BRAUER OBSTRUCTIONS OF FINITE GROUPS OF LIE TYPE IN VIEW OF THE LOCAL LANGLANDS CORRESPONDENCE

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Abstract. In general, it is difficult to characterize the Brauer obstructions of absolutely irreducible representations of a finite group in a uniform manner. For some finite groups of Lie type, one can use their richer structure to calculate the local Brauer obstructions, and a complete characterization of the local obstructions has been known for several classical groups, including general linear groups [24] and special linear groups [32], by using only algebraic means. Despite the significant progress so far, those proofs are ad hoc in nature, as many of the algebraic proofs utilize specific structure of the given group.

Here, we suggest a different approach to the problem using the philosophy of Local Langlands Correspondence. Although inherently more complicated, this approach is more natural, as the Local Langlands Correspondence enables us to transform the problem expressed via representations of discrete series representations of p-adic groups into the analogous problem regarding Weil-Deligne representations. In particular, the representations of a finite Chevalley group are closely related to the depth zero representations of the corresponding p-adic group, and the tame structure of Weil group is significantly simpler. By using the Local Langlands Correspondence, we (re)prove the triviality of ℓ-adic Brauer obstructions of cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$, for $\ell \neq \mathrm{char}\,\mathbb{F}_q$, a fact proven classically in [24]. Furthermore, based upon the construction of depth zero supercuspidal L-packets as in [11], we prove the triviality of ℓ-adic Brauer obstructions of every generic cuspidal representation of $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$ and $\mathrm{PGL}_n(\mathbb{F}_q)$, for $\ell \neq \mathrm{char}\,\mathbb{F}_q$. A similar result will be proven for some generic cuspidal representations of $\mathrm{SO}_{2n}(\mathbb{F}_q)$ and $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.

Contents

1. Introduction

Brauer Obstructions. Let G be a finite group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an absolutely irreducible representation. Given a representation, it is important to know over which field the representation can be defined. Firstly, as G is finite, ρ descends to a representation over $\overline{\mathbb{Q}}$ uniquely up to isomorphism, so we can consider ρ as a representation with values in $\mathrm{GL}_n(\overline{\mathbb{Q}})$. Below $\overline{\mathbb{Q}}$, any field over which ρ is definable must contain the *trace field* of ρ, defined as the extension of $\mathbb{Q}$ generated by the character values of ρ. Note that the trace field of ρ is algebraically characterized as the fixed field of Galois automorphisms $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ where $\rho \cong \rho^\sigma$, the Galois twist of ρ by σ. We will denote the trace field of ρ as $\mathbb{Q}(\rho)$.

The obstruction to the representation descending to its trace field is described by a Galois 2-cocycle, which in this case represents an element of the Brauer group of the trace field, thereby called the *Brauer obstruction* of ρ. The Brauer obstruction $[\psi_\rho] \in H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\rho)), \overline{\mathbb{Q}}^\times) = \mathrm{Br}(\mathbb{Q}(\rho))$ is defined as follows. For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\rho))$, the isomorphism $\rho \cong \rho^\sigma$ is unique up to $\overline{\mathbb{Q}}^\times$ by Schur's lemma. Thus, after choosing isomorphisms between ρ and its isomorphic Galois twists, we get the desired Brauer obstruction in $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\rho)), \overline{\mathbb{Q}}^\times)$, which is well-defined as the cohomology class is independent of the choice of isomorphisms.

In the study of a Brauer obstruction, one often considers the corresponding local Brauer obstruction for each place v in $\mathbb{Q}(\rho)$, as a global Brauer class is completely determined by the associated local Brauer classes. In terms of the definability over a field, ρ is definable over $\mathbb{Q}(\rho)$ if and only if all local Brauer obstructions vanish. Similarly, we can think of the same construction inside $\mathrm{Br}(K)$ for any finite extension $K/\mathbb{Q}(\rho)$, and the triviality of the Brauer class is equivalent to the definability of ρ over K. As the Brauer class inside $\mathrm{Br}(K)$ is the image of $[\psi_\rho]$ by the natural map induced from the inclusion map, its local invariant vanishes if and only if the local degree of the extension $K/\mathbb{Q}(\rho)$ is a multiple of the order of the local Brauer obstructions for each local place. Therefore, if $[\psi_\rho]$ does not vanish, then there is no canonical choice of the field of definition of ρ, as there are many optimal choices of field extensions with given local degrees, if one exists, by Grunwald-Wang theorem.

As the Galois cohomology approach to the Brauer obstructions is abstract and hard to compute by hand, the study of the order of (local) Brauer obstructions has been mostly done with alternative definitions under the guise of *(local) Schur indices*; they are precisely the same as the orders of the local Brauer obstructions. Classically, the Schur indices have been calculated in terms of the two alternative ways, either as the minimum required degree of field extension over the trace field so that the representation becomes realizable, or as the index of the corresponding central simple algebra. These two definitions are in principle more calculable than the Brauer obstruction approach, but still very limited. Nevertheless, the classical approach to Schur indices has been successful in characterizing local obstructions of several classical groups, including general linear groups [24] and special linear groups [32], by using only algebraic means such as group theory and structures of algebras.

Objective. The main purpose of this paper is to present a different perspective on the Brauer obstructions of representations of finite groups of Lie type. The advantage of the Galois cohomological definition of Brauer obstruction is that it applies to the case of smooth irreducible admissible $\overline{\mathbb{Q}}$-representations of locally profinite groups. Thus, if we can promote a representation of a reductive group over a finite field to a depth zero representation of the corresponding p-adic group without harming the cohomological data of Brauer obstructions, we can use the theory of representations of p-adic groups in calculating the Brauer obstructions of the original representation. In particular, we will mainly study *depth zero representations*, which are roughly the representations of p-adic groups arising directly from their analogues over a finite field. The classification and the promotion process of depth zero representations will be examined under our "coefficient-twisting" perspective.

A very plausible strategy to tackle the (depth zero) representation theory of p-adic groups is to work under the philosophy of *Local Langlands Correspondence* (LLC). The Local Langlands Correspondence relates a representation of a reductive p-adic group to a continuous homomorphism from the Weil-Deligne group to the Langlands dual group. The advantage of this approach is that, given some form of the Local Langlands Correspondence, often examining the representation theory of one side is easier than the other side.

In this paper, we use several different already proven versions of Local Langlands Correspondences to deduce triviality of Brauer obstructions of representations of various finite Chevalley groups. First, we use the Local Langlands Correspondence for GL_n, proven in full generality in [20]. This rests on a well-known promotion process for depth zero supercuspidal representations of GL_n that preserves the order of local Brauer obstructions:

Theorem 4. *Let K be a p-adic field, and let k be its residue field. Choose a uniformizer ϖ_K of K. Then, there is a bijective correspondence between the set of cuspidal representations of $\mathrm{GL}_n(k)$ and the set of depth zero irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ with central character sending ϖ_K to 1. Furthermore, this correspondence preserves the order of local Brauer obstructions.*

Furthermore, by using the Local Langlands Correspondence for GL_n, we prove the triviality of ℓ-adic obstructions of depth zero supercuspidal representations, for ℓ not equal to the residue characteristic.

Theorem 7. *Given a uniformizer ϖ_K, for $\ell \neq p$, there is no ℓ-adic Brauer obstruction for any depth zero supercuspidal representation of $\mathrm{GL}_n(K)$ whose central character sends ϖ_K to a rational number.*

The combination of the two theorems implies the desired conclusion, as already proven in [24].

Theorem 8. *For $\ell \neq \operatorname{char} \mathbb{F}_q$, a cuspidal irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ has no ℓ-adic Brauer obstruction.*

Then, we use the "tame regular" Local Langlands Correspondence for quasisplit unramified p-adic groups, proven in [11]. In particular, the construction of the correspondence for semisimple groups is explicit and behaves nicely, and we can in particular get much information on the Brauer obstructions of generic supercuspidal depth zero representations whose corresponding Langlands parameters are elliptic regular. This is the case, for example, when the generic representation occurs as a Deligne-Lusztig representation. In particular, for split classical p-adic groups whose Langlands duals are closed subgroups of special linear groups, we prove a triviality of ℓ-adic Brauer obstructions for some generic depth zero supercuspidal representation.

Theorem 13. *Let G be either SO_{2n+1}, Sp_{2n}, SO_{2n} or PGL_n. Let $\ell \neq p$ be finite primes, and let $q = p^k$ a prime power. Then, every generic cuspidal representation of $G(\mathbb{F}_q)$ which appears as a Deligne-Lusztig representation has a trivial ℓ-adic Brauer obstruction. In particular, every generic cuspidal representation of $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$ and $\mathrm{PGL}_n(\mathbb{F}_q)$ has a trivial ℓ-adic Brauer obstruction.*

The paper is structured as follows. In Section 2, some details of the classical approach to Brauer obstructions will be explored. In Section 3, we will review the details of the promotion process for GL_n, and prove that the promotion process preserves the order of local Brauer obstructions. In Section 4, we will exploit the idea of using the Local Langlands Correspondence, In particular, we prove the triviality of ℓ-adic Brauer obstructions of depth zero supercuspidal representations of GL_n, by using the Local Langlands Correspondence for GL_n. In Section 5, we will use the parametrized version of Local Langlands Correspondence proven in [11]. After discussing the explicit construction of the Local Langlands Correspondence for semisimple groups, we will deduce various results regarding Galois equivariance and Brauer obstructions. Moreover, we will investigate the limitations of using of the Local Langlands Correspondence in this setting.

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2. Algebraic Methods for Calculating Brauer Obstructions

In this section, we will explore the various algebraic ways of understanding Brauer obstructions and how such interpretations give a hands-on way to derive some conclusions about Brauer obstructions. This section will give a flavor of the "classical algebraic" approach to finding the orders of Brauer obstructions of absolutely irreducible representations of finite groups.

In classical references, the notion of Schur index is often more widely used than that of Brauer obstruction. We first define the *Schur index* classically, and show that this notion matches up with other possible definitions, including the order of the Brauer obstruction.

Definition. *Let G be a finite group, and let (V,ρ) be an absolutely irreducible representation of G over a field F of characteristic zero. Let K be a subfield of F. Then, the KG-module $A = \sum_{g\in G} K\rho(g) \subset \mathrm{End}(V)$ is a central simple algebra with center $K(\rho)$* [10, Lemma 70.8]*. We define the order of $[A] \in \mathrm{Br}(K(\rho))$ as the* **Schur index** *of ρ over K, and denote it by $m_K(\rho)$.*

It should be noted that, from above definition, we can discuss the Schur index over both a number field and a non-archimedean local field. For a place v of K, we often call $m_{K_v}(\rho)$ the *v-local Schur index* of ρ over K. For simplicity, the (local) Schur index over $\mathbb{Q}$ would be simply called as the *(local) Schur index*.

There are several equivalent characterizations of Schur index:

Theorem 1. *Let G be a finite group, and let ρ be an irreducible complex representation of G. For a number field K, the following values are the same.*

(1) ("Schur Index") The Schur index $m_K(\rho)$.

(2) ("Period") The period $\sqrt{[D : K(\rho)]}$, where D is the division algebra Brauer equivalent to $A = \sum_{g\in G} K\rho(g)$.

(3) ("Splitting Field") The minimum value of $[L : K(\rho)]$ among fields $L/K(\rho)$ over which an irreducible L-representation ρ' exists so that $\rho' \cong \rho$ as complex representations.

(4) ("Fraction") The minimum value of positive integer m such that $m\chi_\rho$ is the character of a $K(\rho)$-representation, where χ_ρ is the character of ρ.

(5) ("Brauer Obstruction") The order of the Brauer obstruction $[\psi_\rho] \in \mathrm{Br}(K(\rho))$, with ψ_ρ defined as in the introduction.

Proof. That (1) = (2) is the standard result that the period equals the index for global fields [7, Theorem 1.2.4.4]. That (1) = (3) follows from one of corollaries of the Grunwald-Wang Theorem [1, Theorem 10.6] that, given S a finite set of primes of a global field K' and $n_\mathfrak{p}$ a positive integer for each $\mathfrak{p} \in S$, if $K'_\mathfrak{p}$ allows some cyclic extension of degree $n_\mathfrak{p}$, then there is a cyclic extension L'/K' whose degree is the least common multiple of the $n_\mathfrak{p}$'s and that the completions $L'_\mathfrak{P}/K'_\mathfrak{p}$ have degree $n_\mathfrak{p}$ for all $\mathfrak{p} \in S$. In the setting of this problem, the local degrees are suggested by the orders of the local Brauer classes, as the global Brauer class vanishes if the local Brauer classes vanish, and as a field extension of local fields has an effect of multiplication by its extension degree as a map between their Brauer groups (even for ramified extensions!). Moreover, for a real archimedean place, the required local degree is either 1 or 2, as $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. Thus, we can use the Grunwald-Wang Theorem to derive (1) = (3). It is easy to see by definition that (4) divides (3), as the isomorphic L-representation ρ', if seen as a $K(\rho)$-representation, is a direct sum of $[L : K(\rho)]$ copies of ρ, for L is just a $[L : K(\rho)]$-dimensional vector space over $K(\rho)$. That (4) is equal to (3) is proved in [10, 70.5].

