Bott Periodicity

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Chapter 1

Introduction

The challenges to computing higher homotopy groups are well known. For instance, while we can argue $\pi_i S^n$ is trivial for $i < n$ and isomorphic to $\mathbb{Z}$ for $i = n$, not much is known in general when $i > n$.

This makes Raoul Bott's discover of Bott Periodicity in the late 1950s significant. Bott Periodicy demonstrates that the homotopy groups of the unitary group U, orthogonal group O, and symplectic group Sp are periodic and hence all of the homotopy groups are determined just be calculating the first few. Bott Periodicity is a surprising and significant result in algebraic topology. Its contributions include progress to the study of the higher homotopy groups of spheres mentioned above.

Our goal here is to outline the proof for the unitary and orthogonal group cases. While many proofs have been offered since Bott's original discovery, we follow the approach of Milnor presented in his book *Morse Theory* [4]. Appendix A briefly discusses the idea of Morse Theory and relates it to the relevant results needed for the proof.

In the chapter *Unitary Case*, we discuss the statement and outline the proof of Bott Periodicy for the unitary group

$$\pi_{i-1}\,\mathrm{U} \cong \pi_{i+1}\,\mathrm{U}, \quad i \geq 1$$

That is, these homotopy groups are 2-periodic. We establish the isomorphism by introducing some other spaces that let us "link together" isomorphic homotopy groups from $\pi_{i-1}U$ to $\pi_{i+1}U$. Most of these isomorphisms will be the result of some long exact sequences associated to few fiber bundles that we introduce at the beginning of the chapter; however, a few will be a bit more involved to show. In particular, we'll need to introduce a space of geodesics and this is where our results from Morse theory will be useful.

Next, the chapter *Orthogonal Case* discusses Bott Periodicity for the orthogonal group,

$$\pi_i \, \mathrm{O} \cong \pi_{i+8} \, \mathrm{O}, \quad i \geq 0$$

Notice that here the homotopy groups are 8-periodic rather than 2-periodic. The proof is analogous to the real case: we "link" $\pi_i U$ and $\pi_{i+8} U$ via a chain of isomorphisms between homotopy groups of spaces $\Omega_1, \Omega_2, ..., \Omega_8$ that we introduce. Here Ω_1 denotes the space of complex structures, those orthogonal matrices that square to the negative identity, and the higher Ω_i are subspaces of Ω_1. Again, we'll need a key result from the Morse theory, in particular, Theorem 3. To satisfy the conditions of this theorem, we spend some time proving a few lemmas regarding our spaces Ω_i. Again, it'll be important to look at the space of geodesics.

While we aim to keep the presentation largely self-contained, we will occasionally cite without proof results from Riemannian Geometry and some facts about Lie groups in addition to our already mentioned tools from Morse theory. A bibliography is included at the end to direct the interested reader to a more detailed discussion of these ideas. Also, in both the complex and real cases we'll need a result regarding the curvature of a Lie group; an outline of its proof is presented in Appendix B.

Chapter 2

Unitary Case

2.1 Unitary Group

Let $w = (w_1, ..., w_n)$and $z = (z_1, ..., z_n)$ belong to $\mathbb{C}^n$. We have hermitian inner product given by

$$\langle w, z\rangle = \sum_{i=1}^{n} w_i \overline{z_i}$$

Recall an $n\times n$ matrix with complex entries A is unitary if it preserves this form, $\langle Aw, Az\rangle = \langle w, z\rangle$. As our form is hermitian, we have $\langle Aw, z\rangle = \langle w, A^*z\rangle$ where A^* denotes the conjugate transpose of A. Hence our condition for $A \in \mathbb{C}^{n\times n}$ being unitary is equivalent to $AA^* = I_n$.

Let $\mathrm{U}(n)$ denote the *unitary group of degree* n, the group of $n \times n$ unitary matrices of $\mathbb{C}^{n\times n}$ under the operation of group multiplication.

Denoting the columns of A by $\alpha_1, ..., \alpha_n$, we have $\langle \alpha_i, \alpha_j\rangle = \langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \delta_{i,j}$. Hence the columns of a unitary matrix form an orthonormal basis for $\mathbb{C}^n$. One verifies the converse is also true, so A unitary if and only if its columns are orthonormal.

Our goal in this chapter is to describe the proof the Bott Periodicity Theorem for Unitary Groups:
Theorem 1. *Fix $i > 1$, then $\pi_{i+1}\,\mathrm{U}(m) = \pi_{i-1}\,\mathrm{U}(m)$ for m sufficiently large.*

Before beginning the proof of Theorem 1, it will be useful to introduce the notion of a fiber bundle to obtain some first results about the unitary group.

2.2 Fiber Bundles and First Results

A fiber bundle, which we denote by $F \to E \xrightarrow{p} B$ for topological spaces F, E, and B, has projection map $p : E \to B$ such that every point of B has a neighborhood U with homeomorphism $h : p^{-1}(U) \to U \times F$ such that $p = \mathrm{proj}_U(U \times F) \circ h$. For B path connected, such a fiber bundle gives rise to the long exact sequence of homotopy groups

$$... \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to ...$$

See [2] for a more detailed discussion of fiber bundles. We now give some examples that will be useful for the proof of Theorem 1.

Fiber Bundle 1. The inclusion $i : \mathrm{U}(n) \to \mathrm{U}(n+1)$ defined by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

and the map $j : \mathrm{U}(n+1) \to S^{2n+1} \subset \mathbb{C}^{n+1}$ that takes a matrix to its first column give the fiber bundle

$$\mathrm{U}(n) \xrightarrow{i} \mathrm{U}(n+1) \xrightarrow{j} S^{2n+1} \tag{2.1}$$

This induces an exact sequence of homotopy groups

$$... \to \pi_{i+1}\ S^{2m+1} \to \pi_i\ \mathrm{U}(m) \to \pi_i\ \mathrm{U}(m+1) \to \pi_i\ S^{2m+1} \to ...$$

As $\pi_i\ S^{2m+1} = 0$ for $2m + 1 > i$, we have

$$\pi_i\ \mathrm{U}(m) \cong \pi_i\ \mathrm{U}(m+1) \cong \pi_i\ \mathrm{U}(m+2) \cong ... \tag{2.2}$$

whenever $2m + 1 > i$. Thus $\pi_i \, \mathrm{U}(n)$ is independent of n for sufficiently large n. Letting U denote the direct limit of $\mathrm{U}(n)$ as $n \to \infty$, we then have $\pi_i \, \mathrm{U} \cong \pi_i \, \mathrm{U}(n)$ for sufficiently large n. The statement of Bott Periodicity may now be expressed as $\pi_{i-1} \, \mathrm{U} \cong \pi_{i+1} \, \mathrm{U}$ for each $i > 1$.

