/2})$ term to $p(n)$. The same analysis gives a similar bound for J_2, and we have

$$p(n) = \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \omega(h,k)e^{-2\pi inh/k}\int_{K_-}\Psi_k(z)e^{2n\pi z/k^2}dz + O(N^{-1/2})$$

Letting $N \to \infty$, this gives

$$p(n) = i\sum_{k=1}^{\infty} A_k(n)k^{-5/2}\int_{K_-} z^{1/2}\exp\left(\frac{\pi}{12z} + \frac{2\pi z}{k^2}\left(n - \frac{1}{24}\right)\right)dz$$

We change variables $w = 1/z$, $dz = -1/w^2 dw$ to give

$$p(n) = -i\sum_{k=1}^{\infty} A_k(n)k^{-5/2}\int_{1-i\infty}^{1+i\infty} w^{-5/2}\exp\left(\frac{\pi w}{12} + \frac{2\pi}{k^2}\left(n - \frac{1}{24}\right)\frac{1}{t}\right)dt$$

Setting $t = \pi w/12$ and $c = \pi/12$, we have

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n)k^{-5/2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-5/2}\exp\left(t + \frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\frac{1}{t}\right)dt$$

Now in [10], we have the formula

$$I_\nu(z) = \frac{(z/2)^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\nu-1}e^{t+(z^2/4t)}dt$$

For $c > 0, \Re(\nu) > 0$, and I_ν is the modified Bessel function, defined by $I_\nu(z) = i^{-\nu}J_\nu(iz)$, where J_ν is Bessel function of the first kind. Picking $\nu = 3/2$ and $z = \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}$ gives

$$\begin{aligned} p(n) &= 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n)k^{-5/2}\frac{\pi^{-3/2}\left(n - \frac{1}{24}\right)^{-3/4}}{6^{-3/4}k^{-3/2}} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right) \\ &= \frac{2\pi\left(n - \frac{1}{24}\right)^{-3/4}}{24^{3/4}}\sum_{k=1}^{\infty} A_k(n)k^{-1} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right) \end{aligned}$$

Since $I_{3/2}$ is a Bessel function of half odd order, we can reduce

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}}\frac{d}{dz}\left(\frac{\sinh z}{z}\right)$$

With this identity, we finally have

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right)$$

□

5. Second application: Ramanujan congruences

As we have seen, the partition function's close relation to the Dedekind η-function allows the results from the theory of modular forms to be used to prove properties of $p(n)$. As a second example of this correspondance, we give here a proof of the Ramanujan congruences of $p(n)$, obtained using techniques related to modular forms.

The Ramanujan congruences state that for $l \in 5, 7, 11$ and $\delta_l = \frac{l^2-1}{24}$, for every positive integer n, we have the congruence

$$p(ln - \delta_l) \equiv 0 \pmod{l}$$

We will outline a proof by Lachterman, Schayer and Younger, in [7].

5.1. Additional Results from Modular Functions.

We first need a couple more definitions and results from the theory of modular forms. Once again, proofs can be found in any reference on modular forms, such as [9].

Definition 5.1. Klein's modular j-function is defined as

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + \dots$$

Lemma 5.2. *Any weight 0 weakly modular form can be written as a polynomial in the j function.*

We will also need a basic result about modular forms, the valence formula.

Theorem 5.3. *Let $\rho = e^{2\pi i/3}$. If f is a weight k modular form, and $v_\tau(f)$ is the order of the pole or zero at τ, then we have the formula*

$$\frac{k}{12} = v_\infty(f) + v_i(f) + v_\rho(f) + \sum_{\substack{\tau \in H/\Gamma \\ \tau \notin i,\rho}} v_\tau(f)$$

Here, we take the convention that the order of a pole is negative, and the order of a zero is positive.

From the valence formula we can calcuate the dimension of M_k

Lemma 5.4. *Let $m(k)$ be the dimension of the vector space M_k. Then*

$$\tilde{m(k)} = \begin{cases} \lfloor k/12 \rfloor, & k \equiv 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1, & \text{otherwise} \end{cases}$$

5.2. **Proof of Ramanujan Congruences.** Let us now define the following Eisenstein series, corresponding to modular forms of even weight. Denote by $\tilde{E}_k$

$$\tilde{E}_k = \begin{cases} 1, & k \equiv 0 \pmod{12} \\ E_{14}, & k \equiv 2 \pmod{12} \\ E_4, & k \equiv 4 \pmod{12} \\ E_6, & k \equiv 6 \pmod{12} \\ E_4^2, & k \equiv 8 \pmod{12} \\ E_4E_6, & k \equiv 10 \pmod{12} \end{cases}$$

This next theorem, by Choie, Kohnen and Ono [6], will be the key ingredient.

Theorem 5.5. *Let $f \in M_{12n+14}, g \in M_k$. If we denote the constant term in the q expansion of a modular form h by $c(h)$, we have*

$$c\left(\frac{f \cdot g}{\Delta^{n+m(k)}\tilde{E}_k}\right) = 0$$

The following pair of Lemmas are due to Lachterman, Schayer and Younger.