To establish (1) = (5), we note how the two classes $[A]$ and $[\psi_\rho]$ are related in $\mathrm{Br}(K(\rho))$. As an element of $H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/K(\rho)), \overline{\mathbb{Q}}^\times)$, $[A]$ is a cohomology class of a 2-cocycle constructed by fixing an isomorphism $\phi : A \otimes_{K(\rho)} L \xrightarrow{\sim} \mathrm{Mat}_m(L)$ for a splitting field L and measuring how this isomorphism differs (as elements of $\mathrm{Aut}_L(\mathrm{Mat}_m(L))$) after composing $1 \otimes \sigma : A \otimes_{K(\rho)} L \xrightarrow{\sim} A \otimes_{K(\rho)} L$ at the second place by Galois automorphisms $\sigma \in \mathrm{Gal}(L/K(\rho))$. On the other hand, $[\psi_\rho]$ is constructed similarly, with the difference that a corresponding 2-cocycle is constructed from twists $A \otimes_{K(\rho)} L \xrightarrow{\sim} A \otimes_{K(\rho)} L$ at the first place, namely $A \cong A^\sigma$ for $\sigma \in \mathrm{Gal}(L/K(\rho))$. Thus, the product of the two classes is cohomologously trivial, which implies that $[A] = -[\psi_\rho]$. Thus (1) = (5) follows. ☐

Note that the above characterization also applies to the local Schur indices. Indeed, the Brauer class defined in a sense of the Schur index definition above respects the localization map of Brauer groups. Moreover, in the above proof, we showed that $[A] = -[\psi_\rho]$ as Brauer classes, which is more than the Schur index equal to the order of the Brauer obstruction. Finally, the period is equal to the index for a local field since for a non-archimedean local field a Brauer class of order n is split by a unique cyclic unramified extension of degree n [29, XII.3], giving the corresponding division algebra a period-n cyclic algebra structure (and for $\mathbb{R}$ it is obvious).

In particular, the fractional characterization of Schur index gives it some elementary divisibility conditions, and the first characterization of Schur index gives a way to compute it algebraically. For example, by the above characterizations of Schur index, the following are immediate:

Corollary 1. *1. Keeping the notations as above, if ρ' is a nonzero K-representation of G, then $m_K(\rho) \mid \langle \rho, \rho' \rangle$. In particular, $m_{\mathbb{Q}}(\rho) \mid \dim \rho$.*

2. (Brauer-Speiser, [15]) A real-valued irreducible complex character of a finite group has the Schur index over $\mathbb{Q}$ at most 2.

Proof. To prove the first part, it is sufficient to prove for K-irreducible ρ'. Also, we can assume that $a = \langle \rho, \rho' \rangle$ is nonzero. For any $\sigma \in \mathrm{Gal}(K(\rho), K)$, as $\rho'^\sigma = \rho'$, so $\langle \rho^\sigma, \rho' \rangle = a$. Thus, it follows that $\rho' = a \sum_{\sigma \in \mathrm{Gal}(K(\rho)/K)} \rho^\sigma + \theta$ for some representation θ orthogonal to all twists of ρ by elements in $\mathrm{Gal}(K(\rho)/K)$. On the other hand, note that for any Galois extension L/K and an L-representation ϕ, $\sum_{\sigma \in \mathrm{Gal}(L/K)} \phi^\sigma$ is realizable over K. Applying this to $m_K(\rho)\rho$, which is a $K(\rho)$-representation by the fractional characterization of Schur index, we deduce that $\rho'' = m_K(\rho) \sum_{\sigma \in \mathrm{Gal}(K(\rho)/K)} \rho^\sigma$ is a K-representation. As a is nonzero and ρ' is a K-irreducible representation, ρ' is a constituent of ρ''. This implies that $\theta = 0$ and $a \le m_K(\rho)$. However, note also that $m_K(\rho)\rho$ is necessarily $K(\rho)$-irreducible, and $\rho' = a \sum_{\sigma \in \mathrm{Gal}(K(\rho)/K)} \rho^\sigma$ is a K-representation, which is also a $K(\rho)$-representation. Thus, $\langle \rho', m_K(\rho)\rho \rangle \ne 0$ implies that $m_K(\rho)\rho$ is a constituent of ρ', or $a = m_K(\rho)$, as desired.

For the second part, let χ be a real-valued irreducible complex character of a finite group G, and let ρ be the representation of G whose character is χ. Consider $\rho \otimes \rho$, which is an irreducible representation of $G \times G$. Its character is given as $\chi^2(g_1 \otimes g_2) = \chi(g_1)\chi(g_2)$. Note that, through diagonal embedding, $\langle \mathrm{Ind}_G^{G \times G} 1_G, \chi^2 \rangle_{G \times G} = \langle 1_G, \chi^2|_G \rangle_G = \langle \overline{\chi}, \chi \rangle_G = \langle \chi, \chi \rangle_G = 1$ by Frobenius reciprocity. Moreover, $\mathrm{Ind}_G^{G \times G} 1_G$ is realizable over $\mathbb{Q}$. Thus, $m_{\mathbb{Q}}(\chi^2) = 1$. Since the endomorphism algebra of $\rho \otimes \rho$ is the tensor square of that of ρ, it follows that the Brauer obstruction of $\rho \otimes \rho$ is twice the Brauer obstruction of ρ. As $m_{\mathbb{Q}}(\chi^2) = 1$, the Brauer obstruction of ρ is of order at most 2, as desired. ☐

The different characterizations of Schur index give some insight for local Schur indices as well. First of all, it is safe to use the term "p-local Schur index" for a rational prime p, thanks to the Benard's theorem:

Theorem 2 (Benard, [4, Theorem 1])**.** *Let $K/\mathbb{Q}$ be a finite abelian extension. Let p be a place of $\mathbb{Q}$ and let v, w be two places of K over p. If A is a simple component of $K[G]$ (i.e. the endomorphism* *algebra of some K-irreducible representation), then $K_v \otimes_K A$ and $K_w \otimes_K A$ have the same index in* $\mathrm{Br}(K)$*.*

Note that, since a representation of a finite group is realizable over a cyclotomic field containing a root of unity of order the size of the group [28, 12.3], the representation is in particular always realizable over an abelian extension. The term *p-local Schur index* of ρ, denoted as $m_p(\rho)$, will simply mean the order of the Brauer obstruction $[\psi_\rho]$ as an element in $\mathrm{Br}(\mathbb{Q}(\rho)_v)$, for any place v of $\mathbb{Q}(\rho)$ lying over a rational prime p, which is well-defined thanks to the Benard's theorem.

In many small cases, the p-local Schur index can be computed by using mod p modular representation theory. In principle, if we know the representation theory and the mod p representation theory of a finite group, it is often sufficient in characterizing the p-local Schur indices. For example, we have:

Theorem 3 (Benard, [3])**.** *Suppose that an irreducible complex character χ of a finite group G lies in a p-block with cyclic defect groups. If φ is any irreducible Brauer constituent of χ, then $m_p(\chi) = [\mathbb{Q}_p(\chi, \varphi) : \mathbb{Q}_p(\chi)]$.*

As a reminder, a *p-block* is a summand in the unique decomposition of $\overline{\mathbb{F}_p}[G]$ into indecomposable 2-sided ideals. Given a finitely generated $\overline{\mathbb{Q}_p}$-module V, take any lattice (i.e. open compact subgroup) L inside V, and consider the reduction $L \otimes \overline{\mathbb{F}_p}$. The resulting $\overline{\mathbb{F}_p}$-module has composition factors independent of the choice of L thanks to the Brauer-Nesbitt theorem. Furthermore, any composition factor coming from a $\overline{\mathbb{Q}_p}$-irreducible representation belongs to a single p-block [10, Theorem 85.5]. Thus, the notion of a p-block containing an irreducible complex representation is well-defined. For each block, one can define the associated p-subgroup of G called the *defect group* of the block [10, Definition 87.18], which roughly measures the complexity of the block algebra; for example, the defect group is trivial if and only if the block algebra is a full matrix algebra.

In many small hands-on computable cases, defect groups are cyclic, so the above theorem is useful in computing p-local Schur indices in such cases. However, it is hard to get a handle on the general algebra structures of p-blocks as defect groups become complicated. To review the properties of p-blocks with cyclic defect groups and implications on p-local Schur indices, see [16, Chapter VII].

We now briefly explain how these types of results can calculate (local) Schur indices of finite groups of Lie type, for example as in [24]. For a Chevalley group, by considering the involution structures such as transpose-inverse and extending the group by order 2 if necessary, an element is similar to its inverse in the possibly augmented group. As a result, for most cases, the Schur indices must divide 2 [18, Theorem 2.9]. Therefore, for most classical groups, one only needs to check the triviality (or nontriviality) of (local) Schur indices, and this can be achieved by heavily exploiting the divisibility results, in a similar vein as above.

3. Depth Zero Supercuspidal Representations of GL_n

As laid out in the introduction, although one can calculate the Brauer obstructions of several classical groups by only using classical algebraic techniques, it is highly desirable to have a more general perspective on the local Brauer obstructions. It is therefore reasonable, for a Chevalley group G, to try to relate representation of $G(\mathbb{F}_q)$, which are inherently discrete, to more "continuous" objects, namely representations of $G(F)$ for some p-adic field. This should be done by going through some natural promotion process which preserves some data of Brauer obstruction.

Before starting the main discussion, we first have to note how several notions defined in the case of finite group representations translate into the setting of p-adic group representations. First, for a locally profinite group G and an absolutely irreducible smooth representation ρ of G, we can define the *trace field* $\mathbb{Q}(\rho)$ as the fixed field of $\{\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C}) \mid \rho \cong \rho^\sigma\}$, thanks to the Schur's lemma over

$\mathbb{C}$. Similarly, for any algebraic field extension $K/\mathbb{Q}$, we can define the trace field over K, denoted as $K(\rho)$, in the same way. In general, this field might not be algebraic over K, making the notion of Brauer obstruction not applicable for all cases. Nevertheless, if we recall how the notion of trace field was used in the original definition of Brauer obstruction for finite group representations, what we need is one side of the Galois correspondence, namely $\mathrm{Aut}_{K(\rho)}(\mathbb{C}) = \{\sigma \in \mathrm{Aut}_K(\mathbb{C}) \mid \rho \cong \rho^\sigma\}$. This is not a priori automatic. However, if the representation is definable over $\overline{\mathbb{Q}}$, then both groups will contain $\mathrm{Aut}_{\overline{\mathbb{Q}}}(\mathbb{C})$ as normal subgroups, and their quotients will be the same by our usual Galois theory in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus, the original definition of Brauer obstruction translates without change to absolutely irreducible smooth admissible $\overline{\mathbb{Q}}$-representations of locally profinite groups. The other characterization of the Schur index that will be useful in this setting is the splitting field characterization. This is plausible by noting that, even for smooth absolutely irreducible representations, the definition of Brauer obstruction $[\psi_\rho] \in \mathrm{Br}(K(\rho))$ is natural with respect to K and that ρ is definable over $K(\rho)$ if and only if $[\psi_\rho]$ is trivial as an element of $\mathrm{Br}(K(\rho))$; the splitting field characterization is then again the consequence of the Grunwald-Wang theorem.

To continue the well-posedness of the notions of trace field and Brauer obstructions, any representation of a locally profinite group will be defined over $\overline{\mathbb{Q}}$ for the rest of the paper, unless otherwise noted. This is in particular not a restriction, as we will only consider either depth zero representations or complex characters of a torus.