We consider some more fiber bundles that relate the unitary group to other interesting spaces. These will give relations essential to the proof of Theorem 1 outlined shortly.

Fiber Bundle 2. Define the *complex Stiefel manifold*, denoted $V_m(\mathbb{C}^n)$, as the space of m-tuples of orthonormal vectors in $\mathbb{C}^n$.

Note that $\mathrm{U}(n)$ acts transitively on $V_m(\mathbb{C}^n)$ by matrix multiplication. Furthermore, an m-frame of $\mathbb{C}^n$ is fixed by a member of $\mathrm{U}(n)$ that acts non-trivially only on the $n - m$ frame of the orthogonal complement. That is, the action has stabilizer subgroup isomorphic to $\mathrm{U}(n - m)$. Hence we have a fiber bundle

$$\mathrm{U}(n-m) \to \mathrm{U}(n) \to V_m(\mathbb{C}^n). \tag{2.3}$$

The long exact homotopy sequence gives that $\pi_i V_m(\mathbb{C}^n)$ is trivial for all $i < 2(m-n)$.

Remark: In particular, we have $\mathrm{U}(n) \to \mathrm{U}(n+1) \to V_1(\mathbb{C}^{n+1})$. Identifying $V_1(\mathbb{C}^{n+1})$ with S^{2n+1} gives fiber bundle (1.1).

Fiber Bundle 3. Define the *complex Grassmann manifold*, denoted $G_m(\mathbb{C}^n)$, as the collection of m dimensional subspaces of $\mathbb{C}^n$ where $m \leq n$.

Consider the mapping $V_m(\mathbb{C}^n) \to G_m(\mathbb{C}^n)$ that sends an m-frame in $\mathbb{C}^n$ to the m dimensional subspace spanned by it. This map has as fibers the collections of m-frames spanning the same m-dimensional subspace. Representing the vectors of a frame as columns of a matrix, we obtain a $m \times m$ orthonormal matrix. Hence, we have a fiber bundle

$$\mathrm{U}(m) \to V_m(\mathbb{C}^n) \to G_m(\mathbb{C}^n) \tag{2.4}$$

From the associated long exact sequence of homotopy groups,

$$\ldots \to \pi_i V_m(\mathbb{C}^n) \to \pi_i G_m(\mathbb{C}^n) \to \pi_{i-1} U(m) \to \pi_{i-1} V_m(\mathbb{C}^n) \to \ldots$$

we conclude that $\pi_{i-1}\ \mathrm{U}(m) \cong \pi_i\ G_m(\mathbb{C}^n)$ for $i < 2(n-m)$.

Fiber Bundle 4. We define the Special Unitary Group as the subgroup $\mathrm{SU}(m) \subset \mathrm{U}(m)$ consisting of matrices with determinant 1. Then the map $\mathrm{U}(m) \to S^1 \subset \mathbb{C}$ given by taking determinant gives rise to a fiber bundle

$$\mathrm{SU}(m) \to \mathrm{U}(m) \to S^1. \tag{2.5}$$

Since $\pi_i S^1 \cong 0$ for all $i > 1$, from the long exact sequence of homotopy groups

$$\ldots \to \pi_{i+1} S^1 \to \pi_i\, \mathrm{SU}(m) \to \pi_i\, \mathrm{U}(m) \to \pi_i S^1 \to \ldots$$

we conclude that $\pi_i\, \mathrm{SU}(m) \cong \pi_i\, \mathrm{U}(m)$ and hence $\pi_i\, \mathrm{SU} \cong \pi_i\, \mathrm{U}$ for $i > 1$.

2.3 Proof of Bott Periodicity

For sufficiently large m, it follows from the fiber bundles (2.4) and respectively (2.5) that

$$\pi_{i-1}\, \mathrm{U}(m) \cong \pi_i G_m(\mathbb{C}^{2m}) \quad \text{and} \quad \pi_{i+1}\, \mathrm{SU}(2m) \cong \pi_{i+1}\, \mathrm{U}(2m)$$

Hence to prove Theorem 1, we just need

$$\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1}\, \mathrm{SU}(2m)$$

for large m. Rather than prove this directly, we introduce the space of minimal geodesics in $\mathrm{SU}(2m)$ from I to $-I$, denoted Ω^α. Then we argue in two steps that

$$\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_i(\Omega^\alpha) \tag{2.6}$$

and

$$\pi_i(\Omega^\alpha) \cong \pi_{i+1} SU(2m) \tag{2.7}$$

The first step to show the isomorphism 2.6 is straight-forward, while the second step to show the isomorphism 2.7 will require Morse theory. Having shown these, it then follows $\pi_{i-1}\,\mathrm{U}(m) \cong \pi_{i+1}\,\mathrm{U}(2m)$ for sufficiently large m. However, as $\pi_{i-1}\,\mathrm{U}(m)$ stabilizes to $\pi_{i-1}\,\mathrm{U}$ and $\pi_{i+1}\,\mathrm{U}(2m)$ to $\pi_{i+1}\,\mathrm{U}$, we conclude $\pi_{i-1}\,\mathrm{U} \cong \pi_{i+1}\,\mathrm{U}$ as desired.

2.3.1 First Step: $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_i \Omega^\alpha$

Notice $\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_i \Omega^\alpha$ for $i \geq 0$ is an immediate consequence of the following lemma.
Lemma 1. *The spaces $G_m(\mathbb{C}^{2m})$ and Ω^α are homeomorphic.*

Proof. The Lie group $\mathrm{SU}(2m)$ has tangent space $T_I\,\mathrm{SU}(2m)$ that can be identified with the space of $2m \times 2m$ matrices A such that $A + A^* = 0$ and $\mathrm{trace}(A) = 0$. We then have that the exponential map defined by $\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ...$ takes an element of $A \in T_I\,\mathrm{SU}(2m)$ to $\exp(A) \in \mathrm{SU}(2m)$. To verify this, note that for $A \in T_I\,\mathrm{SU}(2m)$ since A is skew-symmetric it is normal and hence $\exp(A)\exp(A)^* = \exp(A + A^*) = I$ and $\det(\exp(A)) = \exp(\mathrm{tr}(A)) = 1$, so indeed $\exp(A) \in \mathrm{SU}(2m)$.

The geodesics beginning at I in $\mathrm{SU}(2m)$ are of the form $\gamma(t) = \exp(tA)$ where $A \in T_I\,\mathrm{SU}(2m)$ and $t \in [0, 1]$. (cf. [4] pp.55-57 and [1] p.88.)