Lemma 5.6. *For $l \in 5, 7, 11$, then for all $n \geq l$, there is an $f_{n,l} \in M_{12n+14}$ such that*

$$f_{n,l} \equiv E_k^l \pmod{l}$$

where k is even, and not congruent to $2 \pmod{2}$*. Here, congruent means that the coefficients of the q expansion are congruent.*

Lemma 5.7. *Let $\tau(n)$ be the n-th coefficient in the q-expansion of Δ^{δ_l}. The Ramanujan congruences are equivalent to*

$$\tau_l(ln) \equiv 0 \pmod{l}$$

for all $n \geq 1$.

Theorem 5.8. *For $l \in 5, 7, 11$ and $\delta_l = \frac{l^2-1}{24}$, for every positive integer n, we have the congruence*

$$p(ln - \delta_l) \equiv 0 \pmod{l}$$

Proof. By the previous proposition, it suffices to show $\tau_l(ln) \equiv 0 \pmod{l}$ for $l \in 5, 7, 11$. We can check that this is true for $n = 1$, so it remains to show this for $n > 1$. Let $t = ln - m(12\delta_l)$. By Lemma 5.6, we can find $f_{t,l} \in M_{12t+14}$ such that $f_{t,l} \equiv E_k^l \pmod{l}$. Then by Theorem 5.5,

$$c\left(\frac{f_{t,l}}{\Delta^{ln}}\Delta^{\delta_l}\right) \equiv c\left(\frac{E_k^l}{\Delta^{ln}}\Delta^{\delta_l}\right) \equiv 0 \pmod{l}$$

Denote the coefficients of $E_k^l/\Delta ln$ by $\lambda(m)$. By using the fact that $(1-x)^l \equiv 1-x^l \pmod{l}$, we ge that $\lambda(m)$ is nonzero only at multiples of l. From there, the constant coefficient equation becomes

$$\sum_{m=0}^{n} \tau_l(lm)\lambda(-lm) \equiv 0 \pmod{l}$$

By induction, $\tau_l(lm) \equiv 0 \pmod{l}$ for all $m < n$, so this implies that $\tau_l(ln) \equiv 0 \pmod{l}$, completing the induction. ∎

6. Further Research on the Partition Function via Modular Forms

Though the partition function is measuring a combinatorial quantity, the two results above show that studying this function through its connection to the η-function and leveraging the machinery of the theory of modular forms can be very fruitful. Indeed, there is much current research centered on using deep results in modular function theory to provide new insight into the partition function.

As an example of these methods, we briefly mention two recent results. The first, by Ahlgren, concerns the existence of congruences for the partition function similar to the Ramanujan congruences, but with different moduli. From [1], a corollary of the main result gives

Theorem 6.1 (Ahlgren, 2000)**.** *If M is a positive integer coprime to 6, then there exist infinitely many arithmetic progressions $An + B$ (none of which is contained in any other), such that*

$$p(An + B) \equiv 0 \pmod{M}$$

Thus, we can find Ramanujan-like congruences for many more moduli. A second result, recently announced by Bruinier and Ono [5], provides an exact formula of $p(n)$ as a finite sum of algebraic numbers. These algebraic numbers are singular moduli of a special Maass form, which can be thought of as a function that obeys the modular relation, but isn't necessarily holomorphic. The Maass form is constructed from the Eisenstein series E_2 (which isn't a modular form, but is in some sense "almost" a modular form), and the η-function, again highlighting the close connection between the partition function and the theory of modular forms.

Further results along these lines, along with an overview of the history of results concerning the partition function and a list of furhter references, can be found in [2].

7. Acknowledgements

I would like to thank Professor Kannan Soundararajan, for first introducing me to the subject of analytic number theory, as well as for patiently directing my reading, and overseeing this honors thesis. Without Professor Soundararajan's guidance and answers to my questions, this work would not have been possible. I would also like to Professor Leon Simon and Professor Brad Osgood in Electrical Engineering, for their mentorship as my undergraduate advisers.

References

[1] Scott Ahlgren. Distribution of the partition function modulo composite integers m. *Mathematische Annalen*, 318:795–803, 2000.
[2] Scott Ahlgren and Ken Ono. Adding and counting: The arithmetic of partitions. *Notices of the AMS*, 48:978–984, 2001.
[3] T. M. Apostol. *Introduction to Analytic Number Theory*. Springer-Verlag, New York, 1990. Undergraduate Texts in Mathematics.
[4] T. M. Apostol. *Modular Functions and Dirichlet Series in Number Theory*. Springer-Verlag, New York, 1990. Graduate Texts in Mathematics, No. 41.
[5] Jan Hendrik Bruinier and Ken Ono. An algebraic formula for the partition function. 2011.

[6] Youngju Choie, Winfried Kohnen, and Ken Ono. Linear relations between modular form coefficients and non-ordinary primes. *Bulletin of The London Mathematical Society*, 37:335–341, 2005.

[7] Samuel Lachterman, Rhiannon Schayer, and Brendan Younger. A new proof of the ramanujan congruences for the partition function. *The Ramanujan Journal*, 15:197–204, 2008. 10.1007/s11139-007-9072-2.

[8] H. Rademacher. On the expansion of the partition function in a series. *Ann. Math*, 44:416–422, 1943.

[9] J.-P. Serre. *A Course in Arithmetic*. Springer-Verlag, New York, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[10] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, 1995. Cambridge Mathematical Library.