To relate finite group representations with p-adic group representations, one has to consider *depth zero* representations of reductive p-adic groups. In general, by [23], the *depth* of a smooth representation of a reductive p-adic group is defined via the Moy-Prasad filtrations. In particular, all depth zero supercuspidal representations of a reductive p-adic group are characterized and constructed through this concept. However, what we truly need is how to safely promote finite group representations to *some* depth zero representations. We also need some sense of bijectivity in the correspondence, as otherwise trace fields might get smaller during the induction process. In other words, there should be no extra Galois automorphism of $\overline{\mathbb{Q}}$ fixing the isomorphism class of the induced representation other than those coming from the original finite group representation. Thus, we will later use a different characterization along the lines of Local Langlands Correspondence for quasisplit groups in concern, not just restricted to general linear groups; see Section 5. For the case of general linear groups, however, the general construction fits well as the construction arising in the general context gives a bijective correspondence. This section will explore the general construction of depth zero supercuspidal representations of GL_n, which predates the more general [23]. Eventually, we will prove that we can safely promote cuspidal representations of finite general linear groups to some classes of depth zero supercuspidal representations of general linear groups over a p-adic field.

In this section, we will only consider cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ for the simplicity of discussion. We will use the general method of obtaining irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ via compactly inducing irreducible characters from compact modulo center subgroups, as explained in [5, §6]. In particular, for the level zero supercuspidal representations, we can take the compact modulo center subgroup to be $K^\times \mathrm{GL}_n(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in K. We give an explicit map as follows. Let π be an irreducible cuspidal representation of $G(k)$. As $\mathrm{GL}_n(\mathcal{O}_K)/(1 + \mathcal{O}_K \mathrm{Mat}_n(\mathcal{O}_K)) \cong \mathrm{GL}_n(k)$, where $\mathfrak{p}_K$ is the maximal ideal of $\mathcal{O}_K$, we can regard π as a representation of $\mathrm{GL}_n(\mathcal{O}_K)$. Extend this to a representation $\widetilde{\pi}$ of $K^\times \mathrm{GL}_n(\mathcal{O}_K)$ by requiring that $\widetilde{\pi}(\omega_K^i g) = \pi(g)$ for a chosen uniformizer ϖ_K and $i \in \mathbb{Z}$ (Note that this definition is well-defined). Finally, compactly induce $\widetilde{\pi}$ to $\mathrm{GL}_n(K)$. Then, $\mathrm{cInd}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)} \widetilde{\pi}$ is a level zero irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ with central character sending ϖ_K to 1, and conversely any such representation arises uniquely as this construction.

We want to show that this bijective process does no harm on the trace fields and the Brauer obstructions. Note that the process $\pi \mapsto \widetilde{\pi}$ preserves the trace field and the Brauer obstruction, as it is constructed group theoretically by composing $\mathrm{GL}_n(\mathcal{O}_K) \to \mathrm{GL}_n(k)$ and trivially extending over $K^\times$. In particular, the process is canonical in a sense that it is independent of the value of the representations. Thus, we only need to show that $\widetilde{\pi} \mapsto \mathrm{cInd}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n}(\widetilde{\pi}) := I(\widetilde{\pi})$ preserves the trace field and the Brauer obstruction. First, notice that as the compact induction is Galois equivariant, the trace field of $I(\widetilde{\pi})$ is a subfield of the trace field of $\widetilde{\pi}$. As the correspondence is bijective, we can conclude that any Galois twist that fixes the trace field of $I(\widetilde{\pi})$ also fixes the isomorphism class of $\widetilde{\pi}$, thereby fixing the trace field of $\widetilde{\pi}$. This shows that the correspondence preserves the trace fields.

Moreover, note that the compact induction used here is also a smooth induction, as the quotient $\mathrm{GL}_n(K)/K^\times \mathrm{GL}_n(\mathcal{O}_K)$ is compact. Recall that both smooth induction and compact induction have canonical maps; namely, for a compact induction of a representation σ of $H \subset G$, there is a canonical map $\alpha_\sigma^c : \sigma \to \mathrm{cInd}_H^G \sigma$ sending $w \mapsto \phi_w$, where ϕ_w is supported in H and $\phi_w(h) = \sigma(h)w$ for $h \in H$; for a smooth induction of a representation σ of $H \subset G$, there is a canonical map $\alpha_\sigma : \mathrm{Ind}_H^G \sigma \to \sigma$ sending $\phi \mapsto \phi(1)$. Thus, for any cuspidal representations π, π' of $\mathrm{GL}_n(\mathbb{F}_q)$, we have a canonical map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{GL}_n(K)}\left(\mathrm{cInd}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)}(\widetilde{\pi}), \mathrm{cInd}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)}\left(\widetilde{\pi'}\right)\right) &\to \mathrm{Hom}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}\left(\widetilde{\pi}, \widetilde{\pi'}\right) \\ &= \mathrm{Hom}_{\mathrm{GL}_n(k)}\left(\pi, \pi'\right), \end{aligned}$$

defined by $f \mapsto \alpha_\pi \circ f \circ \alpha_{\pi'}^c$. This is an isomorphism by using two Frobenius reciprocities, one for smooth induction and one for compact induction. To be more specific, a map $f' : \widetilde{\pi} \to \widetilde{\pi'}$ is the image, via the above canonical map, of an element which sends $\phi_w \in \mathrm{cInd}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)}(\widetilde{\pi})$ to $\phi_{f(w)}$. Thus, the construction of Brauer obstructions for both $\widetilde{\pi}$ and $\mathrm{cInd}_{K^\times \mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n}(\widetilde{\pi})$ would yield the same cohomology class. To summarize, we have shown the following:

Theorem 4. *Let K be a p-adic field, and let k be its residue field. Choose a uniformizer ϖ_K of K. Then, there is a bijective correspondence between the set of cuspidal representations of $\mathrm{GL}_n(k)$ and the set of depth zero irreducible supercuspidal representations of $\mathrm{GL}_n(K)$ with central character sending ϖ_K to 1. Furthermore, this correspondence preserves the order of local Brauer obstructions.*

Remark. It should be noted that there is another argument showing the triviality of the Brauer obstruction for *all* smooth irreducible admissible representations of $\mathrm{GL}_n(K)$ for a p-adic field K, based upon the argument of [33]. This is possible by *the theory of new vectors* available for general linear groups. To define what new vectors are, first let $K_1(t)$ be a subgroup of $\mathrm{GL}_n(\mathcal{O}_K)$ consisted of matrices whose last row is congruent to $(0\ 0\ \cdots\ 0\ 1)$ modulo $\mathfrak{p}_K^t$. Then, for any irreducible smooth admissible representation (π, V) of $\mathrm{GL}_n(K)$, V has a unique nonzero $K_1(f(\pi))$-fixed vector up to constant, where $f(\pi)$ is the conductor of π [8, Theorem 6.3]. This is called the space of *(local) new vectors*. Now the triviality of Brauer obstructions for (π, V) is immediate by the following lemma.

Lemma [33, Lemme I.1]**.** *Let (π, V) be an irreducible complex representation of a group G, and let H be a subgroup of G. For a character $\chi \in \mathrm{Hom}(H, \mathbb{Q}(\pi)^\times)$, suppose the χ-eigenspace of V over H*

$$V^{H,\chi} = \{v \in V \mid \forall h \in H, \pi(h)v = \chi(h)v\}$$

is one-dimensional. Then, there is a G-stable $\mathbb{Q}(\pi)$-subspace $V^0 \subset V$ such that $V = V^0 \otimes_{\mathbb{Q}(\pi)} \mathbb{C}$.

Proof. Pick a nonzero $v \in V^{H,\chi}$, and let V^0 be the vector space generated over $\mathbb{Q}(\pi)$ by $\{\pi(g)v\}_{g \in G}$. As (π, V) is irreducible, to prove that $V = V^0 \otimes_{\mathbb{Q}(\pi)} \mathbb{C}$, we only need to show that the natural map $V^0 \otimes_{\mathbb{Q}(\pi)} \mathbb{C} \to V$ is injective. Suppose not. Then, there must be elements $v_1, \cdots, v_n \in V^0$, linearly independent over $\mathbb{Q}(\pi)$, and $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ such that $\sum_{i=1}^n \lambda_i v_i = 0$. Pick such elements so that n

is minimal. Also, without loss of generality, we can assume that $\lambda_1 = 1$ and $\lambda_2 \notin \mathbb{Q}(\pi)$. As $\mathbb{Q}(\pi)$ is the fixed field of the group $\{\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C}) \mid \pi^\sigma \cong \pi\}$, there must be $\sigma' \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{C})$ such that $\sigma'(\lambda_2) \neq \lambda_2$ whereas $\pi^{\sigma'} \cong \pi$. By Schur's lemma, there is a $\mathbb{C}$-linear map $t : V \to V$ such that $\pi^{\sigma'}$ is a conjugation of π by t, and we have another linear dependence relation $\sum_{i=1}^n \sigma'(\lambda_i)t(v_i)$. On the other hand, as χ is $\mathbb{Q}(\pi)$-valued, $t(v)$ is also in $V^{H,\chi}$. Since $t(v)$ is nonzero as well as $V^{H,\chi}$ is one-dimensional, $t(v)$ is a nonzero scalar multiple of v. After scaling, we can assume that $t(v) = v$. Then, $t(\pi(g)v) = \pi(g)v$ for every $g \in G$, which means that $\sum_{i=1}^n \sigma'(\lambda_i)v_i = 0$. Subtracting it from the original linear dependence relation, we have $\sum_{i=1}^n (\lambda_i - \sigma'(\lambda_i))\, v_i = 0$. As $\lambda_1 = 1 = \sigma'(\lambda_1)$ and $\lambda_2 \neq \sigma'(\lambda_2)$, this contradicts with the minimality of n. □

The theory of local new vectors implies $\dim_{\mathbb{C}} V^{K_1(f(\pi)),1} = 1$, so by the above Lemma, every irreducible smooth admissible representation of $\mathrm{GL}_n(K)$ has a trivial Brauer obstruction. In particular, this gives another proof of triviality of Brauer obstructions for cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$.

4. The Local Langlands Correspondence

One possibility for a more systematic treatment of calculating Brauer obstructions of finite groups of Lie type would be done by using the Local Langlands Correspondence. In this section, we want to observe how the Local Langlands Correspondence can be used heuristically to lay some insight on local Brauer obstructions of $\mathrm{GL}_n(\mathbb{F}_q)$. The Local Langlands Correspondence (LLC) is a conjecture/philosophy about reductive algebraic groups. There are several versions of it, but one possible formulation is as follows.

Conjecture (Local Langlands Correspondence)**.** *Let G be a reductive algebraic group over a p-adic field K. Then, there is a one-to-finite correpondence from the set* $\mathrm{Hom}(W_K', {}^LG)$ *of continuous homomorphisms of the Weil-Deligne group W_K' of K into the Langlands dual group ${}^LG$ of G onto the set of isomorphism classes of smooth admissible irreducible representations of $G(K)$, respecting several desired properties.*

In particular, a finite set of smooth admissible representations of $G(K)$ corresponding to one element of $\mathrm{Hom}(W_K', {}^LG)$ through this correspondence is called an *L-packet.*

Heuristically, the Local Langlands Correspondence can be used in determining the Brauer obstructions of depth zero representations of reductive algebraic groups as the depth zero structure of Weil-Deligne group is significantly simpler than that of reductive p-adic groups. In particular, from the already-established Local Langlands Correspondence for GL_n by [20], we prove the triviality of local Brauer obstructions of cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ at finite places over $\ell \neq p$.

4.1. **The Local Langlands Correspondence for Tori.** Before diving into the Local Langlands Correspondence for general linear groups, we first want to observe how the abelian case—in other words, the case of tori—works in the framework of Local Langlands Correspondence. The Local Langlands Correspondence for tori will be stated and investigated here for later purpose.