We are interested in the minimal geodesics from I to $-I$, so consider A such that $\exp(A) = \gamma(1) = -I$. Since $\exp(TAT^{-1}) = T\exp(A)T^{-1} = -I$, so we may suppose without loss of generality that A is diagonal.

As A is skew-hermitian, it must have diagonal entries that each have real part zero and that sum to zero. Denote these diagonal entries $ia_1, ..., ia_{2m}$. As $\exp(A) = -I$, it follows that each a_i must be of the form $k_i\pi$ for odd k_i. We also have $\sum_i k_i = 0$.

A geodesic $\exp(tA)$ has length $||A|| = \langle A, A\rangle^{1/2} = \pi\sqrt{k_1^2 + ... + k_{2m}^2}$, so a minimal geodesic will have each $k_i = \pm 1$ for $i = 1, ..., 2m$. In particular, since $\sum_i k_i = 0$, we have half of the $k_1, ..., k_{2m}$ equal to 1 and half equal to -1.

Thus the minimal geodesics from $-I$ to I in $SU(2m)$ are of the form $exp(tA)$ where A is a diagonal matrix with m diagonal entries equal to $i\pi$ and m diagonal entries equal to $-i\pi$.

Such an A is uniquely determined by its eigenspace for $i\pi$, an m-dimensional subspace of $\mathbb{C}^{2m}$.

Thus we have a natural homeomorphism between $G_m(\mathbb{C}^{2m})$ and Ω^α. $\square$

2.3.2 Second Step: $\pi_i \Omega^\alpha \cong \pi_{i+1}\,\mathrm{SU}(2m)$

Our second isomorphism, $\pi_i \Omega^\alpha \cong \pi_{i+1}\,\mathrm{SU}(2m)$ for $i \leq 2m$, will take considerably more work. Part of the proof makes use of the idea of the conjugate points and index of a geodesic and the other part uses Morse theory.

Definition 1. *Along a curve γ, we say $\gamma(a)$ and $\gamma(b)$ are conjugate if there is a non-zero Jacobi field J that vanishes at $\gamma(a)$ and $\gamma(b)$. The multiplicity of the conjugate points is the dimension of the vector space of all such Jacobi fields.*

Recall a Jacobi field is one that satisfies $\frac{D^2 J}{dt^2} + R(\frac{d\gamma}{dt}, J)\frac{d\gamma}{dt} = 0$ where $\frac{D}{dt}$ denotes the covarient derivative and R is the curvature.

We now want to introduction the idea of the index of a geodesic. To do so, we quote the Index Theorem, treating it as a definition, which tells us that the index of a geodesic is the number obtained by counting conjugate points along a geodesic.

Theorem 2 (Index Theorem). *The index λ of a geodesic γ from $\gamma(0) = I$ to $\gamma(1) = -I$ is equal to the number of points (counted with multiplicity) such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ. This index λ is always finite.*

A discussion of index and proof of the Index Theorem may be found in [1] pp.242-248.

The proof of Bott Periodicity will require the following essential result from Morse Theory; see Appendix A.

Theorem 3. *For a differential manifold M, if the space of minimal geodesics from p to q is a topological manifold, denoted Ω, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$, then $\pi_i(\Omega) \cong \pi_{i+1}(M)$ for $i = 0, ..., \lambda_0 - 2$*

The next section gives a proof that every non-minimal geodesic γ from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$. Our desired isomorphism 2.7 is then a consequence of Theorem 3. In particular, we consider $M = SU(2m)$, $p = I$, and $q = -I$. Lemma 1 gives that the space of minimal geodesics is homeomorphic to $G_m(\mathbb{C}^{2m})$, so indeed it is a topological manifold. Since every non-minimal geodesic from I to $-I$ has index $\geq 2m+2$, we have that $\pi_i\Omega^\alpha \cong \pi_{i+1}SU(2m)$ for sufficiently large m, concluding the proof of Bott Periodicity for the Unitary group.

2.4 Bound on Index of Non-Minimal Geodesics

Theorem 4. *Every non-minimal geodesic γ from I to $-I$ in $\mathrm{SU}(2m)$ has index $\geq 2m+2$*

Proof. Consider a non-minimal geodesic γ of $\mathrm{SU}(2m)$ from I to $-I$. As in the proof of Lemma 1, we may write it in the form $\gamma(t) = exp(tA)$ where $\gamma(0) = I$ and $\gamma(1) = -I$. Furthermore, we may assume without loss of generality that A is diagonal with entries $k_1\pi i, ..., k_{2m}\pi i$ where k_i odd, $\sum k_i = 0$, and $k_1 \leq k_2 \leq ... \leq k_{2m}$.

Our goal is to find a lower bound on the index of this geodesic by counting conjugate points of $exp(tA)$, $0 \leq t \leq 1$. The following lemma gives us a way to find conjugate points:

Lemma 2. *Conjugate points occur along our geodesic whenever $t = \pi k/\sqrt{e_i} \in [0, 1]$ where k is a non-zero integer and e_i is a positive eigenvalue of $K_V : T_I \mathrm{SU}(2m) \to T_I \mathrm{SU}(2m)$ that maps $W \mapsto R(V, W)V$ for $V = \frac{d\gamma}{dt}(0)$.*

Proof. To see this, note that K_V is self-adjoint:

$$\left\langle K_V(W), W'\right\rangle = \left\langle R(V, W)V, W'\right\rangle = \left\langle R(V, W')V, W\right\rangle = \left\langle W, K_V(W')\right\rangle$$

(see [1], p.91). Hence, we may find an orthonormal basis of eigenvectors $U_1, ..., U_{2m}$ for $T_I SU(2m)$ such that $K_V(U_i) = e_iU_i$.

Recall that on a differential manifold, any vector X tangent to a point $c(t_0)$ along a curve c can be uniquely extended to a parallel vector field $\tilde{X}(t)$ along the curve (see [1], p.52). That is, $\tilde{X}(t_0) = X$ and the covariant derivative $\frac{DX'(t)}{dt}$ is everywhere zero.

In particular, we may extend $V, U_1, ..., U_{2m}$ to parallel vector fields $\tilde{V}, \tilde{U}_1, ..., \tilde{U}_{2m}$ along γ such that $\tilde{V}(0) = V$ and $\tilde{U}_i(0) = U_i$ for $i = 1, ..., 2m$.

Notice $SU(2m)$ is locally symmetric; that is, if X, Y, Z are parallel vector fields, then so is $R(X, Y)Z$. In fact, since $SU(2m)$ is a compact connected Lie group, it is a globally symmetric space, see [3].