Before examining the Local Langlands Correspondence for tori, we obviously need a precise statement of it. We first need to make things clear on which definition of *Langlands dual* we will be using throughout the paper, as there are several different accepted definitions. Let G be a reductive algebraic group over a field K, and let $T \subset G$ be a maximal K-torus. Then, there is the finite Galois splitting field extension K'/K of T, making $G_{K'}$ split as well. On the splitting field, we can define *the root datum* $(X^*, \Phi, X_*, \Phi^\vee)$ as follows: $X^* = \mathrm{Hom}(T_{K'}, \mathbb{G}_m)$, $X_* = \mathrm{Hom}(\mathbb{G}_m, T_{K'})$ are the character and cocharacter lattices, respectively, and Φ and $\Phi^\vee$ are the set of roots and coroots, respectively. Note that *the dual root datum* $(X_*, \Phi^\vee, X^*, \Phi)$ also satisfies the axioms of root datum (cf. [30, 7.4]), and by the Existence Theorem (cf. [30, Theorem 10.1.1]) and the Isomorphism Theorem (cf. [30, Theorem 9.6.2]), there exists uniquely up to isomorphism a complex reductive algebraic group $\widehat{G}$ (and a maximal torus $\widehat{T} \subset \widehat{G}$ depending on the choice of $T \subset G$) whose associated root datum is $(X_*, \Phi^\vee, X^*, \Phi)$. We define *the Langlands dual group* ${}^LG = \mathrm{Gal}(K'/K) \ltimes \widehat{G}$, where the action of $\mathrm{Gal}(K'/K)$ on $\widehat{G}$ follows from the $*$-*action* [31, 2.3], defined as follows. Given a root basis Δ of a set of positive roots Φ^+ of G, consider its image through the natural Galois action of $\mathrm{Gal}(K'/K)$ induced from base change morphism on the character lattice X^*. For $\sigma \in \mathrm{Gal}(K'/K)$, as $\sigma(\Phi^+)$ is also a positive system of roots, there uniquely exists an element $w_\sigma \in N_G(T)(K')/T(K')$ of the Weyl group such that $w_\sigma(\sigma(\Phi^+)) = \Phi^+$ and accordingly $w_\sigma(\sigma(\Delta)) = \Delta$. Let the $*$-action $\rho : \mathrm{Gal}(K'/K) \to \mathrm{Aut}(R, \Delta)$ be defined as $\sigma \mapsto w_\sigma \circ \sigma$, where R is the root datum. By the Isomorphism theorem, up to the choice of pinning, $\mathrm{Aut}(R, \Delta)$ embeds into $\mathrm{Aut}(\widehat{G})$, which is the desired action of $\mathrm{Gal}(K'/K)$ on $\widehat{G}$. Note that the choice of pinning can possibly conjugate the homomorphism, so ${}^LG$ is well-defined up to $\widehat{G}(\mathbb{C})$-conjugation. It should be noted that in this paper the Langlands dual groups will only be used when G is either a torus or a quasi-split group, and for both cases the $*$-action is just the natural Galois action.

Now we try to precisely formulate the Local Langlands Correspondence for tori, proven by Langlands in [22]. Let T be a torus over a p-adic field K. First, the algebraic group side is quite clear, as the smooth admissible irreducible representations of $T(K)$ must be 1-dimensional. Thus, one side of the correspondence should be the set of continuous group homomorphisms $T(K) \to \mathbb{C}^\times$. On the Galois side, notice that, for the abelian cases, one can work with Weil groups instead of Weil-Deligne groups, as the corresponding nilpotent matrix automatically has to be zero. A representation we are concerned about in the Galois side is a continuous homomorphism $\phi : W_K \to {}^LT$ which is compatible with the Galois actions. More precisely, the condition is that the map $\mathrm{pr}_1 \circ \phi : W_K \to \mathrm{Gal}(K'/K)$ is the natural surjection, where pr_1 is the projection map to the first factor. This condition forces that $\phi|_{W_{K'}}$ should factor through $\widehat{T}$, and this implies that ϕ factors through $W_{K'}^{\mathrm{ab}}$. Thus, ϕ should factor through $W_{K'/K}$, the extension $1 \to K'^\times \to W_{K'/K} \to \mathrm{Gal}(K'/K) \to 1$ corresponding to the canonical class in $H^2(\mathrm{Gal}(K'/K), K'^\times)$. Letting the Weil group $W_{K'/K}$ act on $\widehat{T}$ through $\mathrm{pr}_1 \circ \phi$, $\mathrm{pr}_2 \circ \phi : W_{K'/K} \to \widehat{T}$ becomes a continuous 1-cocycle. Also, conjugation of ϕ by an element of $\widehat{T}$ changes $\mathrm{pr}_2 \circ \phi$ precisely by a 1-coboundary. Thus, $H^1_c(W_{K'/K}, \widehat{T})$ is tautologically bijective with the inner conjugacy classes of continuous group homomorphisms $W_{K'/K} \to {}^LT$. We can now state the Local Langlands Correspondence for tori as follows.

Theorem 5 (Local Langlands Correspondence for Tori, [22, Theorem 2(a)])**.** *There is a canonical isomorphism between $H^1_c(W_{K'/K}, \widehat{T})$ and $\mathrm{Hom}_c(T(K), \mathbb{C}^\times)$, the set of continuous group homomorphisms $T(K) \to \mathbb{C}^\times$.*

What does it mean for the correspondence to be canonical? We will justify the canonicity by examining the case of split tori and observing how the general case and the split case are compatible with each other.

Without proving the bijectivity, we construct the desired correspondence. First, note that given the character group $L = \mathrm{Hom}(T_{K'}, \mathbb{G}_m)$ of $T_{K'}$, we can recover $T(K') = \mathrm{Hom}_{\mathbb{Z}}(L, K'^\times)$ and $T(K) = T(K')^{\mathrm{Gal}(K'/K)} = \mathrm{Hom}_{\mathbb{Z}}(L, K'^\times)^{\mathrm{Gal}(K'/K)}$. In terms of the dual lattice $\widehat{L}$, $T(K') = K'^\times \otimes \widehat{L}$ and $T(K) = (K'^\times \otimes \widehat{L})^{\mathrm{Gal}(K'/K)}$. In [22, pp. 241], it is proved that, the restriction map

$$\mathrm{Res} : H_1(W_{K'/K}, \widehat{L}) \to H_1(C_{K'}, \widehat{L})^{\mathrm{Gal}(K'/K)}$$

is an isomorphism, where $C_{K'} = K'^\times$ as we are only dealing with the local case. Notice also that $H_1(K'^\times, \widehat{L}) = K'^\times \otimes \widehat{L}$ as the action is trivial. Thus, the restriction map matches $T(K)$ with $H_1(W_{K'/K}, \widehat{L})$. From the natural pairing

$$H^1_c(W_{K'/K}, \widehat{T}) \times H_1(W_{K'/K}, \widehat{L}) \to H_0(W_{K'/K}, \mathbb{C}^\times) = \mathbb{C}^\times,$$

associated with the natural pairing $\widehat{T} \times \widehat{L} \to \mathbb{C}^\times$, we can construct a continuous character $T(K) \to \mathbb{C}^\times$ provided an element of $H^1_c(W_{K'/K}, \widehat{T})$. The facts that both the restriction map above and the correspondence are isomorphisms can be proven at the level of local class field theory, and the reader is referred to [22].

The above correspondence in the split case is very easy. Suppose that the torus T splits over K, so we can take $K' = K$. Starting from $\phi : W_K \to (\mathbb{C}^\times)^r$, it should factor through W_K^{ab}, and the local class field theory gives the local Artin map $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$, which is an isomorphism. Thus, we are given $\phi \circ \mathrm{Art}_K : K^\times \to (\mathbb{C}^\times)^r$. Let the character corresponding to the i-th coordinate be denoted as ψ_i. Then the pairing in turn gives a character $\Psi : \mathbb{G}_m^r(K) \to \mathbb{C}^\times$ which is defined as $\Psi(a_1, \cdots, a_r) = \psi_1(a_1) \cdots \psi_r(a_r)$. Later, we will see that the $r = 1$ case of this correspondence is indeed the Local Langlands Correspondence for GL_1; the correspondence is given by composing with the local Artin isomorphism.

The canonicity of the correspondence can be justified as follows. Using the notation as above, we have the following commutative diagram

$$\begin{tikzcd} H^1_c(W_{K'/K}, \widehat{T}) \arrow[r, "\mathrm{LLC}"] \arrow[d, hook, "\mathrm{Res}"'] & \mathrm{Hom}_c(T(K), \mathbb{C}^\times) \arrow[d, hook] \\ H^1_c(W_{K'/K'}, \widehat{T}) \arrow[r, "\mathrm{LLC}"] & \mathrm{Hom}_c(T(K'), \mathbb{C}^\times) \end{tikzcd}$$

where the left vertical map is induced from the inclusion $W_{K'/K'} = K'^\times \hookrightarrow W_{K'/K}$, and the right vertical map is induced from the Galois averaging map $T(K') \to T(K)$. The elements in the bottom which are in the image of the vertical inclusions are characterized as the $\mathrm{Gal}(K'/K)$-invariant elements. Put in other words, for a K-torus T with splitting field K', the two Local Langlands Correspondence maps for T, as a K-torus and as a (split) K'-torus, are the same on the common domain $H^1_c(W_{K'/K}, \widehat{T})$.

4.2. **The Local Langlands Correspondence for** GL_n**.** From now on, ℓ is a prime different from p, unless otherwise noted. In proving the triviality of local Brauer obstructions of $\mathrm{GL}_n(\mathbb{F}_q)$, we will use the Local Langlands Correspondence for GL_n, proven by Michael Harris and Richard Taylor in [20]. To specifically observe how the Local Langlands Correspondence of GL_n gives an insight on determining ℓ-adic Brauer obstructions of GL_n, we refer to how the blueprint for proving Local Langlands Correspondence map of GL_n was first laid out. In his celebrated article [6], Henri Carayol conjectured decompositions of representations coming from vanishing cycle sheaves and ℓ-adic cohomology of Drinfeld's rigid-analytic upper half space, respectively, which realize both the Jacquet-Langlands correspondence and the Local Langlands Correspondence simultaneously. Roughly speaking, specifically for the vanishing cycle case, for each piece with a fixed central character, he conjectured the decomposition of form

$$\bigoplus \pi \otimes \mathrm{JL}^{-1}(\pi^\vee) \otimes r_\ell(\pi),$$

where the sum is over the discrete representations of GL_n over a p-adic field with a given central character, JL is the Jacquet-Langlands correspondence and r_ℓ is the normalized Local Langlands Correspondence, twisted by an unramified character so that it becomes equivariant under Galois automorphisms of the coefficient field $\overline{\mathbb{Q}_\ell}$. Since the vanishing cycle sheaf representation is realized over $\mathbb{Q}_\ell$, any of its $\mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$-twist is canonically isomorphic to itself. Thus, by considering the isotypic parts with central charcater trivial on the fixed uniformizer ϖ_K, which is all we need for our problem, we can calculate the Brauer obstruction of π in terms of the Brauer obstructions of $\mathrm{JL}^{-1}(\pi^\vee)$ and $r_\ell(\pi)$.

We now summarize the precise construction. Let k be the residue field of K. For any $n \geq 1$, there is a unique one-dimensional formal $\mathcal{O}_K$-module $\Sigma_{K,n}$ over $\overline{k}$ of $\mathcal{O}_K$-height g. For the functor sending an Artinian local $\mathcal{O}_K$-algebra with residue field $\overline{k}$ to the set of isomorphism classes of deformations of $\Sigma_{K,n}$ to A, there is a complete noetherian local $\mathcal{O}_K$-algebra $R_{K,n}$ with residue field $\overline{k}$ prorepresenting it. Also, $\mathrm{End}_{\mathcal{O}_K}(\Sigma_{K,n}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the division algebra, denoted $D_{K,n}$, with center K that goes to $1/n$ under the invariant map. It is shown in [14] that, for any integer $m \geq 0$, there is a finite flat $R_{K,n}$-algebra $R_{K,n,m}$ over which the universal deformation of $\Sigma_{K,n}$ has a universal Drinfeld level p^m-structure.

Let $\Psi^i_{K,\ell,n,m}$ be the i-th vanishing cycle sheaf of $(\mathrm{Spf}\, R_{K,n,m})^{\mathrm{an}}$ with coefficients in $\overline{\mathbb{Q}_\ell}$, and let $\Psi^i_{K,\ell,n} = \varinjlim_m \Psi^i_{K,\ell,n,m}$. The obvious actions of $\mathrm{GL}_n(K)$, $\mathcal{O}^\times_{D_{K,n}}$ and I_K, the inertia group of K, on the tower $\{\Psi^i_{K,\ell,n,m}\}_m$ gives rise to the action ([6, 1.4]) of $A_{K,n}$ on $\Psi^i_{K,\ell,n}$, where $A_{K,n}$ is the group of triples $(\gamma, \delta, \sigma) \in \mathrm{GL}_n(K) \times D^\times_{K,n} \times W_K$ such that $v_K(\det \gamma) = v_K(\det \delta) + v_K(\sigma)$. This action is in fact admissible [20, Lemma II.2.8]. Therefore, for any irreducible admissible representation ρ of $D^\times_{K,n}$ over $\overline{\mathbb{Q}_\ell}$, $\Psi^i_{K,\ell,n}(\rho) = \mathrm{Hom}_{\mathcal{O}^\times_{D_{K,n}}}(\rho, \Psi^i_{K,\ell,n})$ is an admissible $\mathrm{GL}_n(K) \times W_K$-module.