Therefore $K_{\tilde{V}}(\tilde{U}_i) = R(\tilde{V}, \tilde{U}_i)\tilde{V}$ is a parallel vector field along γ extending $K_V(U_i) = e_iU_i$. By uniqueness, $K_{\tilde{V}}(\tilde{U}_i) = e_i\tilde{U}_i$.

Consider $W_i = sin(\sqrt{e_i}t)U_i$ for $e_i > 0$. Then

$$\frac{D^2W_i}{dt^2} + K_V(W_i) = \frac{d^2}{dt^2}(sin(\sqrt{e_i}t))U_i + sin(\sqrt{e_i}t)e_iU_i = 0$$

and so W_i is a Jacobi Field which vanishes at multiples of $\pi/\sqrt{e_i}$. The completes the proof as we then have conjugate points along our geodesic at all times $t \in [0, 1]$ that are a multiple of $\pi k/\sqrt{e_i}$. □

We now calculate the positive eigenvalues of $K_V : T_I\,\mathrm{SU}(2m) \to T_I\,\mathrm{SU}(2m)$. In Appendix B, we prove Lemma 7 that states $R(A, W)A = \frac{1}{4}[[A, W], A]$ where $[X, Y]$ denotes $XY - YZ$.

Thus for matrix $W = \Big(w_{j,l}\Big)$, that is the matrix with value $w_{j,l}$ in the j^{th} row and l^{th} column, we have

$$K_V(W) = \frac{1}{4}[[A, W], A] = \frac{\pi}{4} i \left[\Big((k_j - k_l)w_{j,l}\Big), A\right] = \frac{\pi^2}{4}\Big((k_j - k_l)^2 w_{j,l}\Big)$$

Fixing j_0, l_0 and defining $X = \Big(x_{j,l}\Big)$ by $x_{j_0,l_0}, x_{l_0,j_0} = 1$ and otherwise $x_{j,l} = 0$, we have $K_V(X) = \frac{\pi^2}{4}(k_{j_0} - k_{l_0})^2 X$.

Thus, we have positive eigenvalues $\frac{\pi^2}{4}(k_j - k_l)^2$ whenever $k_j \neq k_l$. Lemma 2 gives conjugate points at $t = \frac{2}{k_j - k_l}, \frac{4}{k_j - k_l}, \frac{6}{k_j - k_l}, \ldots$. Hence there are $\frac{k_j - k_l}{2} - 1$ conjugate points for $t \in (0, 1)$ for fixed k_j, k_l.

By the Index Theorem, we have index at least

$$\sum_{k_j \neq k_l} (\frac{k_j - k_l}{2} - 1) = \sum_{k_j < k_l} (k_j - k_l - 2).$$

As our geodesic is non-minimal, we have $\sum_{i=1}^{2m} k_i = 0$ with not all $k_i = \pm 1$. Thus we must have one of the following cases:

Case 1: m of the k_i are positive and m are negative. Then there is at least one ≥ 3 and one ≤ 3 so we have index at least

$$\sum_1^{m-1}(3 - (-1) - 2) + \sum_1^{m-1}(1 - (-3) - 2) + (3 - (-3) - 2) = 4m.$$

Case 2: at least $m + 1$ of the k_i are positive. Then at least one of our negative k_i is ≤ -3. Hence index at least

$$\sum_1^{m+1}(1 - (-3) - 2) = 2(m + 1).$$

Case 3: at least $m + 1$ of the k_i are negative. Then at least one positive k_i is ≥ 3. Hence index at least

$$\sum_1^{m+1}(3 - (-1) - 2) = 2(m + 1).$$

In every case we have index $\geq 2m + 2$. This completes the proof of Theorem 3. $\square$

2.5 Calculating Homotopy Groups

Bott Periodicity then tells us that $\pi_i\,\mathrm{U}$ is completely determined by $\pi_0\,\mathrm{U}$ and $\pi_1\,\mathrm{U}$. From equation 2.2 above, we have that

$$\pi_0\,\mathrm{U} \cong \pi_0\,\mathrm{U}(1) \quad \text{and} \quad \pi_1\,\mathrm{U} \cong \pi_1\,\mathrm{U}(1)$$

As U(1) may be identified with the unit circle in $\mathbb{C}$, we conclude

$$\pi_n\,\mathrm{U} \cong \begin{cases} 0 & n \equiv 0 \ (mod\ 2) \\ \mathbb{Z} & n \equiv 1 \ (mod\ 2) \end{cases}$$

Chapter 3

Orthogonal Case

3.1 Orthogonal Group

The *orthogonal group of degree* n, denoted $\mathrm{O}(n)$, is the group of real $n \times n$ matrices that preserve the inner product $\langle x, y \rangle = \sum x_i y_i$ for $x, y \in \mathbb{R}^n$, that is $\langle x, y \rangle = \langle Ax, Ay \rangle$. Equivalently, $\mathrm{O}(n)$ consists of all real $n \times n$ matrices A such that $A^* A = I$.

Note that $\mathrm{O}(n)$ is a subgroup of $\mathrm{U}(n)$.

3.2 Bott Periodicity and Outline of Proof

We start with a sketch of the proof of Bott Periodicity for the Orthogonal group:
Theorem 5. *For $i \geq 0$, $\pi_i \mathrm{O} \cong \pi_{i+8} \mathrm{O}$.*

Here O denotes the direct limit of $\mathrm{O}(n)$. To see that the homotopy groups of $\mathrm{O}(n)$ stabilize, note that similar to fiber bundle 2.1 from the preceding section, we have a fiber bundle,

$$\mathrm{O}(n) \to \mathrm{O}(n+1) \to S^n$$

Since $\pi_k S^n \cong 0$ for $0 < k < n$, it follows from the long exact sequence of homotopy groups

$$... \to \pi_{k+1} S^n \to \pi_k \mathrm{O}(n) \to \pi_k \mathrm{O}(n+1) \to \pi_k S^n \to ...$$

that $\pi_k \mathrm{O}(n) \cong \pi_k \mathrm{O}(n+1)$ for $n > k+1$.

Notice the statement of Bott Periodicity for the Orthogonal group is analogous to that for the Unitary group, but is now 8-periodic rather than 2-periodic. Also, the proof will be analogous to that of the complex case. Recall that we proceeded there by introducing the special unitary group and showed that its space of minimal geodesics is a smooth manifold with i^{th} homotopy group isomorphic to both the $(i-1)^{th}$ and $(i+1)^{th}$ homotopy groups of U. Similarly, here we introduce the space of complex structures, $\Omega_1(n)$, and examine the space of minimal geodesics on it. However, to show the 8-periodic structure of the theorem, it will be necessary to extend the idea of a complex structure to structures $\Omega_2(n), \Omega_3(n), ...$

We then introduce three lemmas that argue these are all smooth manifolds and describe their geodesics. From these lemmas, Theorem 3 above will let us deduce $\pi_i \Omega_k(n) \cong \pi_{i-1}\Omega_{k+1}(n)$. A further relationship identifying $\Omega_{8k}(n)$ with the orthogonal group lets us conclude $\pi_i \, \mathrm{O}(n) \cong \pi_{i+8} \, \mathrm{O}(n)$ for large n. Then, just as $\pi_i \Omega$ "linked" $\pi_{i-1} \, \mathrm{U}$ with $\pi_{i+1} \, \mathrm{U}$, are able to "link" $\pi_i \, \mathrm{O}(n)$ and $\pi_{i+8} \, \mathrm{O}(n)$ with the appropriate homotopy groups of $\Omega_1, ..., \Omega_8$. Finally, as the homotopy groups for $\mathrm{O}(n)$ stabilize, we may then pass to the limit to conclude $\pi_i \, \mathrm{O} \cong \pi_{i+8} \, \mathrm{O}$.

3.3 Complex Structures

Definition 2. *We call J a complex structure on $\mathbb{R}^n$ if $J \in \mathrm{O}(n)$ and $J^2 = -I_n$. Denote the space of complex structures $\Omega_1(n) \subset \mathrm{O}(n)$.*

Considering an orthonormal matrix A in block-diagonal form, it's immediate that n must be even for A to be a complex structure.

For instance, when $n = 4$ we have complex structures

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; J_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

These J_1, J_2, and J_3 correspond to the quaternions i, j, and k, respectively. Indeed, one verifies $J_a J_b = -J_b J_a$ for $1 \leq a < b \leq 3$. Similarly, assuming from now on that n is divisible by a sufficiently large power of 2 allows us to fix complex structure $J_1, ..., J_{k-1}$ on $\mathbb{R}^n$ such that $J_a J_b = -J_b J_a$ for $1 \leq a < b \leq k-1$.

Definition 3. *Define $\Omega_k(n)$ to be the space of complex structures that anti-commute with fixed $J_1, ..., J_{k-1}$.*

So we have $\Omega_k(n) \subset \Omega_{k-1}(n) \subset ... \subset \Omega_1(n) \subset \mathrm{O}(n)$, which motivates the notation $\Omega_0(n) = \mathrm{O}(n)$. It is a consequence of Lemma 4 below that $\Omega_k(n)$ is non-empty for n divisible by a sufficiently large power of 2.

Just as we defined O, the direct limit of $\mathrm{O}(n)$, it will be useful to consider the direct limit of $\Omega_k(n)$ for each fixed k as $n \to \infty$. To justify that the direct limit as $n \to \infty$ exists, notice that there is a natural inclusion from $\phi : \Omega_k(n) \to \Omega_k(n + n')$ given as follows: Fix complex structures $J_1', ..., J_k'$ on $\mathbb{R}^{n'}$ and let $J_1, ..., J_k$ denote the complex structures on $\mathbb{R}^n$ that determine $\Omega_k(n)$. Then for any $J \in \Omega_k(n)$, define $\phi(J)$ by $J \oplus J_k'$. Notice this is a complex structure on $\mathbb{R}^n \oplus \mathbb{R}^{n'}$ and $J \oplus J_k'$ anticommutes with $J_i \oplus J_i'$ for $i = 1, ..., k-1$. Hence for each k we have a well-defined direct limit of $\Omega_k(n)$ as $n \to \infty$ which we will denote by Ω_k.

3.4 First Results

The following three lemmas give some first results about the structures $\Omega_k(n)$.

Lemma 3. *Each $\Omega_k(n)$ is a smooth, totally geodesic submanifold of $O(n)$.*

Proof. We may find a neighborhood of I_n in $\mathrm{O}(n)$ with all points in the form $\exp(A)$ where A skew-symmetic and small, i.e. has norm near 0. Likewise, for each $J \in \Omega_k(n)$ we may find a neighborhood of J in $\mathrm{O}(n)$ with all points in the form $Jexp(A)$ where A is small and skew-symmetric.

We argue that considering this neighborhood's intersection with $\Omega_k(n)$ puts linear conditions on A. That is, locally $\Omega_k(n)$ takes the values $J\exp(A)$ as A varies over some linear subspace of our tangent space.

To see this, notice $J\exp(A)$ belongs to $\Omega_k(n)$ if and only if both (i) $J\exp(A)$ is a complex structure and (ii) $J\exp(A)$ anti-commutes with $k-1$ fixed complex structures $J_1, ..., J_{k-1}$.

Notice (i) is equivalent to $J^{-1}AJ + A = 0$, and hence $AJ = -JA$, by the relation

$$\exp(J^{-1}AJ)\exp(A) = J^{-1}\exp(A)J\exp(A) = -(J\exp(A))^2 = I$$

while (ii) is equivalent to $J_i^{-1}AJ_i - A = 0$, and hence $AJ_i = J_iA$, for $1 \leq i \leq k-1$ by

$$\exp(J_i^{-1}AJ_i)\exp(-A) = J_i^{-1}\exp(A)J_i\exp(-A) = -(J_i\exp(A))^2 = I$$

These determine linear conditions on A. Therefore, $\Omega_k(n)$ is a smooth, totally geodesic submanifold of $O(n)$. □

Lemma 4. *For fixed J_l, the space of minimal geodesics from J_l to $-J_l$ in $\Omega_l(n)$ is homeomorphic to $\Omega_{l+1}(n)$, for $0 \leq l < k$.*

Proof. Fix all anti-commuting complex structures $J_1, ..., J_{l-1}$. These determine $\Omega_l(n)$. Fixing also some $J_l \in \Omega_l(n)$ determines $\Omega_{l+1}(n)$. Our proof will proceed in two parts.

For the first part, we consider $J \in \Omega_{l+1}(n)$ and associate it with a minimal geodesic as follows:

Define $A = J_l J$, then $A^2 = -JJ_lJ_lJ = -I$ so A is a complex structure. Furthermore, for $i < l$, $AJ_i = -J_lJ_iJ = J_iA$ and $AJ_l = J = -J_lA$. We claim $\gamma(t) = J_l exp(\pi t A)$, for $t \in [0,1]$, is a minimal geodesic in $\Omega_l(n)$ from J_l to $-J_l$.

First note that $J_l \exp(\pi t A) = \exp(\pi t J_l A J_l^{-1}) J_l = \exp(-\pi t A) J_l$, hence $\gamma(t)^2 = -I$ for any $t \in [0,1]$. Furthermore, $(J_l exp(\pi t A))J_i = J_l J_i \exp(\pi t J_i^{-1} A J_i) = -J_i(J_l \exp(\pi t A))$. So indeed $\gamma(t) \in \Omega_l(n)$ for all $t \in [0,1]$.