How does this connect with the construction of r_ℓ? Let $[\Psi_{K,\ell,n}(\rho)]$ be the virtual $\mathrm{GL}_n(K) \times W_K$-module defined as $[\Psi_{K,\ell,n}(\rho)] = \sum_{i=0}^{g-1} (-1)^{n-1-i} [\Psi^i_{K,\ell,n}(\rho)]$. Then, if we denote the Jacquet-Langlands correspondence in a sense of Deligne, Kazhdan and Vigneras [12] as JL, in the Grothendieck group of $\mathrm{GL}_n(K) \times W_K$-modules, $[\Psi_{K,\ell,n}(\mathrm{JL}^{-1}(\pi^\vee))] = [\pi \otimes r_\ell(\pi)]$ for any irreducible supercuspidal representation π of $\mathrm{GL}_n(K)$. Finally, the Local Langlands Correspondence map $\mathrm{rec}_{K,n}$ is defined as $\mathrm{rec}_{K,n}(\pi) = r_\ell(\pi^\vee \otimes (|\ | \circ \det)^{(1-n)/2})$.

Even though the result of Harris and Taylor is not precisely as strong as the original formulation of Carayol, it is natural to expect the following result.

Theorem 6. *If π is an irreducible supercuspidal $\overline{\mathbb{Q}}$-representation of $\mathrm{GL}_n(K)$ such that the $W_K \times D^\times_{K,n}$-representation $r_\ell(\pi) \otimes \mathrm{JL}^{-1}(\pi^\vee)$ has a trivial ℓ-adic Brauer obstruction, then π also has a trivial ℓ-adic Brauer obstruction.*

Proof. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$. Then we have to prove that π, seen as an ℓ-adic representation, has trivial Brauer obstruction. From now on, every Galois action-related concept would implicitly be ℓ-adic (e.g. Brauer obstruction).

Let $\mathcal{U}^i_{K,\ell,n} = \mathrm{Ind}_{A_{K,n}}^{\mathrm{GL}_n(K) \times D^\times_{K,n} \times W_K} \Psi^i_{K,\ell,n}$. In [25] it is shown that, at least for irreducible supercuspidal representations, the situation is very nice so that one can expect the Carayol's conjecture to be true, i.e. the corresponding isotypic pieces $\Psi^i_{K,\ell,n}$'s and $\mathcal{U}^i_{K,\ell,n}$'s have no "cancellations." In precise terms, for π any irreducible supercuspidal representation of $\mathrm{GL}_n(K)$,

$$\mathcal{U}^i_{K,\ell,n}(\mathrm{JL}^{-1}(\pi^\vee)) = \begin{cases} \pi \otimes r_\ell(\pi) & i = n-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{U}^i_{K,\ell,n}(\tau)$ is a τ-isotypic piece of $\mathcal{U}^i_{K,\ell,n}$, for τ an irreducible representation of $D^\times_{K,n}$; note that by Frobenius reciprocity we have a canonical isomorphism $\mathcal{U}^i_{K,\ell,n}(\tau) \cong \Psi^i_{K,\ell,n}(\tau)$. Note that $\Psi^i_{K,\ell,n}$ and $\mathcal{U}^i_{K,\ell,n}$ are representations with $\overline{\mathbb{Q}}_\ell$-coefficients that can be also thought as the extension of scalars of representations over $\mathbb{Q}_\ell$. As both r_ℓ and JL are Galois equivariant bijections (cf. [26, Section 6]), π, $r_\ell(\pi)$ and $\mathrm{JL}^{-1}(\pi^\vee)$ all have the same trace field. Thus, the actions of π-fixing $\overline{\mathbb{Q}_\ell}$-automorphisms on $\mathrm{JL}^{-1}(\pi^\vee)$-isotypic piece of $\mathcal{U}^{n-1}_{K,\ell,n}$, which is just $\pi \otimes \mathrm{JL}^{-1}(\pi^\vee) \otimes r_\ell(\pi)$, satisfies the cocycle condition. Thus, $[\psi_{\pi,\mathbb{Q}(\rho)_v}] + [\psi_{\mathrm{JL}^{-1}(\pi^\vee)\otimes r_\ell(\pi),\mathbb{Q}(\rho)_v}] = 0$ as elements of $\mathrm{Br}(\mathbb{Q}(\rho)_v)$ for a place v of $\mathbb{Q}(\rho)$ over ℓ. In particular, π has a trivial ℓ-adic Brauer obstruction if and only if $\mathrm{JL}^{-1}(\pi^\vee) \otimes r_\ell(\pi)$ has a trivial ℓ-adic Brauer obstruction. $\square$

As the Local Langlands Correspondence preserves the depth [2, Proposition 4.2], by analyzing the depth zero structures of $D_{K,n}^\times$ and W_K, we can calculate the Brauer obstructions of depth zero irreducible representations of W_K and $D_{K,n}^\times$ relatively easily. Combining with the above theorem, we can deduce the following desired result.

Theorem 7. *Given a uniformizer ϖ_K, for $\ell \neq p$, there is no ℓ-adic Brauer obstruction for any depth zero supercuspidal representation of $\mathrm{GL}_n(K)$ whose central character sends ϖ_K to a rational number.*

Proof. By Theorem 6, it is sufficient to prove that $r_\ell(\pi) \otimes \mathrm{JL}^{-1}(\pi)^\vee$ has a trivial ℓ-adic Brauer obstruction for supercuspidal depth zero π. Let ω_π be the central character of π. As π is of depth zero, ω_π is completely determined by the value at ϖ_K, which we will denote as $c \in \mathbb{Q}^\times$. Obviously, r_ℓ, an unramified character twist of $\mathrm{rec}_{K,n}$, preserves the depth. Also, both correspondences characterizes central characters rather explicitly. Namely,

$$r_\ell(\pi) = \mathrm{rec}_{K,n}(\pi^\vee \otimes (|\ | \circ \det)^{(1-n)/2}) = \mathrm{rec}_{K,n}(\pi)^\vee \otimes (|\ | \circ \mathrm{Art}_K^{-1})^{(1-n)/2},$$

where $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ is the local Artin map, and $\det \mathrm{rec}_{K,n}(\pi) = \mathrm{rec}_{K,1}(\omega_\pi) = \omega_\pi \circ \mathrm{Art}_K^{-1}$, so that $\det(\mathrm{rec}_{K,n}(\pi)(\Phi_K)) = c$ for the corresponding geometric Frobenius $\Phi_K \in W_K$. Moreover, $\mathrm{JL}^{-1}(\pi)$ and π have the same central characters. By analyzing the representation theory of quotients of $D_{K,n}^\times$ and W_K by wild inertia groups, we shall deduce the theorem.

If we denote P_K as the wild inertia group in W_K, then $W_K/P_K = \prod_{p' \neq p} \mathbb{Z}_{p'} \rtimes \mathbb{Z}$ where the $\mathbb{Z}$ is generated by Φ_K and the action of it on $\prod_{p' \neq p} \mathbb{Z}_{p'}$ is defined as $\Phi_K^{-1} \tau \Phi_K = \tau^q$ for $\tau \in \prod_{p' \neq p} \mathbb{Z}_{p'}$. Since $r_\ell(\pi)$ is finite dimensional, we can think $r_\ell(\pi)$ as an n-dimensional irreducible (since π is supercuspidal) representation of $G = (\mathbb{Z}/m\mathbb{Z}) \rtimes \mathbb{Z}$ where $(m, p) = 1$. Based on the discussion about central character, we can think this G-representation as a product $\rho \otimes \psi$, where $\det \rho(\Phi_K) = c$ and $\psi(\tau^a \Phi_K^b) = q^{b(n-1)/2}$, where τ represents a generator of $\mathbb{Z}/m\mathbb{Z}$. Since some power of Φ_K is in the center of G by the conjugacy relation, it follows that, by Schur's lemma, ρ can further be thought as a representation of $H = (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/s\mathbb{Z})$ for some $s \in \mathbb{N}$; specifically, s is twice the order of q in $(\mathbb{Z}/m\mathbb{Z})^\times$.

Now we take a look at the tame structure of $D_{K,n}^\times$. If we let $\mathcal{O}_{D_{K,n}}$ and $\mathfrak{p}_{D_{K,n}}$ be the ring of integers and the maximal ideal, respecitvely, then in particular $\mathrm{JL}^{-1}(\pi)^\vee$ should vanish on $1+\mathfrak{p}_{D_{K,n}}$. We can take a (q^n-1)-st root of unity α and a uniformizer $\omega_{D_{K,n}}$ such that $\alpha \mapsto \omega_{D_{K,n}}^{-1} \alpha \omega_{D_{K,n}}$ generates $\mathrm{Gal}(K[\alpha]/K)$, $\omega_{D_{K,n}}^n$ is a uniformizer of K and $D_{K,n}^\times/(1+\mathfrak{p}_{D_{K,n}})$ is the group generated by $\omega_{D_{K,n}}$ and α (cf. [9, Section 1]). Thus, $\mathrm{JL}^{-1}(\pi)^\vee$ is a character twist of ρ', an irreducible (since π is irreducible discrete series) representation of $(\mathbb{Z}/(q^n-1)\mathbb{Z}) \rtimes (\mathbb{Z}/a\mathbb{Z}) = \langle \alpha, \omega_{D_{K,n}} \rangle$ for some $a \in \mathbb{N}$ (necessarily $a|n$) and $\omega_{D_{K,n}}^{-1} \alpha \omega_{D_{K,n}} = \alpha^j$ for some $(j, q^n-1) = 1$ such that $\rho'(\omega_{D_{K,n}}^n) = 1$. The twisting character ψ' defined by $\mathrm{JL}^{-1}(\pi)^\vee = \rho' \otimes \psi'$ sends $\omega_{D_{K,n}}^n$, a uniformizer of K, to c^{-1}.

The Jacquet-Langlands correspondence is explicitly described for supercuspidal depth zero representations as follows. For a depth zero supercuspidal representation π' of $\mathrm{GL}_n(K)$, consider the corresponding cuspidal representation $\overline{\pi'}$ of $\mathrm{GL}_n(\mathbb{F}_q)$. By Deligne-Lusztig theory, it is the Deligne-Lusztig representation of a character $\theta : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$. As $\mathcal{O}_{D_{K,n}}^\times/(1+\mathfrak{p}_{D_{K,n}}) \cong \mathbb{F}_{q^n}^\times$, we can consider the corresponding character $\theta : \mathcal{O}_{D_{K,n}}^\times \to \mathbb{C}^\times$. Extend it over $K^\times \mathcal{O}_{D_{K,n}}^\times$ by specifying the value of a uniformizer to be that of the central character of π', and induce it to $D_{K,n}^\times$; that is $\mathrm{JL}^{-1}(\pi')$. In this process, note that the dimension of $\mathrm{JL}^{-1}(\pi')$ is $[D_{K,n}^\times : K^\times \mathcal{O}_{D_{K,n}}^\times] = n$. Thus, both $r_\ell(\pi)$ and $\mathrm{JL}^{-1}(\pi)^\vee$ are n-dimensional.

For both cases, the problem basically boils down to knowing irreducible representations of semidirect product of cyclic groups. By using Mackey theory, we can completely describe such irreducible representations, as done in [28, 8.2]. In particular, an n-dimensional irreducible representation φ of $(\mathbb{Z}/|\alpha|\mathbb{Z}) \rtimes (\mathbb{Z}/|\beta|\mathbb{Z}) = \langle \alpha, \beta \rangle$ is of form

$$\varphi(\alpha) = \begin{pmatrix} \zeta_u & 0 & 0 & \cdots & 0 \\ 0 & \zeta_u^j & 0 & \cdots & 0 \\ 0 & 0 & \zeta_u^{j^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \zeta_u^{j^{n-1}} \end{pmatrix}, \varphi(\beta) = \begin{pmatrix} 0 & 0 & 0 & \cdots & \zeta_{u'} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where u divides $|\alpha|$, ζ_u, $\zeta_{u'}$ are u-th, u'-th roots of unity, respectively, n is the order of j modulo u and $\beta^{-1}\alpha\beta = \alpha^j$. We will temporarily denote by $\zeta_{u'}$ the top-right entity of $\varphi(\beta)$. Specifically, for $\mathrm{JL}^{-1}(\pi^\vee)$, this root of unity is 1, as $\rho'(\omega_{D_{K,n}})^n$ is the identity matrix. For $r_\ell(\pi)$, the corresponding root of unity is $(-1)^{n-1}$, as the matrix $r_\ell(\pi)(\beta)$ should be of determinant 1.