Next, as $A^2 = -I$ and $AA^* = I$, we have $A = -A^*$ so A is skew-symmetric and thus $exp(\pi t A)$ defines a geodesic in $\mathrm{O}(n)$. Also, changing our basis for A we may obtain a matrix of the form $C = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & A_k \end{pmatrix}$ where each A_i denotes a block $\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$ where a_i positive. As $A^2 = -I$ we have $C^2 = -I$ and hence $a_i = 1$ for all i. Thus

$$\exp(\pi t A_i) = \sum_{k \geq 0} \frac{1}{k!} A_i^k = \begin{pmatrix} cos(\pi t) & sin(\pi t) \\ -sin(\pi t) & cos(\pi t) \end{pmatrix}$$

and hence $\exp(\pi T A T^{-1}) = -I$. Therefore $\gamma(t)$ is a geodesic in $\Omega_l(n)$, connecting $\gamma(0) = J_l$ and $\gamma(1) = -J_l$ as desired.

Finally, minimality follows since for any B of composed of diagonal blocks $\begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}$ where $exp(\pi B) = -I$ we must have all b_i odd integers. Then, since $\langle B, B \rangle = 2 \sum_i b_i^2$, we see minimality is equivalent to $|b_i| = 1$ for each b_i.

For the second part of the proof, we consider a minimal geodesic $\gamma(t)$ from $\gamma(0) = J_l$ to $\gamma(1) = -J_l$ in $\Omega_l(n)$ and show that it may be associated with an element of $\Omega_{l+1}(n)$. We may express a geodesic $\gamma(t)$ in the form $J_l exp(\pi t A)$ for some skew-symmetric A. Notice this is analogous to how we expressed the geodesics of $\mathrm{SU}(2m)$ in the unitary case of the proof.

We will associate each geodesic $J_l exp(\pi t A)$ to an element of $\Omega_{i+1}(n)$ by showing $J_l A \in \Omega_{l+1}(n)$.

Changing basis we may express A in the block form B where, by minimality, all non-zero entries have absolute value 1. It follows $B^2 = -I$ and hence $A^2 = -I$ so A is a complex structure.

Furthermore, by (i) in Lemma 3 above we have $AJ_l = -J_l A$. Hence

$$(J_l A)^2 = -AJ_lJ_lA = -I, \quad (J_lA)J_l = -J_l(J_lA)$$

And from (ii) above we obtain $AJ_i = J_iA$ for $i < l$, so

$$(J_lA)J_i = J_lJ_iA = -J_i(J_lA)$$

Thus $J_l A \in \Omega_{l+1}(n)$, as desired. □

Lemma 5. *For each $k \geq 0$, there is a real-valued function g_k such that:*

(i) any non-minimal geodesic from J to $-J$ in $\Omega_k(n)$ has index at least $g_k(n)$ and

(ii) $g_k(n)$ tends to infinity as $n \to \infty$.

Proof. The case $k = 0$ is analogous to the proof of Theorem 3 in the complex case. For it, we may denote the matrix corresponding to the non-minimal geodesic $exp(\pi t A)$ for some skew-symmetric A such that $exp(\pi A) = -I$. In some orthonormal basis we have that A is of the form $\begin{pmatrix} A_1 & 0 & ... & 0 \\ 0 & A_2 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & A_m \end{pmatrix}$ with blocks down the diagonal $A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$ for odd integers $0 < a_1 \leq a_2 \leq ... \leq a_m$. As before, we calculate index of $exp(t\pi A)$ by counting the conjugate points, which by Lemma 2 are determined by the eigenvalues of $K_{\pi A}$. Recall that $K_{\pi A} W = \frac{\pi^2}{4}[[A, W], A]$, which then implies that $\pi^2(a_i + a_j)^2/4$ for all $i \neq j$ are eigenvalues for $K_{\pi A}$. Hence by Lemma 2, we have conjugate points at $\frac{2}{a_i+a_j}, \frac{4}{a_i+a_j}, ...$ for each $a_i \neq a_j$. There are $\frac{a_i+a_j-1}{2}$ such values between 0 and 1, so by the Index Theorem, we have index at least

$$\sum_{a_i \neq a_j} \frac{a_i + a_j - 1}{2} = \sum_{a_i < a_j} a_i + a_j - 1$$

As our geodesic is non-minimal, we must have some $a_i \neq 1$, so in particular $a_m \geq 3$. Thus we have index at least $\sum_{i=1}^{m-1}(1 + 3 - 1) = 2m - 2 = n - 2$. Hence $g_0(n) = n - 2$ satisfies (i) and (ii) of the lemma.

The cases when $k > 0$ are similar but more involved; we outline the the proof for $k \not\equiv 2 \pmod 4$. For a more detailed discussion, see [4], pp.143-148.

When $k \not\equiv 2$ $(mod\ 4)$, we consider geodesics of the form $J \exp(t\pi A)$ from J to $-J$ where A is skew, anti-commutes with J, and commutes with $J_1, ..., J_{k-1}$.

We want to study the eigenvectors of $K_{\pi A}$ where $K_{\pi A} W = \frac{\pi^2}{4}[[A, W], A]$. First, as above we decompose $\mathbb{R}^n$ into "eigenspaces" $M_1 \oplus M_2 \oplus ... \oplus M_s$ where each M_i is closed under the action of $J_1, .., J_{k-1}, J, A$ and has no proper, non-trivial subspace that is also closed under these actions. One may argue that all of the M_i are isomorphic with each other and have dimension 2^d, the smallest power of two greater than $k + 1$. Hence $n = 2^d s$.

There are two imaginary eigenvalues for each restriction $A|_{M_i}$ which we denote $\pm i a_i$. We find that $K_{\pi A}$ has eigenvalues $\frac{\pi^2}{4}(a_i + a_j)^2$ for $a_i \neq a_j$. Then arguing analogously to the $k = 0$ case, we conclude that we have index at least $n/2^d - 1$. Note that for fixed k, this tends to infinity as $n \to \infty$. □

3.5 Concluding Bott Periodicity

With these three lemmas, it follows from Theorem 3 in the preceding section that $\pi_i\Omega_k(n) \cong \pi_{i-1}\Omega_{k+1}(n)$ for sufficiently large n. Passing to the direct limit as $n \to \infty$ we conclude:

Theorem 6. $\pi_h\Omega_0 \cong \pi_{h-1}\Omega_1 \cong \pi_{h-2}\Omega_2 \cong ... \cong \pi_1\Omega_{h-1}$

The final ingredient to prove Bott Periodicity is to show that Ω_k are 8-periodic, i.e. $\Omega_0 \cong \Omega_8$. One may describe each $\Omega_k(n)$ explicitly in terms of Ω_{k-1} for $k = 1, 2, ..., 8$. Refer to [4], pp.138-141 to see this done in detail. In particular, it follows:

Lemma 6. *The space* $\Omega_8(16n)$ *is diffeomorphic to orthogonal group* $\mathrm{O}(n)$.

Passing to the limit, we conclude O is diffeomorphic to Ω_8. Finally, since Theorem 3 gives

$$\pi_h\Omega_8 \cong \pi_{h+1}\Omega_7 \cong ... \cong \pi_{h+7}\Omega_1 \cong \pi_{h+8}\Omega_0,$$

we have, as desired,

$$\pi_h \, \mathrm{O} \cong \pi_{h+8} \, \mathrm{O} \, .$$

This finishes our description of the proof of Bott Periodicity for the Orthogonal group. Now one just needs to calculate $\pi_0 \, \mathrm{O}, ..., \pi_7 \, \mathrm{O}$ to completely determine $\pi_n \, \mathrm{O}$ for any $n \in \mathbb{N}$. It then turns out:

$$\pi_n \, \mathrm{O} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \equiv 0, 1 \ (mod\ 8) \\ \mathbb{Z} & n \equiv 3, 7 \ (mod\ 8) \\ 0 & n \equiv 2, 4, 5, 6 \ (mod\ 8) \end{cases}$$

We conclude by calculating the first few of these homotopy groups.

3.6 Calculating Homotopy Groups

Since $\pi_k \, \mathrm{O}(n)$ stabilizes to $\pi_k \, \mathrm{O}$, we proceed in the following calculations by determining $\pi_k \, \mathrm{O}(n)$ for large n.

3.6.1 $\pi_0 O$

Consider $\mathrm{O}(n)$, for fixed n even. Any $A \in \mathrm{O}(n)$ can be expressed as $A = I'B$ where $I' = \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & I \end{smallmatrix} \right)$ and $det\ B = 1$.

As B is an orthogonal matrix, we may change basis such that it is of the form $C = \left(\begin{smallmatrix} R_1 & 0 & ... & 0 \\ 0 & R_2 & ... & 0 \\ & & ... & \\ 0 & 0 & ... & R_m \end{smallmatrix} \right)$ where each R_k is of the form $\left(\begin{smallmatrix} cos\ \theta_k & -sin\ \theta_k \\ sin\ \theta_k & cos\ \theta_k \end{smallmatrix} \right)$ for some θ_k.

By considering the path $(1-t)\theta_k$ for $t \in [0,1]$, we have a path in $O(n)$ from C to I. And thus also a path in $\mathrm{O}(n)$ from A to I'.

Clearly, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & I \end{smallmatrix}\right)$ are not in the same path component of $\mathrm{O}(n)$ as they have differing determinants.

Hence $O(n)$ has two path connected components, so $\pi_0 \mathrm{O}(n)$ stabilizes to $\pi_0 \mathrm{O} \cong \mathbb{Z}/2\mathbb{Z}$.

3.6.2 $\pi_1 \mathrm{O}$

For large n, we have $\pi_1 O(n) \cong \pi_0 \Omega_1(n)$. We count the path components of $\Omega_1(n)$.

First consider $n = 2$. For any $A \in \Omega_1(2)$, for some $T \in \mathrm{O}(2)$, we have $TAT^{-1} = \left(\begin{smallmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{smallmatrix}\right)$ for some θ. Then as $A^2 = -I$, we may solve for θ as $\frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. Thus TAT^{-1} is either $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ or $B' = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. In fact, these are similar $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) B \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = B'$. So $A = SBS^{-1}$ for some $S \in O(2)$

Likewise, for n even, any $A \in \Omega_1(n)$ is of the form $S \left(\begin{smallmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ 0 & 0 & \dots & B \end{smallmatrix}\right) S^{-1}$ where $S \in \mathrm{O}(n)$.

When $\det S = 1$, we showed above that we have a path in $\mathrm{O}(n)$ from $S \to I$ and hence a path in $\Omega_1(n)$ from A to $\left(\begin{smallmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ 0 & 0 & \dots & B \end{smallmatrix}\right)$.

When $\det S = -1$, we have a path in $\mathrm{O}(n)$ from $S \to \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \end{smallmatrix}\right)$ and hence a path in $\Omega_1(n)$ from A to $\left(\begin{smallmatrix} B' & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ 0 & 0 & \dots & B \end{smallmatrix}\right)$.

Furthermore, we cannot have B connected by a path to B' in $\Omega(2)$, else we'd have I and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ connected by a path in $\mathrm{O}(2)$–a contradiction. Similarly, $\left(\begin{smallmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ 0 & 0 & \dots & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} B' & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ 0 & 0 & \dots & B \end{smallmatrix}\right)$ are not connected by a path in $\Omega_1(n)$. So we have precisely two path components. Thus $\pi_1 O \cong \pi_0 \Omega_1 \cong \mathbb{Z}/2\mathbb{Z}$.

3.6.3 $\pi_2 \mathrm{O}$

We make use of the relation $\pi_2 \mathrm{O}(n) \cong \pi_0 \Omega_2(n)$ for large n. To study $\Omega_2(n)$, it will be useful to assume 4 divides n and make use of the quaternions. First, consider the case $n = 4$.

Each quaternion $q = a + bi + cj + dk$ may be expressed as a 4×4 real matrix, or equivalently, the 2×2 complex matrix

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

Hence, corresponding to the quaternions i, j, k we have, respectively, the complex matrices

$$\mathrm{I} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathrm{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{K} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Each belongs to $\Omega_1(4)$. Consider $\Omega_2(4)$ as the subgroup of matrices that anti-commute with I. Consider such a $\mathrm{L} \in \Omega_2(4)$. The relation $\mathrm{LI} = -\mathrm{IL}$ gives that the diagonal entries of L are zero. Then since L^2 is the negative identity, we may conclude that L is of the form $\left(\begin{smallmatrix} 0 & a+bi \\ -a+bi & 0 \end{smallmatrix}\right)$ for some reals a, b such that $a^2 + b^2 = 1$. Hence, L denotes multiplication by the quaternion $aj + bk$.