Thus, the both representations are n-dimensional irreducible representations of $(\mathbb{Z}/|\alpha|\mathbb{Z}) \rtimes \mathbb{Z}$ of form

$$\varphi'(\alpha) = \begin{pmatrix} \zeta_u & 0 & 0 & \cdots & 0 \\ 0 & \zeta_u^j & 0 & \cdots & 0 \\ 0 & 0 & \zeta_u^{j^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \zeta_u^{j^{n-1}} \end{pmatrix}, \varphi'(\beta) = \begin{pmatrix} 0 & 0 & 0 & \cdots & z_{\varphi'} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

for some $z_{\varphi'} \in \mathbb{C}^\times$. Note that by comparing the central characters we get $z_{r_\ell(\pi)} = (-1)^{n-1}q^{n(n-1)/2}c$ and $z_{\mathrm{JL}^{-1}(\pi^\vee)} = c^{-1}$, so in particular both values are rational numbers. Also, as $\mathrm{JL}^{-1}((\cdot)^\vee)$ and $r_\ell(\cdot)$ are bijective Galois equivariances, π, $r_\ell(\pi)$ and $\mathrm{JL}^{-1}(\pi^\vee)$ should have the same trace fields. Note also that, in the above notation of φ', if $z_{\varphi'}$ is rational, the trace field is $\mathbb{Q}(\zeta_u + \cdots + \zeta_u^{j^{n-1}})$, which is the unique subfield of $\mathbb{Q}(\zeta_u)$ with degree $\mathbb{Q}(\zeta_u)/\mathbb{Q}(\zeta_u + \cdots + \zeta_u^{j^{n-1}})$ being n. Thus, both $\mathrm{JL}^{-1}(\pi^\vee)$ and $r_\ell(\pi)$ have trace field F which is a subfield of some cyclotomic field $\mathbb{Q}(\zeta_u)$ with $\mathbb{Q}(\zeta_u)/F$ degree n. Fixing a surjective character $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_u)/F) \to \mathbb{Z}/n\mathbb{Z}$, the endomorphism algebras of $r_\ell(\pi)$ and $\mathrm{JL}^{-1}(\pi^\vee)$ are cyclic algebras $(\chi, 1/z_{r_\ell(\pi)})$ and $(\chi, 1/z_{\mathrm{JL}^{-1}(\pi^\vee)})$, respectively, inside $\mathrm{Br}(F)$. Let $\mathfrak{l}$ be any prime of the trace field F over $\ell \neq p$. Then, by the Brauer-Hasse-Noether theorem, the element in $\mathrm{Br}(F_\mathfrak{l})$ corresponding to $r_\ell(\pi) \otimes \mathrm{JL}^{-1}(\pi^\vee)$ has the Hasse invariant $\frac{k}{n}$, where k is the order $\mathrm{ord}_{\varpi_{F_{\mathfrak{l},n}}}\left(1/z_{r_\ell(\pi)}z_{\mathrm{JL}^{-1}(\pi^\vee)}\right)$ of uniformizer $\varpi_{F_{\mathfrak{l},n}}$ of degree n unramified extension $F_{\mathfrak{l},n}/F_\mathfrak{l}$. But as $q^{n(n-1)/2}$ is an integer power of $p \neq \ell$, it follows that $k = 0$, or that the Hasse invariant of $r_\ell(\pi) \otimes \mathrm{JL}^{-1}(\pi^\vee)$ in $\mathrm{Br}(F_\mathfrak{l})$ is 0. This implies that there is no ℓ-adic Brauer obstruction of $r_\ell(\pi) \otimes \mathrm{JL}^{-1}(\pi^\vee)$.

□

Since 1 is a rational number, together with Theorem 4, Theorem 7 implies the intended conclusion:

Theorem 8. *For $\ell \neq \mathrm{char}\,\mathbb{F}_q$, a cuspidal irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ has no ℓ-adic Brauer obstruction.*

Remark. The role of the Jacquet-Langlands in this framework is crucial, as every irreducible admissible representation of $\mathrm{GL}_n(K)$ has a trivial ℓ-adic Brauer obstruction whereas some irrreducible continuous representations of W_K do not, even in the tame case. We will construct such representations as follows. Note that an irreducible symplectic representation has a nontrivial archimedean Brauer obstruction. Thus, for any irreducible representation of the tame Weil group, if n is even and the top-right root of unity is -1, then the representation has a nontrivial archimedean Brauer obstruction. Rewriting what was done above, a symplectic irreducible representation φ of tame

Weil group should factor through $G' = (\mathbb{Z}/m\mathbb{Z}) \rtimes (\mathbb{Z}/2n\mathbb{Z})$, where $(m, q) = 1$, n is even and n is the order of q in $(\mathbb{Z}/m\mathbb{Z})^\times$. By the abuse of notation, let φ be the irreducible representation of G'. Since $|G'|/\deg\varphi = 2m$ is coprime to p if p is odd, in particular φ is in a p-block with a trivial defect group [10, Theorem 86.3]. This means that, by Theorem 3, the p-local Schur index of φ must be 1. However, if $[\mathbb{Q}(\varphi) : \mathbb{Q}] = \varphi(m)/n$ is odd, the sum of invariants of archimedean local Brauer classes should be one half, so there must be some finite place away from p that cancel the nontrivial invariants from archimedean places. For example, if $n = 2, m = 3, q = 5, K = \mathbb{Q}_5$, then there is a symplectic representation W_K with nontrivial Brauer obstruction over $\ell = 3$.

5. Depth Zero Supercuspidal L-packets

In the previous section, we have observed that the Local Langlands Correspondence for GL_n gives a fruitful result on the Brauer obstructions of representations of general linear groups, at least at finite places away from p. To extend this perspective to other classical groups, it is crucial to have some form of canonicity in the version of Local Langlands Correspondence in hand. However, unlike general linear groups, in general a Langlands parameter is related to more than one representation on the algebraic group side, called *L-packets*. Thus, to use the Local Langlands Correspondence in more general perspective, we need a parametrization of L-packets.

In [11], the Local Langlands Correspondence for some depth zero parameters is established for pure inner forms of quasisplit unramified p-adic groups, using a parametrization in the sense we want. Even though, in most cases, this construction does not exhaust all depth zero supercuspidal representations, this construction has nice behavior for generic supercuspidal representations, and the correspondence can be used to obtain some information on generic cuspidal representations of semisimple classical groups. In particular, unlike the case of general linear groups, the correspondence is Galois equivariant for semisimple groups. The Galois equivariance, with the characterization of generic representations, will give triviality of Brauer obstructions of some generic cuspidal representations of other classical groups, most notably odd orthogonal groups $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$. We will present the general construction of depth zero supercuspidal L-packets, mainly following [11] and [19].

5.1. The Local Langlands Correspondence in General Position.

We first explain what we should expect for the Local Langlands Correspondence and in particular depth zero supercuspidal L-packets. Before formulating the required version of the Local Langlands Correspondence, we have to make several preliminary remarks.

Pure Inner Forms. The formulation of Local Langlands Correspondence in this section, following [11], will consist of simultaneous parametrizations of all pure inner forms of the given quasisplit connected reductive group. Recall that a *pure inner form G_ω* of a quasisplit connected reductive K-group G for a nonarchimedean p-adic field K is any twist of G arising from the image of $\omega \in H^1(K, G)$ by the map $H^1(K, G) \to H^1(K, \mathrm{Aut}_{G/K})$ induced by conjugation. Also, recall the Kottwitz' isomorphism:

Theorem 9 (Kottwitz, [11, Corollary 2.4.3])**.** *With the notations as above, $H^1(K, G)$ and the component group $\pi_0(Z(^LG))$ of the center of the dual group ^LG are naturally isomorphic.*

From now on, we will restrict ourselves to the case of G being semisimple, since in such cases $Z(^LG)$ becomes finite abelian. Note that, even in this simpler setting, two different elements in $H^1(K, G)$ might induce the K-isomorphic inner forms. The Local Langlands Conrrespondence in this setting will therefore partition the L-packet of square-integrable representations of pure inner forms of G into several subsets, parameterized over several elements of $H^1(K, G)$.

Elliptic Langlands Parameters. The version of the Correspondence will involve *elliptic Langlands parameters.*

Definition. *An* **elliptic Langlands parameter** *is a homomorphism* $\varphi : W_K \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ *such that the following conditions hold.*

1. The parameter φ *is trivial on an open subgroup of* I_K*.*

2. The element $\varphi(\mathrm{Frob}_K) = (\theta, f)$*, where* θ *generates* $\mathrm{Gal}(K'/K)$ *and* $f \in \widehat{G}$ *is semisimple.*

3. The restriction $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ *is algebraic.*

4. (Ellipticity) The image of φ *is not contained in a proper Levi subgroup of* ${}^L G^0$*. Equivalently,* $(Z_{\widehat{G}}(\varphi))^0 = Z({}^L G)^0$*, where* $Z_{\widehat{G}}(\varphi)$ *is the centralizer of* φ *in* $\widehat{G}$*.*

The first three are the standard definitions of Langlands parameter, whereas the last condition is the extra ellipticity condition. The ellipticity is added to make the parametrization of inner forms easier: as $Z({}^L G)$ is in the center of $Z_{\widehat{G}}(\varphi)$, by the ellipticity of our parameter, $Z({}^L G) \hookrightarrow Z_{\widehat{G}}(\varphi)$ will induce a homomorphism $\pi_0(Z({}^L G)) \to A_\varphi := \pi_0(Z_{\widehat{G}}(\varphi))$. Thus, an irreducible representation $\rho \in \mathrm{Irr}(A_\varphi)$ will induce an element $\omega_\rho \in H^1(K, G)$ by the Kottwitz' isomorphism.

The following version of the Local Langlands Correspondence predicts a bijection between $\widehat{G}$-conjugacy classes of pairs (φ, ρ) and $G(K)$-conjugacy classes of (u, π), where φ is an elliptic Langlands parameter, $\rho \in \mathrm{Irr}(A_\varphi)$, $u \in Z^1(K, G)$, and π is an irreducible square-integrable representation of $G_u(K)$. To be more precise:

Conjecture [11, 3.5]**.** *The set of* $G(\overline{K})$*-conjugacy classes of pairs* (u, π) *of a continuous* 1*-cocycle* $u : \mathrm{Gal}(\overline{K}/K) \to G(\overline{K})$ *and an irreducible square-integrable representation* π *of* $G_u(K)$ *are partitioned into finite sets* $\Pi(\varphi, \omega)$ *for each* $\widehat{G}$*-conjugacy class of* (φ, ρ) *for elliptic Langlands parameters* φ *and* $\rho \in \mathrm{Irr}(A_\varphi)$*, such that* $\Pi(\varphi, \omega)$ *is consisted of* $G(\overline{K})$*-conjugacy classes of* $(u, \pi(\varphi, \rho))$*, where* u *is a* 1*-cocycle inside the class* ω*,* $\omega_\rho = \omega$ *and* $\pi(\varphi, \rho)$ *is a square-integrable irreducible representation of* $G_\omega(K)$*. This partition has several properties, among which are the following.*

1. $\pi(\varphi, \rho)$ *is depth zero if and only if* φ *is tame (in other words,* φ *is trivial on the wild inertia group).*

2. $\pi(\varphi, 1)$ *is generic, and if* G *has connected center, then this is the only generic representation of* $\amalg_{\omega \in H^1(K,G)} \Pi(\varphi, \omega)$*.*

In [11], $\pi(\varphi, \rho)$ is constructed for a *tame regular semisimple elliptic Langlands parameter (TRSELP)* φ, which is defined as follows.