Furthermore, it is clear that any 2×2 complex matrix M denoting a quaternion of the form $cj+dk$ for $c, d \in \mathbb{R}$ and $c^2+d^2 = 1$ belongs to $\Omega_2(4)$. This is equivalent to $sin(\theta)j+cos(\theta)k$ for some $\theta \in [0, 2\pi)$. Hence any two elements of $\Omega_2(4)$ may be expressed as $sin(\theta_1)j + cos(\theta_1)k$ and $sin(\theta_2)j + cos(\theta_2)k$ and are connected by a path in $\Omega_2(4)$ by varying θ from θ_1 to θ_2.

For higher multiples of 4, we consider $\Omega_2(4k)$ determined by the complex structure $\left(\begin{smallmatrix} \mathrm{I} & \dots & 0 \\ & \dots & \\ 0 & \dots & \mathrm{I} \end{smallmatrix}\right)$ then proceed analogously. We likewise find $\Omega_2(4k)$ is path connected and hence $\pi_2 \mathrm{O} \cong \pi_0\Omega_2 \cong 0$.

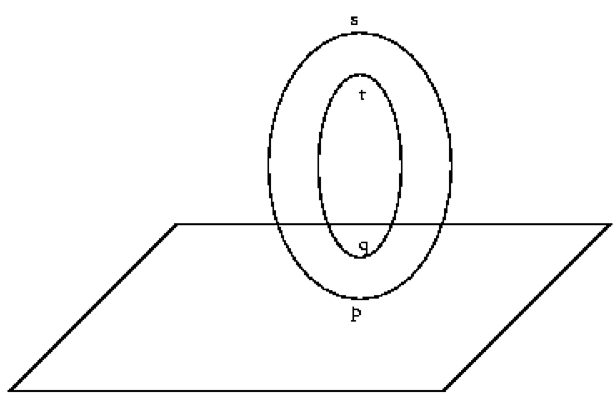
Appendix A

Discussion of Morse Theory

There are a number of proofs for Bott Periodicity. Here, we followed the approach of Milnor appearing in the final part of [4], which depends on the Morse Theory developed earlier in Milnor's book.

The basic idea of Morse Theory is to connect the structure of a differential manifold M with the behavior of a real valued function, called a Morse function, defined on M. The condition to be a Morse function is easily satisfied; the "typical" function $f : M \to \mathbb{R}$ is Morse, and if not it may be made Morse by a small perturbation. A key result of Morse theory then is that the homotopy type of M is determined entirely by the behavior of f at its non-degenerate critical points (i.e. critical points where the Hessian of f is non-singular). In particular, we assign to each critical point an index λ determined by the Hessian of f at that point. Then M has the homotopy type of a CW-complex with a cell of dimension λ for each critical point.

For instance, consider a torus T in suspension over a plane as in the figure.



We may assign a height function $h : T \to \mathbb{R}$ that indicates the minimal distance from each point of the torus to the plane. The labeled points, p, q, t, and s, are the critical points of h. We assign an index to each critical point according to the number of independent directions one can move and have h decrease. Thus,

$$\text{index}(p) = 0, \quad \text{index}(q) = 1, \quad \text{index}(t) = 1, \quad \text{index}(s) = 2.$$

The result mentioned above then tells us that T should have the homotopy type of a CW-complex with one 0 cell, two 1 cells, and one 2 cell. Indeed, this is the familiar representation for the torus.

In fact, the result may be strengthened. Consider the section of T from height $h(p)$ to some height less than $h(q)$. Notice this has the homotopy type of a 0 cell which is the index of p. Likewise the section from $h(p)$ to some height between $h(q)$ and $h(t)$ has the homotopy type of a 1 cell attached to a 0 cell which are the indices of p and q. And so on.

Using the language of Riemannian Geometry, there is an analogous result regarding the path space of geodesics on a Riemannian manifold N. Just as we considered non-degenerate critical points and assigned to each an index, we consider the non-minimal geodesics connecting points p and q in N and assign to each one an index λ according to our Index Theorem. Recall this is by counting the conjugate points along the geodesic. Then, analogous to the above result describing the topology of M, we have the Fundamental Theorem of Morse Theory which states that the path space between p and q in N has the homotopy type of a CW-complex which contains one cell of dimension λ for each geodesic of index λ.

Milnor uses the Fundamental Theorem as a critical tool in his proof for Theorem 3. See [4] pp.118-123.

Recall the statement of Theorem 3:

For a differential manifold M with $p, q \in M$, if the space of minimal geodesics from p to q is a topological manifold, denoted Ω, and if every non-minimal geodesic from p to q has index $\geq \lambda_0$, then $\pi_i(\Omega) \cong \pi_{i+1}(M)$ for $i = 0, ..., \lambda_0 - 2$.

In the proof of Bott Periodicity for the Unitary group, this is what enabled us to conclude $\pi_i \Omega^\alpha \cong \pi_{i+1} SU(2m)$, the most involved to prove of the isomorphisms connecting $\pi_{i-1}U(m)$ with $\pi_{i+1}U(m)$ for large m.

Also, Theorem 3 was essential to the proof of Bott Periodicity for the Orthogonal group, where it tied together Lemmas 3, 4, and 5 to give us the isomorphisms $\pi_i \Omega_k(n) \cong \pi_{i-1}\Omega_{k+1}(n)$, for large n, letting us conclude $\pi_{i+8}\Omega_0(n) \cong \pi_i \Omega_8(n)$.

Appendix B

Curvature of a Lie Group

Lemma 7. *The curvature of a Lie Group may be described by,*

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$$

Proof. Our curvature is defined by $R(X,Y)Z = -\nabla_X(\nabla_Y Z) + \nabla_X(\nabla_Y Z) + \nabla_{[X,Y]}Z$ where ∇ denotes the Riemannian connection (cf. [1], p. 55).

Since we are considering a Lie group, we have for V in the tangent space at I that $\nabla_V V = 0$ (cf. [1],p. 81). Hence for X, Y in the tangent space at I,

$$\nabla_X Y + \nabla_Y X = \nabla_X X + \nabla_X Y + \nabla_Y X + \nabla_Y Y = \nabla_{X+Y}(X+Y) = 0$$

and by definition

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

So we obtain $\nabla_X Y = \frac{1}{2}[X,Y]$. Plugging this into our definition of curvature yields

$$R(X,Y)Z = \frac{1}{4}[Y,[X,Z]] - \frac{1}{4}[X,[Y,Z]] + \frac{1}{2}[[X,Y],Z]$$

$$= \frac{1}{4}\left([[Z,X]Y] + [[Y,Z],X] + [[X,Y],Z]\right) + \frac{1}{4}[[X,Y],Z] = \frac{1}{4}[[X,Y],Z]$$

Here we used the Jacobi Identity $[[Z,X]Y] + [[Y,Z],X] + [[X,Y],Z] = 0$ which can be easily verified directly. □

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