Definition. *An elliptic Langlands parameter* $\varphi : W_K \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ *is* **tame regular semisimple** *if the following two conditions hold.*

1. (Tameness) φ *is trivial on the wild inertia group* $P_K \subset W_K$*.*

2. (Regularity) The centralizer of $\varphi(I_K)$ *in* ${}^L G^0$ *is a (maximal) torus.*

In particular, the regularity condition implies that φ *is trivial on* $\mathrm{SL}_2(\mathbb{C})$*.*

If $\widehat{G}$ is semisimple, then the ellipticity condition implies that $Z_{\widehat{G}}(\varphi)$ is finite, and that $A_\varphi = Z_{\widehat{G}}(\varphi)$. The tameness condition implies that $\varphi(I_K) = \langle s \rangle$ for some s in a maximal torus $\widehat{T}$ in $\widehat{G}$. Finally, if we let w be the corresponding element $\varphi(\mathrm{Frob}_K) \in N_{{}^L G}(\widehat{T})$ in the quotient group ${}^L W := N_{{}^L G}(\widehat{T})/\widehat{T} = \mathrm{Gal}(K'/K) \ltimes \widehat{W}$, where $\widehat{W}$ is the Weyl group of $\widehat{T}$ in $\widehat{G}$, then the above conditions imply that $s^w = s^q$ and that $\widehat{T}^w$ is finite. The finiteness of $\widehat{T}^w$ also implies that, for a character lattice $X = X_*(\widehat{T}) = \mathrm{Hom}(\widehat{T}, \mathbb{C}^\times) = \mathrm{Hom}(\mathbb{G}_m, X)$, $\mathrm{Irr}(A_\varphi)$ is characterized by the co-invariants of w in X, or $\mathrm{Irr}(A_\varphi) = X/(1 - w)X$. Conversely, given such (s, w), we can construct a TRSELP φ. Thus, the above process of choosing $s \in \widehat{T}$ and $w \in {}^L W$ from a TRSELP φ is a bijection between $\widehat{G}$-conjugacy classes of TRSELPs and $\widehat{W}$-conjugacy classes of pairs $(s, w) \in \widehat{T} \times {}^L W$.

5.2. **Construction of Tame Regular L-packets.** We now explain how to construct the L-packets of a TRSELP φ. For a simpler discussion, from now on we assume that G is semisimple. Moreover, the Local Langlands Correspondence established in [11] also requires that G is unramified, meaning that the splitting field K'/K of G is unramified. As observed above, from a TRSELP φ, we get an element $s \in \widehat{T}$ whose centralizer is as small as possible ($\widehat{T}$), and $w \in {}^L W$ with $\widehat{T}^w$ finite. We now abstractly define an anisotropic torus T_w which also splits over K'. Note first that T has a Galois action which is trivial on I_K as G is unramified, and that Frob_K acts by a generator of $\mathrm{Gal}(K'/K)$. We define T_w to be a torus which splits over K' with Galois action trivial on I_K and Frob_K acting by w. Note that the ellipticity implies that $\mathrm{Hom}(\mathrm{GL}_1, T_w)^{\mathrm{Frob}_K} = 0$, so T_w is indeed anisotropic over K.

Now we want to modify the parameter φ to another parameter $\varphi' : W_K \to {}^L T_w := \langle w \rangle \ltimes \widehat{T}$ which explicitly contains both w and s. More precisely, we define φ' by $\varphi'|_{I_K} = \varphi|_{I_K}$ and $\varphi'(\mathrm{Frob}_K) = w \ltimes 1$. This is indeed a homomorphism, and therefore by the Local Langlands Correspondence for tori, we can associate a character $\chi_\varphi : T_w(K) \to \mathbb{C}^\times$.

For $\rho \in \mathrm{Irr}(A_\varphi) = X/(1-w)X$, we now construct $\pi(\varphi, \rho)$ as a representation of G_{ω_ρ}, given the character $\chi_\varphi : T_w(K) \to \mathbb{C}^\times$ of the abstract torus T_w. Given $\lambda \in X$, consider the affine transformation $t_\lambda w : x \mapsto \lambda + wx$. This is an element of the affine Weyl group W_{aff}. The ellipticity implies that $(1-w)^{-1}\lambda$ is the unique fixed point of $t_\lambda w$ in the affine apartment $X \otimes \mathbb{R}$. After conjugation, let this fixed point be contained in the affine chamber $C = \{x \mid 0 < \langle \alpha, x \rangle < 1 \text{ for all } \alpha \in \Phi^+\}$ for a fixed choice of positive roots Φ^+. If we denote the stabilizer of C as Ω and its normal complement in W_{aff} as W^o, then as $W_{\mathrm{aff}} = X \rtimes W = W^o \rtimes \Omega$, we have a unique factorization $t_\lambda w = v_\lambda \omega_\lambda$ for $v_\lambda \in W^o, \omega_\lambda \in \Omega$.

We finally have to note the relationship between Ω and inner forms of G. Note that the projection $X \hookrightarrow W_{\mathrm{aff}} \to \Omega$ induces an isomorphism $\Omega \cong \mathrm{Irr}(Z(\widehat{G}))$, thereby the isomorphism

$$\Omega_{\mathrm{Gal}(K'/K)} \cong \mathrm{Irr}\left(Z(\widehat{G})^{\mathrm{Gal}(K'/K)}\right) = \mathrm{Irr}(Z({}^L G)),$$

where $\Omega_{\mathrm{Gal}(K'/K)}$ is the set of co-invariants of $\mathrm{Gal}(K'/K)$. Therefore, $\omega_\lambda \in \Omega$ corresponds to the restriction of $\lambda : \widehat{G} \to \mathbb{C}^\times$ to $Z(\widehat{G})$. As $x_\lambda := (1-w)^{-1}\lambda$ is a fixed point of ω_λ, we can therefore consider the corresponding parahoric subgroup $P_{\omega_\lambda, x_\lambda} \subset G_{\omega_\lambda}(K) =: G_\lambda(K)$ by Bruhat-Tits theory, and in fact is a vertex in the building $\mathcal{B}(G_\lambda(K))$ [11, Lemma 4.4.1]. The parahoric subgroup $P_\lambda := P_{\omega_\lambda, x_\lambda}$ has a reduction

$$1 \to P_\lambda^+ \to P_\lambda \to \overline{P_\lambda}(k) \to 1$$

upon the choice of integral model where $\overline{P_\lambda}$ is the connected reductive group over k, the residue field of K, with root datum determined by G and x_λ. Specifically, the set of positive roots of $\overline{P_\lambda}$ is $\{\alpha \in \Phi^+ \mid \langle \alpha, x_\lambda \rangle \in \mathbb{Z}\}$, where Φ^+ is the set of positive roots of G. Also, by Lang's theorem, there is an element $p_\lambda \in G(K')$ fixing x_λ such that the Lang map $f_{p_\lambda} : T_\omega \to T_\lambda$ is a K-isomorphism, with $T_\lambda(K) \subset P_\lambda$.

Now we are ready to define $\pi(\varphi, \rho)$. Restricting the above reduction exact sequence to $T_\lambda(K) \subset P_\lambda$, we get the corresponding inclusion of maximal torus $\overline{T_\lambda}(k) \subset \overline{P_\lambda}(k)$ as the associated finite k-groups. Let $R_\lambda := \pm R_{\overline{T}_\lambda, \chi_\lambda}^{\overline{P}_\lambda}$ be the Deligne-Lusztig representation, with χ_λ seen as a character of $\overline{T}_\lambda$. We can see this as a representation of P_λ and, finally we can define $\pi(\varphi, \rho)$ as $\mathrm{cInd}_{P_\lambda}^{G_\lambda(K)} R_\lambda$.

5.3. **Brauer Obstructions of Some Generic Representations.** Now we investigate the effect of Galois twist on coefficients of Langlands parameters. We will keep using the notation of the previous section.

Theorem 10. *Let $\ell \neq p$, and fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_\ell}$. Given a TRSELP $\varphi : W_K \to {}^L G$, for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$, define φ^σ to be a homomorphism such that $\varphi^\sigma|_{I_K} = \sigma \circ \varphi|_{I_K}$ and $\varphi^\sigma(\mathrm{Frob}_K) =$*

$\varphi(\mathrm{Frob}_K)$ *by the isomorphism* $N_{\widehat{G}^\sigma}(\widehat{T}^\sigma) \cong \widehat{W} \cong N_{\widehat{G}}(\widehat{T})$. *Then,* $\pi(\varphi^\sigma, \rho) = \pi(\varphi, \rho)^\sigma$ *for any* $\rho \in \mathrm{Irr}(A_\varphi)$.

Proof. Note that the only effect of Galois twist on the TRSELP φ which further permeates through the construction of $\pi(\varphi, \rho)$ is the effect on $\varphi'|_{I_K} = \varphi|_{I_K}$, as the rest is forgotten afterwards. In particular, the construction of anisotropic tori and parahoric subgroup $T_\lambda(K) \subset K_\lambda \subset G_\lambda(K)$, which depends on $w \in {}^LW$ and ρ, is unaffected by the Galois twist. Also, the construction of virtual Deligne-Lusztig representation $R^{\overline{P}_\lambda}_{\overline{T}_\lambda, \chi_\lambda}$ is Galois equivariant at finite places $\ell \neq p$, and the sign giving the true Deligne-Lusztig representation is determined by dimensions of G and its maximal torus, so the signs are also the same once G is fixed. Thus, the construction of (true) Deligne-Lusztig representation R_λ is Galois equivariant at finite places $\ell \neq p$. Therefore, the only construction to worry about is the parameter $\varphi'^\sigma : W_K \to {}^LT_w$ and thereby the character $\chi_{\varphi^\sigma} : T_w(K) \to \mathbb{C}^\times$. On the other hand, note that the Local Langlands Correspondence for tori is Galois equivariant precisely in the sense that $\chi_{\varphi^\sigma} = \sigma \circ \chi_\varphi$; the split case is obviously Galois equivariant as it is a result of composing with Art_K^{-1}, and the general case is realized as a special case of the split case, by the previous remark on the canonicity of Local Langlands Correspondence for Tori. Therefore, $\pi(\varphi^\sigma, \rho) = \pi(\varphi, \rho)^\sigma$. □

The Galois equivariance only ensures that the trace fields cannot be bigger. On the other hand, we want to deduce a result of form that for some $\pi(\varphi, \rho)$'s, the trace fields are the same as those of φ, when localized at finite places $\ell \neq p$. The first thing to look at is to search for a Galois equivariant bijective correspondence. Indeed, the tame Local Langlands Correspondence proved in [11] is bijective. However, it is a bijective correspondence between conjugacy classes, and therefore in general $\pi(\varphi, \rho)$ has strictly smaller trace field than φ. In particular, conjugating by $\widehat{G}$ mixes A_φ, and thereby mixes ρ up. For example, if $G = \mathrm{SO}_{2n+1}$ is the split orthogonal K-group, then $A_\varphi = (\mathbb{Z}/2\mathbb{Z})^s$ for some s, and $\widehat{G}$-conjugation permutes the components of A_φ.

On the other hand, such a mixing does not happen when ρ is trivial, and the conjugacy class therefore contains a single isomorphism class. In other words, $\varphi \mapsto \pi(\varphi, 1)$ is a genuine bijective Galois equivariant correspondence between the set of TRSELPs and the set of depth zero supercuspidal generic representations of $G(K)$ appearing in some L-packets we have constructed above. As a bijective Galois equivariant map preserves trace fields, we conclude that this correspondence $\varphi \mapsto \pi(\varphi, 1)$ preserves trace fields. Moreover, note that the whole correspondence is constructive. To be more precise, as φ is definable over a field $F \supset \mathbb{Q}$, then φ' is definable over F. Moreover, the construction of χ_φ relies on the perfect pairing $\widehat{T} \times \widehat{L} \to \mathbb{C}^\times$, where $\widehat{L}$ is the dual lattice of the character group of T. This pairing is definable over $\mathbb{Q}$, so in turn χ_φ is definable over F. Thus, R_λ and eventually $\pi(\varphi, 1)$ is also definable over F. This applies to the localizations, and thus we have proved the following.

Theorem 11. *The correspondence* $\varphi \mapsto \pi(\varphi, 1)$ *preserves trace fields at finite places* $\ell \neq p$. *Moreover, the order of the* ℓ*-adic Brauer obstruction of* $\pi(\varphi, 1)$ *divides the order of the* ℓ*-adic Brauer obstruction of* φ. *In particular, if* φ *has a trivial* ℓ*-adic Brauer obstruction, then* $\pi(\varphi, 1)$ *also has a trivial* ℓ*-adic Brauer obstruction.*

Note that, as mentioned above, the depth zero supercuspidal L-packets we have constructed are, in general, far from covering every depth zero supercuspidal representation of $G(K)$, and this also applies to the generic representations. On the other hand, Theorem 6.8 and 10.7 of [13] with Lemma 6.1.2 of [11] implies that, if G is of adjoint type, then every depth-zero generic supercuspidal representation of G appears as $\pi(\varphi, 1)$ for some TRSELP φ. Among the classical groups, in particular SO_{2n+1} is in this case, and therefore we can examine the Brauer obstructions of all depth zero generic supercuspidal representations of $\mathrm{SO}_{2n+1}(K)$. In particular, we can prove the following.

Theorem 12. *Let K be a p-adic field, and k be its residue field. Let $\ell \neq p$ be a finite prime.*

(i) Every depth zero generic supercuspidal representation of $\mathrm{SO}_{2n+1}(K)$ has a trivial ℓ-adic Brauer obstruction.

(ii) Every generic cuspidal representation of $\mathrm{SO}_{2n+1}(k)$ has a trivial ℓ-adic Brauer obstruction.

Proof. To show (i), by Theorem 11, we only need to show that a TRSELP $\varphi : W_K \to {}^L(\mathrm{SO}_{2n+1}(K)) = \mathrm{Sp}_{2n}(\mathbb{C})$ has a trivial ℓ-adic Brauer obstruction. A homomorphism $\varphi : W_K \to \mathrm{Sp}_{2n}(\mathbb{C})$ is a TRSELP if and only if $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_s$ for pairwise non-isomorphic irreducible symplectic representations $\varphi_1, \cdots, \varphi_s$ of the tame Weil group W_K/P_K. This precisely amounts to our previous discussion of finite-dimensional irreducible representations of the tame Weil group. Thus, each φ_i should be of form

$$\varphi_i(\alpha) = \begin{pmatrix} \zeta_u & 0 & 0 & \cdots & 0 \\ 0 & \zeta_u^j & 0 & \cdots & 0 \\ 0 & 0 & \zeta_u^{j^2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \zeta_u^{j^{n-1}} \end{pmatrix}, \varphi_i(\mathrm{Frob}_K) = \begin{pmatrix} 0 & 0 & 0 & \cdots & z_i \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

for some $z_i \in \overline{\mathbb{Q}_\ell}^\times$, where α denotes an element of I_K/P_K. As φ_i is symplectic, the unit determinant condition forces $z_i = -1$. Therefore, by the Brauer-Hasse-Noether theorem, the Hasse invariant of the cyclic algebra corresponding to φ_i in $\mathrm{Br}(K_{\mathfrak{l}})$ is 0, where $\mathfrak{l}$ is any prime over ℓ. This implies that φ_i's, and as a result φ, have trivial ℓ-adic Brauer obstructions, as desired.

To show (ii), we have to prove that there is a promotion process from generic cuspidal representations of $\mathrm{SO}_{2n+1}(k)$ to generic supercuspidal depth zero representations of $\mathrm{SO}_{2n+1}(K)$ which preserves the Brauer obstructions. However, note that from the construction we have a bijective correspondence

$$\left\{ \begin{array}{c} \text{Generic cuspidal representations} \\ \text{of } \mathrm{SO}_{2n+1}(k) \text{ arising as} \\ \text{Deligne-Lusztig representations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Depth zero generic supercuspidal} \\ \text{representations of } \mathrm{SO}_{2n+1}(K) \\ \text{arising as } \pi(\varphi, 1) \text{ for some TRSELP } \varphi \end{array} \right\}$$

by the compact induction $\mathrm{cInd}_{K_1}^{\mathrm{SO}_{2n+1}(K)}$, with a fixed good parahoric group K_1 corresponding to $1 \in H^1(K, \mathrm{SO}_{2n+1}(K))$. As this is also a smooth induction, by the same reason as the proof of Theorem 4, this preserves the trace fields as well as the orders of local Brauer obstructions. Thus, (i) implies (ii). □

We now want to use the similar strategy to other semisimple classical p-adic groups, namely $\mathrm{Sp}_{2n}(K)$, $\mathrm{SO}_{2n}(K)$, $\mathrm{PGL}_n(K)$ and $\mathrm{SL}_n(K)$. Note that, for $G = \mathrm{Sp}_{2n}$, SO_{2n} or PGL_n, the dual group $\widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C}), \mathrm{SO}_{2n}(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$ is also naturally embedded in $\mathrm{SL}_N(\mathbb{C}) \subset \mathrm{GL}_N(\mathbb{C})$ for $N = 2n+1, 2n, n$, respectively. Therefore, the corresponding TRSELP's have trivial ℓ-adic Brauer obstructions. Moreover, the promotion process involving a compact and smooth induction is a Galois equivariant bijection, so the finite group representations promote without harming the order of Brauer obstructions. Note that, for Sp_{2n} and SO_{2n}, the disconnected center attributes to the phenomenon that not every generic representation shows up in our L-packets. Nevertheless, exactly same argument as above will conclude that the generic cuspidal representations which also arise as Deligne-Lusztig representations have trivial ℓ-adic Brauer obstructions.

Thus, we have the following all-encompassing conclusion.

Theorem 13. *Let G be either SO_{2n+1}, Sp_{2n}, SO_{2n} or PGL_n. Let $\ell \neq p$ be finite primes, and let $q = p^k$ a prime power. Then, every generic cuspidal representation of $G(\mathbb{F}_q)$ which appears as a Deligne-Lusztig representation has a trivial ℓ-adic Brauer obstruction. In particular, every generic cuspidal representation of $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$ and $\mathrm{PGL}_n(\mathbb{F}_q)$ has a trivial ℓ-adic Brauer obstruction.*

Proof. Note that, as PGL_n is centerless, every generic cuspidal representation of $\mathrm{PGL}_n(\mathbb{F}_q)$ arises as a Deligne-Lusztig representation. All the other statements are proved above. $\square$

5.4. **Remarks on Limitations of the Method.** Note that we have not explored another case of semisimple classical group, namely SL_n. This is because its Langlands dual, $\mathrm{PGL}_n(\mathbb{C})$, does not embed into $\mathrm{GL}_n(\mathbb{C})$. Nevertheless, one can hope to get a similar result by analyzing the local Schur indices of projective representations of W_K.

However, it is not in general true that a generic cuspidal representation of $\mathrm{SL}_n(\mathbb{F}_q)$ has trivial ℓ-adic Brauer obstructions, for $\ell \neq p$. Even this phenomenon happens in one of the simplest cases, $\mathrm{SL}_2(\mathbb{F}_q)$. Namely, for a non-quadratic non-real character $\chi : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$, one can construct the Deligne-Lusztig representation $\pi(\chi)$ for $\mathrm{GL}_2(\mathbb{F}_q)$, and it remains irreducible even after restricting to $\mathrm{SL}_2(\mathbb{F}_q)$. This is a generic cuspidal representaion. However, for $q = 13$ and $\ell = 7$, such representation has a nontrivial ℓ-adic Brauer obstruction.

The above phenomenon can be vaguely explained as follows. Even though we start with a fixed character χ of $\mathbb{F}_{q^2}^\times$, there are after all $q-1$ distinct characters of $\mathbb{F}_{q^2}^\times$ which ends up with the same representation of $\mathrm{SL}_2(\mathbb{F}_q)$. In the same vein, we will briefly investigate how L-packets mix within themselves by going through the case of special linear groups.

Many results about the Local Langlands Correspondence of SL_n can be derived from the Local Langlands Correspondence of GL_n with only pure group theory input, as done in [17] and [21]. Let $\widetilde{\pi}$ be an irreducible admissible representation of $\mathrm{GL}_n(K)$. Then, the irreducible consistuents of $\mathrm{Res}^{\mathrm{GL}_n(K)}_{\mathrm{SL}_n(K)} \widetilde{\pi}$ are multiplicity-free, and the irreducible constituents are all $\mathrm{GL}_n(K)$-conjugate. Let $X(\widetilde{\pi})$ be the group of characters ω of $\mathrm{GL}_n(K)$ such that $\widetilde{\pi} \otimes \omega \cong \widetilde{\pi}$. Fix a vector space $V_{\widetilde{\pi}}$ on which $\widetilde{\pi}$ acts; for $\omega \in X(\widetilde{\pi})$, there exists a nonzero operator $I_\omega \in \mathrm{End}(V_{\widetilde{\pi}})$, unique up to a nonzero constant, such that $\widetilde{\pi} \circ I_\omega = I_\omega \circ (\widetilde{\pi} \otimes \omega)$. Fortunately, for the case of $\mathrm{SL}_n(K)$, $I_{\omega_1} \circ I_{\omega_2} = I_{\omega_2} \circ I_{\omega_1}$, which would not be true for other inner forms of $\mathrm{SL}_n(K)$. Let $S(\widetilde{\pi})$ be the subgroup of $\mathrm{Aut}(V_{\widetilde{\pi}})$ generated by I_ω's and $\mathbb{C}^\times$. Then, there is a natural surjection $S(\widetilde{\pi}) \to X(\widetilde{\pi})$, denoted as ω by abuse of notation. We then define the semidirect product $S(\widetilde{\pi}) \rtimes \mathrm{GL}_n(K)$ by $\omega(I)(g) = gI$ for $g \in \mathrm{GL}_n(K)$ and $I \in S(\widetilde{\pi})$. Then, $S(\widetilde{\pi}) \rtimes \mathrm{GL}_n(K)$ acts naturally on $V_{\widetilde{\pi}}$, and in particular $S(\widetilde{\pi}) \times \mathrm{SL}_n(K)$ acts naturally on $V_{\widetilde{\pi}}$. If $V_{\widetilde{\pi}}$ is seen as a representation of $S(\widetilde{\pi}) \times \mathrm{SL}_n(K)$, then it decomposes as

$$V_{\widetilde{\pi}} \cong \oplus_{\pi \in \Pi(\widetilde{\pi})} \rho_\pi \otimes \pi,$$

where $\Pi(\widetilde{\pi})$ is the set of L-packets corresponding to $\widetilde{\pi}$. This decomposition is multiplicity free and canonical, as $\rho_\pi \cong \rho_{\pi'}$ if and only if $\pi \cong \pi'$, and each ρ_π is an irreducible finite-dimensional representation of $S(\widetilde{\pi})$.

There is no canonical parametrization of $\Pi(\widetilde{\pi})$, but as the decomposition of $V_{\widetilde{\pi}}$ is canonical, so one might hope to derive some results by analyzing the effect of Galois twist on the decomposition. As a Galois twist of π fixing itself might change the isomorphism class of $\widetilde{\pi}$ (necessarily by a character twist), we would want to know the relation between ρ_π and ρ'_π, where ρ'_π is the corresponding representation of $S(\widetilde{\pi}')$ where $\widetilde{\pi}' \otimes \omega_0 \cong \widetilde{\pi}$. Then, we can choose an intertwining operator $I_{\omega_0} \in \mathrm{Hom}(V_{\widetilde{\pi}}, V_{\widetilde{\pi}'})$ such that

$$\widetilde{\pi}' \otimes \omega_0 \circ I_{\omega_0} = I_{\omega_0} \circ \widetilde{\pi}.$$

If we define an isomorphism $i_{\omega_0} : S(\widetilde{\pi}') \to S(\widetilde{\pi})$ by $i_{\omega_0}(I') = I_{\omega_0}^{-1} \circ I' \circ I_{\omega_0}$ for $I' \in S(\widetilde{\pi}')$, then we have $\rho'_\pi \cong \rho_\pi \circ i_{\omega_0}$ [2, Lemma 2.9]. As the 2-cocycle corresponding to the commutator relations between the intertwining operators is trivial,

$$V_{\widetilde{\pi} \otimes \omega} \cong \oplus_{\pi \in \Pi(\widetilde{\pi})} \rho_\pi \otimes \pi,$$

as $S(\widetilde{\pi}) \times \mathrm{SL}_n(K)$-representations for any $\omega \in X(\widetilde{\pi})$. This already tells us the limitation we have predicted above; as there are identical pieces with multiplicity, the effect of Galois twist cannot be nailed down by a naive multiplicity-free argument used for the case of general groups. In particular, for the aforementioned example $\pi(\chi)$ of $\mathrm{SL}_2(\mathbb{F}_q)$ in the beginning of the section, $X(\pi(\chi))$ is consisted of $q-1$ elements, and this really results in a nontrivial ℓ-adic Brauer obstruction.

Remark. It should be noted that, by using classical methods, the local Brauer obstructions of irreducible representations of $\mathrm{SL}_n(\mathbb{F}_q)$ are completely characterized in [32].

The method used in this paper does not help for examining p-adic or infinite Brauer obstructions. There are some cases where a global Brauer obstruction is nontrivial precisely due to the nontriviality at archimedean places and finite places over p. For example, for q a square of an odd prime, all faithful characters of $\mathrm{Sp}_4(\mathbb{F}_q)$ have global and real Schur indices 2, and the only finite prime where the local index may differ from 1 is p ([27],[18, Theorem 7]).

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